Growth and Redistribution: The Hedging Perspective

Larry Samuelson, Jakub Steiner

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Overview

investment portfolio problem

- assets
- hedging
- rebalancing

macroeconomic problem

- people
- wealth allocation
- redistribution

Kelly's betting

- information-theoretic investment portfolio optimization
- optimality conditions formulated in terms of redistribution

our contribution

- we add macro-motivated constraints
- extend the solution using large deviations theory

1 Kelly's Betting

2 Macroeconomic Reinterpretation

3 Analysis

Kelly '56

assets $i \in I$, discrete time t

each asset i has a gross return $r(i, \omega_t)$ in each period t

- iid states $\omega_t \sim q^0(\omega) \in \Delta(\Omega)$

stationary portfolio $p(i) \in \Delta(I)$

- share $p(i)$ of wealth invested in each asset i each morning

Objective

Kelly maximizes long-run wealth along typical histories:

- in which the empirical distribution of ω_t approximates $q_0(\omega)$.

growth-optimal portfolio

maximizes long-run growth rate of wealth:

$$\max_{p(i)} \mathrm{E}_{q^0(\omega)} \ln \left(\sum_i p(i) r(i, \omega) \right)$$

it outgrows any other portfolio with probability 1

Example

$\Omega = I$

$\omega \sim q^0(\omega)$

exogenous return function: $r(i, \omega) = \text{const.} \times \mathbb{1}_{\omega=i}$

optimal allocation: $p^*(i) = q^0(i)$

in contrast, maximization of expected return leads to a.s. bankruptcy

Redistribution

maintaining a portfolio requires periodical redistribution:

- from winners to losers

Kelly's FOC:

- no systematic redistribution

Outcome Distribution

def: conditional outcome distribution

share of wealth in asset i at the end of a period with $\omega_t = \omega$:

$$o_p(i \mid \omega) := \frac{p(i)r(i,\omega)}{\sum_j p(j)r(j,\omega)}.$$

def: outcome distribution

share of wealth in i at the end of a random period:

$$o_p(i) := \mathrm{E}_{q^0(\omega)}\, o(i \mid \omega).$$

Kelly's Optimality Condition

First Order Condition

Optimal portfolio eliminates systematic redistribution of investments:

$$p^*(i) = o_{p^*}(i).$$

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Macroeconomic Reinterpretation

individuals $i \in I$ with random gross returns $r(i, \omega_t)$

social planner maximizes long-run growth rate of aggregate wealth

we add two novel features:

- a constraint on the endowment distribution $p(i)$
- constrained control over the return function $r(i, \omega)$

set $\mathcal{P}$ of feasible policies $\big(p(i), r(i, \omega)\big)$

- needs not be a product set

Main Result

Proposition

Optimal p^* minimizes KL-divergence from the outcome distribution

$$p^*(i) \in \underset{p(i) \in \mathcal{A}^*}{\arg\min}\, \mathrm{KL}\left(o_{p^*,r^*}(i) \parallel p(i)\right),$$

where $\mathcal{A}^*$ is the set of feasible p given the optimized returns $r^*(i,\omega)$.

recall: $\mathrm{KL}\left(o(i) \parallel p(i)\right) = \mathrm{E}_{o(i)} \ln \frac{o(i)}{p(i)}$ is a pseudo-distance

