On the Dual Approach to Recursive Optimization

Messner - Pavoni - Sleet

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Outline

- Introduction and relationship to literature.
- (Brief) summary of the paper.
- My comments.
- Conclusion.

Introduction

- Great paper: I learned a lot.
- Contribution of this paper:
    - Provide recursive dual formulations for a (large) class of dynamic (incentive) problems.
    - Extend/generalize Marcet and Marimon (2011) and Messner, Pavoni and Sleet (RED, 2012).

Discussion

- Why do we need recursive formulations?
  - Facilitate characterization.
  - Enable quantitative analysis.
- Why do we need to use the dual problem?
  - Makes characterization easier.
  - Makes quantitative analysis easier.

Marcet and Marimon (1999 - 2011)

- Consider a problem with limited commitment.
- Rewrite it as a recursive problem with capital, current shock and (up-to-date) sum of Lagrange multipliers being the state.
- Assumptions avoiding technical difficulties.

Messner, Pavoni, Sleet (RED, 2012)

Sequential primal	$\Leftrightarrow$	Recursive primal
		$\Updownarrow$
~~Sequential dual~~		Recursive dual

- Equivalence between sequential primal and recursive dual under somewhat restrictive conditions:
  - Environment without backward-looking state variables (capital), i.e. not a generalization of Marcet-Marimon.
  - Linearity in forward looking constraints (incentive constraints, limited commitment constraints etc.).
- Provide convergence results for the Bellman operator in the recursive dual problem.

This Paper

$$\begin{array}{ccc} \text{Sequential primal} & \Leftrightarrow & \text{Recursive primal} \\ \Updownarrow & & \\ \text{Sequential dual} & \Leftrightarrow & \text{Recursive dual} \end{array}$$

- Equivalence between sequential primal and recursive dual under very general conditions:
  - An environment with backward-looking state variables.
  - General form of forward looking constraints (including non-separability across states).
- Provide contraction results for the Bellman operator in the recursive dual $\Rightarrow$ uniqueness and speed of convergence.

Brief Outline

- Sequential primal $\rightarrow$ Recursive dual
- Steps:
  1. Sequential primal with constraints.
  2. Rewrite as a Lagrangian, rewrite in sup-inf form.
  3. Consider the sequential dual, i.e. inf-sup.
  4. Decompose, rewrite as recursive dual.

Super-simple Example

2 period problem with capital and limited commitment:

- Lender has y in period 1 and 2. Can eat it or 'lend' it to entrepreneur, who can use it as capital to produce. Initial capital is zero, no depreciation, irreversibility. Entrepreneur can walk away with installed capital and current borrowing.

Step 1: Lender's problem:

$$
\begin{aligned}
V^* = \sup_{c_1^\ell, c_2^\ell, c_1^e, c_2^e, k_1, k_2} \quad & u^\ell(c_1^\ell) + u^\ell(c_2^\ell) \\
f(k_1) + (y - k_1) - c_1^\ell - c_1^e &\geq 0 \\
f(k_2) + (y - k_2 + k_1) - c_2^\ell + c_2^e &\geq 0 \\
u^e(c_1^e) + u^e(c_2^e) - v_1(k_1) &\geq 0 \\
u^e(c_2^e) - v_2(k_2) &\geq 0
\end{aligned}
$$

Denote $a = (c_1^\ell, c_2^\ell, c_1^e, c_2^e, k_1, k_2) \in A = \Re_+^6$.

Step 2: Rewrite as Lagrangian:

Denote $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \Lambda = \Re^4_+$.

$$
\begin{aligned}
L(a, \lambda) = \quad & u^\ell(c_1^\ell) + u^\ell(c_2^\ell) + \\
& \lambda_1[f(k_1) + (y - k_1) - c_1^\ell - c_1^e] + \\
& \lambda_2[f(k_2) + (y - k_2 + k_1) - c_2^\ell + c_2^e] + \\
& \lambda_3[u^e(c_1^e) + u^e(c_2^e) - v_1(k_1)] + \\
& \lambda_4[u^e(c_2^e) - v_2(k_2)]
\end{aligned}
$$

In sup-inf notation:

$$V^* = \sup_{a \in A} \inf_{\lambda \in \Lambda} L(a, \lambda)$$

Intuition: infinite penalization if constraints violated.

