Economic Growth Theory

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Lecture notes*

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Contents

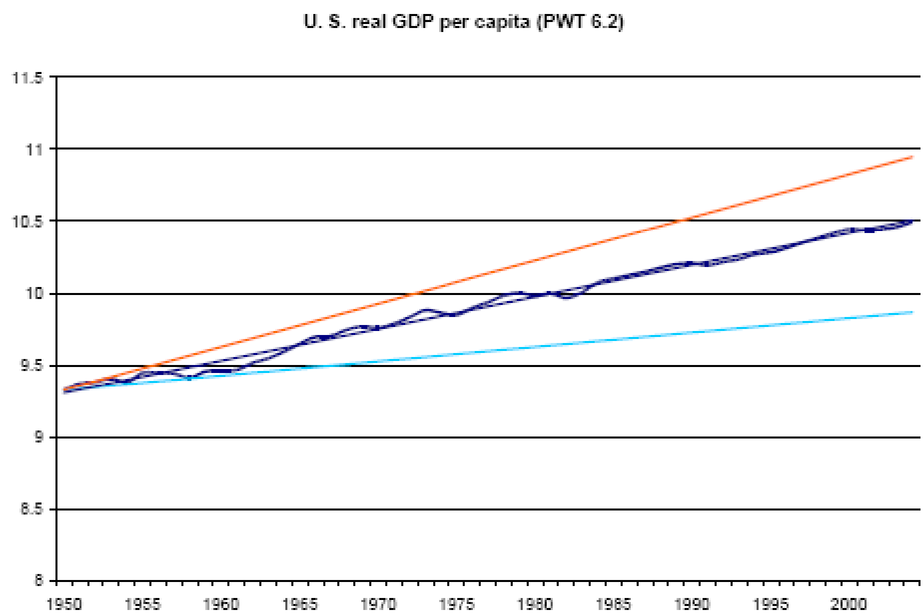
*These notes may contain typos/mistakes and are subject to changes/updates during our course. Please keep track if there are any.

Empirical observations; Kaldor stylized facts of growth; Neoclassical production function; The Solow-Swan model

Growth matters over long time periods

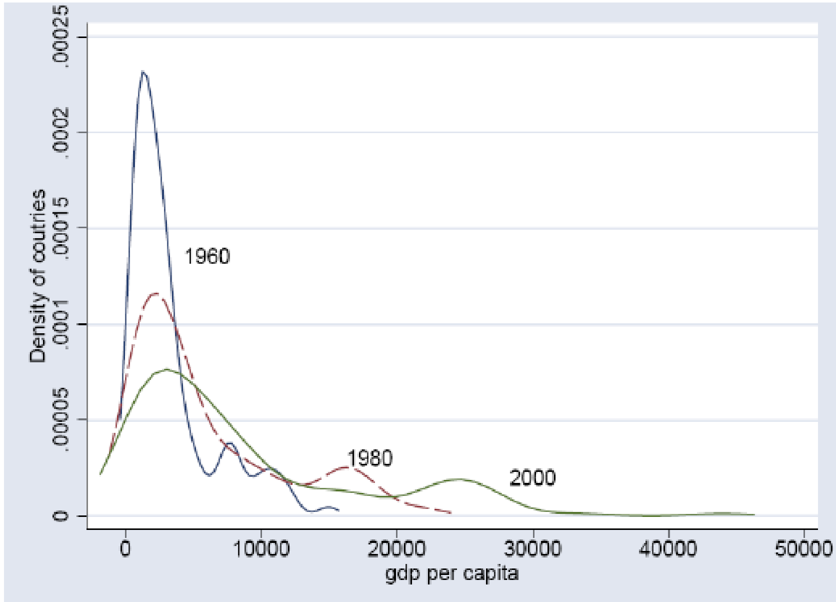
1. For the welfare of a single country–example: postwar growth experience of the U.S.

   - GDP per capita was 11,233 $US in 1950. This increased steadily to 36,098 $US by 2004, hence increasing by a factor of around 3 in 50 years. This is due to an average annual growth of 2.2%.
   - If it had grown at 1% only, then GDP per capita would have increased by a factor of 1.7 by 2004
   - If it had grown at 3%, then GDP per capita would have increased by a factor of 5 by 2004

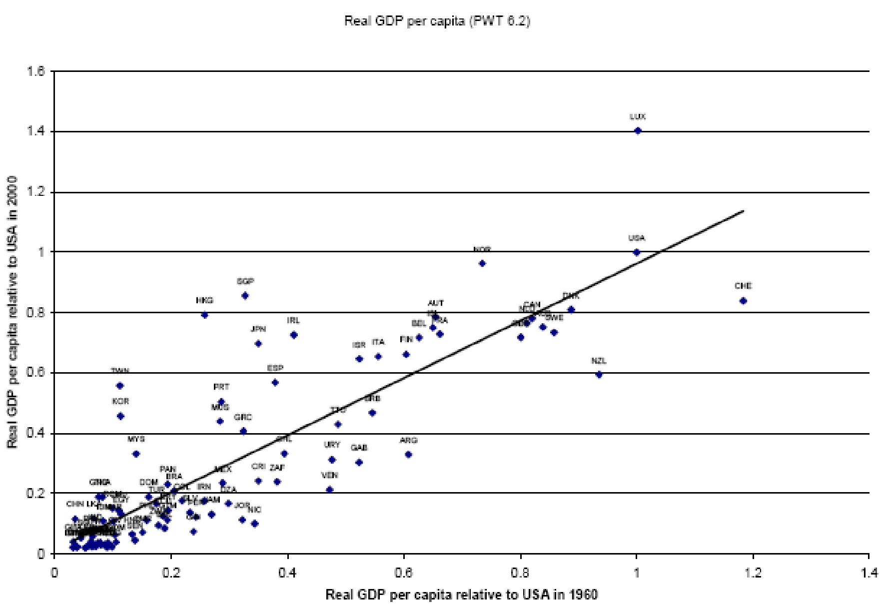


1. For the welfare across countries–explore the properties of the distribution of PPP-adjusted GDP per capita across the available set of countries.

