Growth and Likelihood

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Stochastic Growth

stochastic growth: multiplication of random variables

in economics

- wealth is a product of random returns

in statistics

- likelihood of sample is a product of likelihoods of random data points

growth-rate maximization in both cases

Consistency Principles

Optimal policy seeks consistency with outcomes it generates.

in economics

- meritocracy: consistency between wealth and merit shares

in predictive coding

- consistency between prediction and sensory information

Literature

economics

- redistribution enhances growth
  - neoclassical explanation: concave returns
  - we: redistribution is a hedge against productivity shocks

- meritocracy: should wealth be a function of
  - initial conditions, output, luck?

information theory

- Kelly's betting

machine learning

- variational methods: Bayes rule as an optimization
- we: a growth-based proof

On the Notation

probabilities $p(x)$ and likelihoods $p(y \mid x)$ induce

$$\begin{aligned} p(x, y) &= p(x)p(y \mid x) \\ p(y) &= \sum_x p(x, y) \\ p(x \mid y) &= \frac{p(x, y)}{p(y)} \end{aligned}$$

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probabilities $q(x)$ and generalized likelihoods $q(y \mid x) \geq 0$ induce

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\end{aligned}
$$

properly normalized distributions in **bold**

Model

individuals $i \in I$, discrete time

stationary allocation $\mathbf{q}(i)$

- each i receives share $\mathbf{q}(i)$ of the aggr. wealth each morning

gross return $r(i, \omega_t) \geq 0$

- iid shocks $\omega_t \sim \mathbf{p}_0(\omega)$
- each t, wealth of each individual i is multiplied by $r(i, \omega_t) \geq 0$

planner controls allocation to maximize long-run growth rate

$$\max_{\mathbf{q}(i) \in \mathcal{Q}} \mathrm{E}_{\mathbf{p}_0} \ln \left(\sum_i \mathbf{q}(i) r(i, \omega) \right)$$

set $\mathcal{Q}$: constraints on inequality

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Example

$\omega \sim \mathbf{p}_0 \in \Delta(I)$

returns $q(\omega \mid i) = \mathbb{1}_{\omega=i}$

unconstrained allocation: $\mathbf{q}(i) \in \mathcal{Q} = \Delta(I)$

optimal allocation: $\mathbf{q}^*(i) = \mathbf{p}_0(i)$

equivalent to Kelly's betting

Merit Distribution

share of aggr. wealth produced by individual i in a period with shock ω

$$\frac{\mathbf{q}(i)q(\omega \mid i)}{\sum_j \mathbf{q}(j)q(\omega \mid j)} := \mathbf{q}(i \mid \omega)$$

definition

merit distribution: share of aggr. wealth produced by i in a random period

$$\mathbf{m}_{\mathbf{q}}(i) = \mathrm{E}_{\mathbf{p}_0(\omega)}\, \mathbf{q}(i \mid \omega)$$

Naive Merit Principle

Proposition

Growth-maximizing allocation $\mathbf{q}^*$ minimizes KL-divergence from the induced merit:

$$\mathbf{q}^*(i) \in \underset{\mathbf{q}(i) \in \mathcal{Q}}{\arg\min}\, \mathrm{KL}\left(\mathbf{m}_{\mathbf{q}^*}(i) \,\|\, \mathbf{q}(i)\right).$$

recall: KL-divergence is a pseudo-distance between two distributions

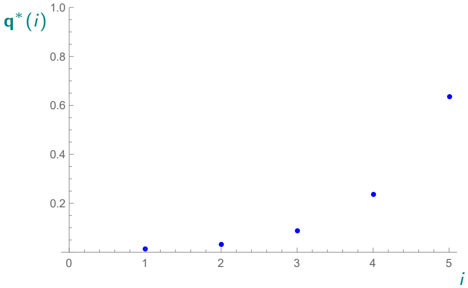
Example

no uncertainty

individual $i = 1, \ldots, 5$ has a return i

an inequality constraint $\mathrm{H}(\mathbf{q}(i)) \geq 1$

growth-optimal allocation



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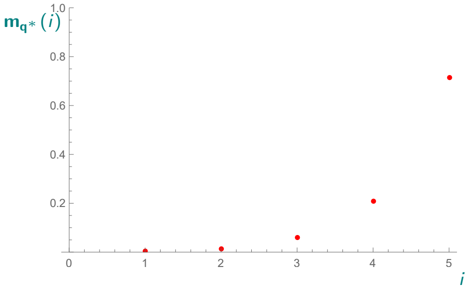