note the myopicity: the outcome distribution is fixed to o_{p^*,r^*}

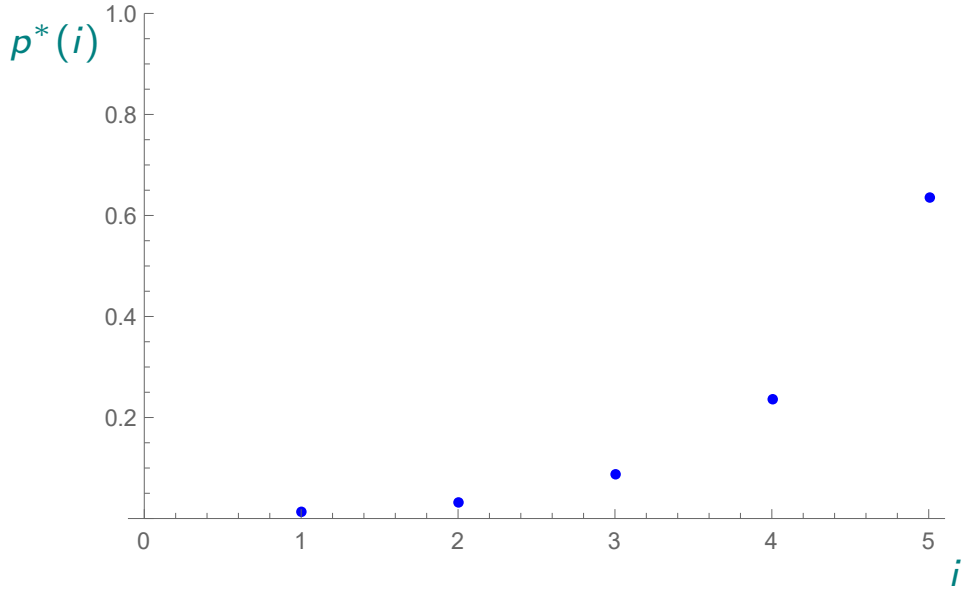
Example

no uncertainty

individual $i = 1, \ldots, 5$ has a return i

inequality constraint $\mathrm{H}\left(p(i)\right) \geq 1$

growth-optimal endowment distribution



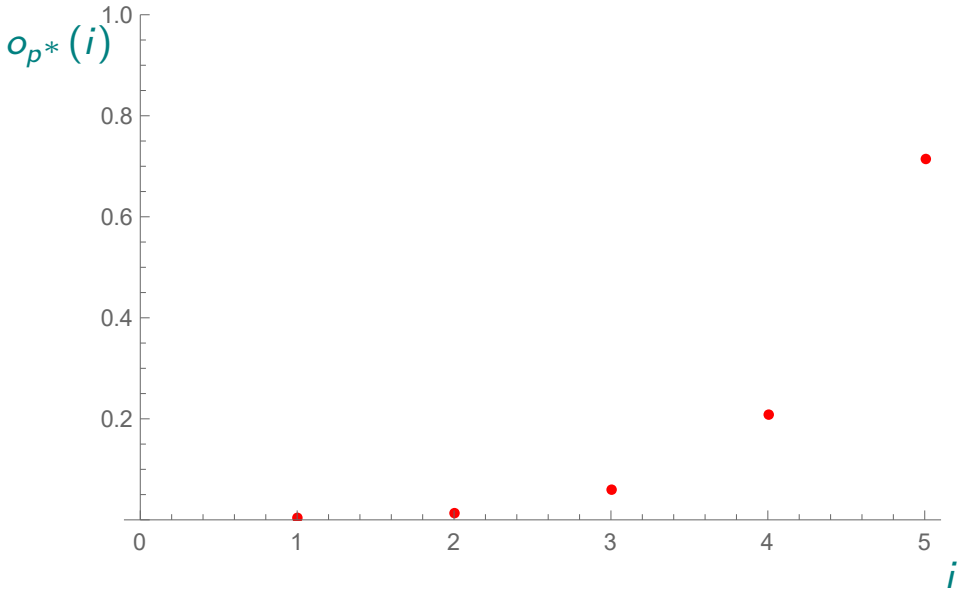
Example

no uncertainty

individual $i = 1, \ldots, 5$ has a return i

inequality constraint $\mathrm{H}\left(p(i)\right) \geq 1$

induced outcome distribution



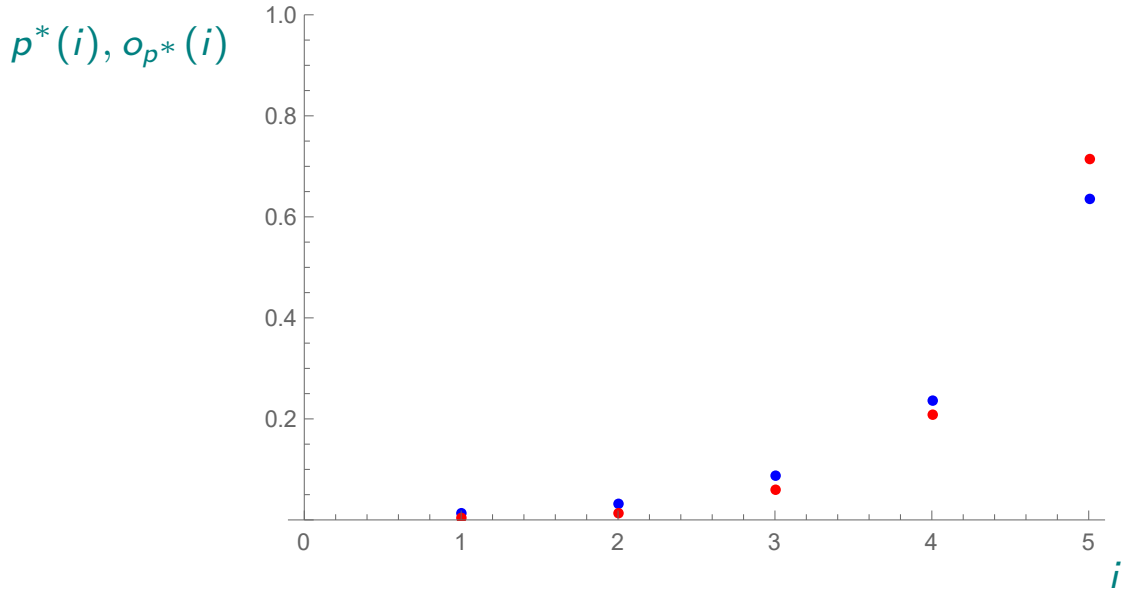
Example

no uncertainty

individual $i = 1, \ldots, 5$ has a return i

inequality constraint $\mathrm{H}\left(p(i)\right) \geq 1$

the 'closest' feasible endowment distribution to o_p^* coincides with p^*



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Change of the Notation

relabel the return function $r(i, \omega)$ as $p(\omega \mid i)$

- generalized (non-normalized) conditional distribution

define (generalized) joint distribution

$$p(i, \omega) = p(i)p(\omega \mid i)$$

Auxiliary Problem

Proposition

Policy $p^*(i,\omega)$ maximizes the growth rate if and only if solves

$$\min_{o(i,\omega),p(i,\omega)} \quad \mathrm{KL}\Big(o(i,\omega) \parallel p(i,\omega)\Big)$$

$$\text{s.t.} \quad o(i,\omega) \in \Delta(I \times \Omega)$$

$$o(\omega) = q^0(\omega)$$

$$p(i,\omega) \in \tilde{\mathcal{P}},$$

together with some $o^*(i,\omega)$.