Step 3: Rewrite in inf-sup form:

$$D^* = \inf_{\lambda \in \Lambda} \sup_{a \in A} L(a, \lambda)$$

- This is the sequential dual problem.

Step 4: First separate period 1 and period 2.

$$D^* = \inf_{\lambda \in \Lambda} \sup_{c_1^\ell, c_1^e, k_1} u^\ell(c_1^\ell) + \lambda_1[f(k_1) + (y - k_1) - c_1^\ell - c_1^e] + \lambda_2 k_1 + \lambda_3[u^e(c_1^e) - v_1(k_1)] + \sup_{c_2^\ell, c_2^e, k_2} u^\ell(c_2^\ell) + \lambda_2[f(k_2) + y - k_2 - c_2^\ell + c_2^e] + \lambda_3 u^e(c_2^e) + \lambda_4[u^e(c_2^e) - v_2(k_2)]$$

- Period 1 multipliers: λ_1, λ_3.
- Period 2 multipliers: λ_2, λ_4.
- Define $\phi = \lambda_2$. Summarizes how period 2 multiplier λ_2 affects period 1 problem. Forward looking?
- Define $\mu = \lambda_3$. Summarizes how period 1 multiplier λ_3 affects period 2 problem. Backward looking?

Now rewrite recursively:

$$
\begin{aligned}
D_1 :=& \inf_{\lambda_1,\lambda_3,\phi,\mu} \sup_{c_1^\ell, c_1^e, k_1} u^\ell(c_1^\ell) + \lambda_1[f(k_1) + (y - k_1) - c_1^\ell - c_1^e] + \\
& \phi k_1 + \lambda_3[u^e(c_1^e) - v_1(k_1)] + D_2(\mu, \phi) \\
& \text{s.t.} \qquad \mu = \lambda_3 \\
D_2(\mu, \phi) =& \inf_{\lambda_2,\lambda_4} \sup_{c_2^\ell, c_2^e, k_2} u^\ell(c_2^\ell) + \lambda_2[f(k_2) + y - k_2 - c_2^\ell + c_2^e] + \\
& \mu u^e(c_2^e) + \lambda_4[u^e(c_2^e) - v_2(k_2)] \\
& \text{s.t.} \qquad \phi = \lambda_2
\end{aligned}
$$

Bellman principle: $D^* = D_1$.

Further Results

- Do all this in a much more general ∞ horizon environment.
- Contraction result for the Bellman operator in the recursive dual problem, i.e. D^* unique and have convergence.
- Directly relate recursive dual with sequential primal.
- Provide a method to check if no sufficiency, i.e. if cannot show solution to recursive dual solves sequential primal.

Comments

- What economic problems have not had a (tractable) recursive formulation prior to this paper?
- Examples in MPS (RED, 2012):
  - Limited commitment models, see Marcet and Marimon (2011) for a Lagrangian approach. In primal form with promised value as state see e.g. Ljungqvist and Sargent textbook.
  - Private info with i.i.d. shocks: Thomas and Worrall (1990).
  - Private info with persistent shocks: Fernandes and Phelan (2000) primal approach. Kapicka (2013) uses the first order approach (in the primal). Fukushima and Waki (2013).

Comments

- What is the benefit of using this method over other methods?
- Theoretically: very general, less restrictive assumptions, easier to characterize the set of feasible states.
- Quantitatively:
  - Computation time, robustness? We do not know:
    Solve a sample problem and compare speed and precision.
  - Contractive methods robust, but tend to be slow and imprecise (value function iteration in the standard growth model or incomplete market model a la Aiyagari).
  - Rate of convergence $\rho < 1$, growth model converges at rate β.
  - Could improve upon this method using insights from quantitative work in that literature (e.g. Howard algorithm, iterating on the policy functions using the Euler equation)?

Minor Comments

- Would like an application with:
    - backward and forward looking constraint and shocks,
    - to guide through the theory.
- Are Assumptions 1 and 2 strong? They seem to be important for the contraction results.

Conclusion

- Great theoretical step forward.
- Importance for quantitative analysis?
    - Time will tell as in Marcet and Marimon case.