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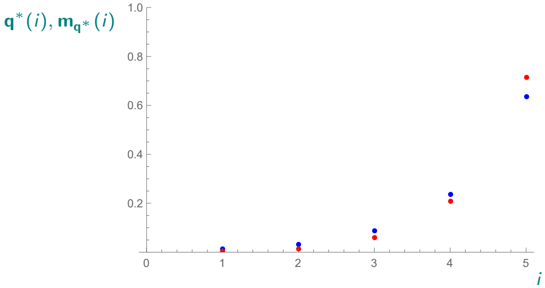
an inequality constraint $\mathrm{H}(\mathbf{q}(i)) \geq 1$

naive meritocracy returns the growth-optimal allocation



Naiveté

planner doesn't minimize the wedge between allocation and merit

- the principle ignores endogeneity of merit

Peter Andre: Shallow Meritocracy

- people don't incorporate indirect effects into their merit judgements

Extension of KL-divergence

$$\mathrm{KL}(\mathbf{p} \parallel \mathbf{q}) := \sum_x \mathbf{p}(x) \ln \frac{\mathbf{p}(x)}{\mathbf{q}(x)}$$

a map $\Delta(X) \times \Delta(X) \to \mathbb{R}_+$

the distribution "most consistent" with q is the normalization of q

$$\frac{q(x)}{\sum_{x'} q(x')} \in \arg\min_{\mathbf{p}} \mathrm{KL}(\mathbf{p} \parallel q)$$

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Joint Optimization

Theorem

Allocation $\mathbf{q}^*(i)$ maximizes the growth rate if and only if solves

$$\min_{\mathbf{p}(i,\omega),\mathbf{q}(i)} \quad \mathrm{KL}\left(\mathbf{p}(i,\omega) \parallel q(i,\omega)\right)$$

$$\text{s.t.} \quad \mathbf{p}(\omega) = \mathbf{p}_0(\omega)$$

$$\mathbf{q}(i) \in \mathcal{Q},$$

together with some $\mathbf{p}^*(i,\omega)$.

Additionally, $\mathbf{p}^*(i)$ is the merit distribution induced by $\mathbf{q}^*(i)$.

interpretation of $\mathbf{p}(i,\omega)$?

Growth Rate as a Consistency Optimization

Donsker and Varadhan's variational formula

Lemma

For any $q : X \to \mathbb{R}_{++}$,

$$\ln \sum_x q(x) = -\min_{\mathbf{p}} \mathrm{KL}(\mathbf{p} \parallel q).$$

1. what are these distributions $\mathbf{p}$?
2. why $\mathbf{p}$ consistent with q?

Proof

set up a growth process

$$y_t = \left(\sum_x q(x) \right)^t$$

the expression of interest is its growth rate

$$\ln \sum_x q(x) = \frac{1}{t} \ln y_t$$

sum over sequences

$$y_t = \sum_{(x_1,\ldots,x_t)} \prod_{t'} q(x_{t'})$$

the summands depend only the empirical distribution $\mathbf{p}$ of the sequence

Proof

\# of sequences with an empirical distribution $\mathbf{p}$ is $\approx \exp[\mathrm{H}(\mathbf{p})t]$

$$y_t = \sum_{\mathbf{p}} \prod_{x} q(x)^{\mathbf{p}(x)t} \exp[\mathrm{H}(\mathbf{p})t]$$

process y_t is a sum of exponential growths

$$y_t = \sum_{\mathbf{p}} \exp\left[-\mathrm{KL}(\mathbf{p} \parallel q)t\right]$$

the exponential function with the highest exponent dominates

$$y_t \approx \exp\left[-\min_{\mathbf{p}} \mathrm{KL}(\mathbf{p} \parallel q)t\right]$$

Q&A

1. what are these distributions $\mathbf{p}$?
   - each $\mathbf{p}$ corresponds to an empirical distribution of a sequence
   - the original process is a sum of growths across all $\mathbf{p}$
   - the fastest growth dominates
2. why $\mathbf{p}$ consistent with q grows fast?
   - concentrate on x with high $q(x)$
   - but, be random to keep # of sequences high
   - the optimal compromise $\mathbf{p}^*$ matches q

Back to Economic Growth

recall the growth rate expression

$$\sum_{\omega} \mathbf{p}_0(\omega) \ln \left(\sum_i q(i, \omega) \right)$$

apply the lemma for each ω

Dynasties of Dollars

a dynasty originates in \$1 at $t = 1$

it moves from one individual to another

it multiplies by the return of its current owner

define path $\mathbf{p}(i, \omega)$ as

- the empirical frequency of a dynasty being in hands of i and state ω

all paths s.t: $\mathbf{p}(\omega) = \mathbf{p}_0(\omega)$ coexist

wealth of dynasties with path $\mathbf{p}(i, \omega)$ grows at rate,

$$- \mathrm{KL}\left(\mathbf{p}(i, \omega) \parallel q(i, \omega)\right)$$

Allocation Optimization

growth-maximizing allocation solves

$$\min_{\mathbf{q}(i)} \quad \mathrm{KL}\left(\mathbf{p}^*(i,\omega) \parallel q(i,\omega)\right)$$