The outcome distribution $o_{p^*}(i)$ induced by the policy p^* is given by the marginalization of $o^*(i,\omega)$.

Dynasties of Dollars

interpretation of $o(i, \omega)$

a dynasty originates in \$1 at $t = 1$

it is assigned to individuals $i_t \sim p(i)$

it multiplies by the return of its current owner

define outcome (distribution) $o(i, \omega)$ as:

- the empirical frequency of a dynasty being in hands of i and state ω

all $o(i, \omega)$ s.t: $o(\omega) = q^0(\omega)$ arise

(A)typical Outcomes

typical outcome:

- $o(i, \omega) = p(i)q_0(\omega)$

atypical outcomes:

- $o(i, \omega) \neq p(i)q_0(\omega)$
- disproportionably often in hands of some individuals
- or an empirical correlation between i and ω

each atypical outcome $o(i, \omega)$ experienced by only few dynasties

Wealth of an Outcome

fix a policy p

define wealth of outcome o as the total wealth of all dynasties achieving it

wealth of outcome o grows at the rate:

$$-\mathrm{KL}\left(o(i,\omega) \parallel p(i,\omega)\right) + \text{const.}$$

the fastest growing outcome dominates

Discussion

large deviations play a natural role in stochastic growth

- rare histories
- some achieve an extraordinary growth and dominate

trade-off between rarity and growth rate of a deviation

formal equivalence to cognitive trade-offs

- Robson, Steiner, Samuelson '23: rational inattention
- Millner '25: ambiguity aversion