$$\text{s.t.} \quad \mathbf{q}(i) \in \mathcal{Q}.$$

Allocation Optimization

growth-maximizing allocation solves

$$\min_{\mathbf{q}(i)} \quad \left\{ \mathrm{KL}\left(\mathbf{p}^*(i) \parallel \mathbf{q}(i)\right) + \sum_i \mathbf{p}^*(i)\, \mathrm{KL}\left(\mathbf{p}^*(\omega \mid i) \parallel q(\omega \mid i)\right) \right\}$$

$$\text{s.t.} \quad \mathbf{q}(i) \in \mathcal{Q}.$$

Learning the Growth-Maximizing Allocation

start with an arbitrary interior allocation $\mathbf{q}(i)$

compute induced merit $\mathbf{m}_{\mathbf{q}}(i)$

update to the "most fair" allocation $\mathbf{q}'(i) \in \mathcal{Q}$ given merit $\mathbf{m}_{\mathbf{q}}(i)$

iterate

this converges to the optimal policy (for convex $\mathcal{Q}$)

- Csiszár & Tusnády '84

Learning: Growth Perspective

Berk'66, White'82

sample $(x_1, \ldots, x_t)$ from $\mathbf{p}(x)$

likelihood of sample under a hypothesis $\mathbf{q}(x)$ grows at rate

$$-\mathrm{KL}(\mathbf{p} \parallel \mathbf{q}) - \mathrm{H}(\mathbf{p})$$

$\Rightarrow$ a statistician converges to hypothesis $\mathbf{q}^* \in \arg\min_{\mathbf{q}} \mathrm{KL}(\mathbf{p} \parallel \mathbf{q})$

Predictive Coding

a system

- samples a signal ω from $\mathbf{p}_0(\omega)$
- seeks to form belief about a cause i of the signal ω
- knows likelihoods $\mathbf{q}(\omega \mid i)$
- entertains a set $\mathcal{Q}$ of priors $\mathbf{q}(i)$

chooses the best fit

$$\mathbf{q}^*(i) \in \arg\min_{\mathbf{q}(i) \in \mathcal{Q}} \mathrm{KL}\left(\mathbf{p}_0(\omega) \parallel \mathbf{q}(\omega)\right)$$

this can be justified by

- White's or Berk's asymptotic results on learning, or
- minimization of surprise

finally, forms belief $\mathbf{q}^*(i \mid \omega)$ by Bayes law

Generative and Recognition Models

generative model $\mathbf{q}(i, \omega)$: system's internal model of the world

recognition model $\mathbf{p}(i, \omega)$: system's interpretation of the signal

- arbitrary belief $\mathbf{p}(i \mid \omega)$ upon observing ω
- signals ω are sampled from $\mathbf{p}_0(\omega)$, thus $\mathbf{p}(\omega) \equiv \mathbf{p}_0(\omega)$
- $\mathbf{p}(i, \omega) = \mathbf{p}_0(\omega)\mathbf{p}(i \mid \omega)$

generative and recognition models may differ

- but a good pair is as consistent as possible

Variational Characterization

Corollary

The best fit solves

$$\min_{\mathbf{p}(i,\omega),\mathbf{q}(i)} \quad \mathrm{KL}\left(\mathbf{p}(i,\omega) \,\|\, \mathbf{q}(i,\omega)\right)$$

$$\text{s.t.} \quad \mathbf{p}(\omega) = \mathbf{p}_0(\omega)$$

$$\mathbf{q}(i) \in \mathcal{Q}.$$

proven by a variational argument in machine learning

we provide a growth-based proof

The Connection

optimization of growth rates of multiplicative random processes

- aggregate wealth is a product of random returns

$$\prod_t \left(\sum_i \mathbf{q}(i) q(\omega_t \mid i) \right)$$

- likelihood of a sample is a product of likelihoods of data points

$$\prod_t \left(\sum_i \mathbf{q}(i) \mathbf{q}(\omega_t \mid i) \right)$$

economic growth

- allocation and returns q
- path $\mathbf{p}$

predictive coding

- generative model $\mathbf{q}$
- recognition model $\mathbf{p}$

Approximate Bayes-Consistency

analogue of the naive merit principle

misspecification $\Rightarrow$ empirical mean of posteriors over causes $\neq$ prior

$$\mathsf{E}_{\mathbf{p}_0(\omega)}\, \mathbf{p}^*(i \mid \omega) \neq \mathbf{q}^*(i)$$

Proposition

Optimal generative distribution $\mathbf{q}^*(i)$ maximizes consistency with the recognition distribution $\mathbf{p}^*(i) = \mathsf{E}_{\mathbf{p}_0(\omega)}\, \mathbf{p}^*(i \mid \omega)$:

$$\mathbf{q}^*(i) \in \arg\min_{\mathbf{q}(i) \in \mathcal{Q}} \mathrm{KL}\left(\mathbf{p}^*(i) \parallel \mathbf{q}(i)\right).$$

Summary

we established equivalence between

- economic growth and
- predictive coding

unified consistency principles that apply to both

$\Rightarrow$ a fairness principle in the economic context

growth-based approach as an alternative to the variational approach