   - The distribution has shifted to the right due to growth in per capita income in most countries.
   - The dispersion has increased. There is indication of stratification of countries/ "twin peaks" which can indicate that there is no convergence between high income and low income countries



1. The differences in real GDP per capita remain rather persistent over time. Important exception: East Asian countries.



Kaldor stylized facts of growth

1. Per capita output grows over time and its growth rate does not tend to decline
2. Physical capital per worker grows over time
3. The physical capital to output ratio is nearly constant
4. Rate of return on capital is nearly constant
5. The shares of capital and labor in national income are nearly constant
6. The growth rate of output per worker within a country may exhibit short periods of acceleration

Across countries - the growth rates of output per worker differ

Neoclassical production function

- $Y = F(K, L)$, where K is the capital input, and L is the labor input
  - Constant Returns to Scale (CRTS): $F(\lambda K, \lambda L) = \lambda Y; \forall \lambda > 0$
  - F is increasing and concave function. That is $F_K, F_L > 0$ and $F_{KK}, F_{LL} < 0$,where $F_X = \partial F / \partial X$ for $\forall X$
    - $F_K, F_L > 0$ and $F_{KK}, F_{LL} < 0 \Rightarrow$ diminishing returns to inputs
  - Inada conditions: $\lim_{K \to 0} F_K = \lim_{L \to 0} F_L = +\infty$, $\lim_{K \to +\infty} F_K = \lim_{L \to +\infty} F_L = 0$
  - Essentiality of all inputs: $F(0, L) = F(K, 0) = 0$ (this follows from $F'' < 0$ and CRTS)

Solow-Swan model - (Solow, 1956; Swan, 1956)

Main assumptions

- Neoclassical production function
- One sector model of growth
  - $Y = C + I$ (resource constraint), where Y is the aggregate output, C is the aggregate consumption, and I is the aggregate investment
    * the investment I covers the depreciated amount of capital and generates new capital, i.e., $I = \dot{K} + \delta K$, where $\dot{K} = dK/dt$ is the new capital, δ is the depreciation rate, and δK is the depreciated amount of capital
- Closed economy
  - Closed economy insures that $I = S$, where S is are the aggregate savings
- Exogenous savings rate
  - $S = I = sY; s \in (0, 1)$

Further assumptions

- Exogenous rate of population growth, $L(t) = L(0)e^{nt}$
  - $\dot{L}/L = n$, where $\dot{L}/L$ is the growth rate of population
- (To keep it simple) No technological progress

From $Y = C + I$ and $I = \dot{K} + \delta K$ it follows that

$$\dot{K} = Y - C - \delta K.$$

This is the standard law of motion of capital. Further, from $Y = C + I$ and $I = sY$ it follows that $C = (1 - s)Y$. Therefore, the law of motion of capital can be rewritten as

$$\dot{K} = sY - \delta K.$$

The further analysis we will perform in per-capita terms. Denote the macroeconomic aggregates in per-capita terms by small letters,

$$\begin{aligned} k &: = \frac{K}{L}; c := \frac{C}{L}, \\ y &: = \frac{Y}{L} = F\left(\frac{K}{L}, 1\right) = f(k). \end{aligned}$$

From these definitions it follows that

$$\begin{aligned}
&f'(k) > 0; f''(k) < 0, \\
&\dot{k} = d\left(\frac{K}{L}\right)/dt = \frac{\dot{K}}{L} - nk,
\end{aligned}$$

where $f'(k) = df(k)/dk; f''(k) = d^2 f(k)/dk^2$. Therefore, the law of motion of capital in per-capita terms is given by

$$\dot{k} = sf(k) - (n + \delta) k.$$

Market equilibrium

Household (HH) - *representative*

- HH offers labor L and holds assets W
  - The marginal unit of labor earns income w and the assets earn returns r
- Evolution of asset holdings, W: $\dot{W} = rW + wL - C$

  In per-capita terms: $\dot{\varpi} = (r - n)\varpi + w - c$
- The means of savings is lending to other HH and/ or owning capital
- There is no optimization problem given that the savings rate is exogenously fixed

Firm - *representative*

- Perfect competition in input and output markets
- It chooses labor, L, and capital, K, to maximize its profits within every period: $\pi = F(K, L) - RK - wL$, given the wage, w, and the rental rate, R, for the services of a unit of capital
- First order conditions (optimal rules) are:

$$\begin{aligned}
[K]: \quad & \frac{\partial \pi}{\partial K} = 0 \Leftrightarrow F_K = f'(k) = R \\
& \frac{\partial F(K, L)}{\partial K} = \frac{\partial F(K/L, 1)}{\partial K/L} = f'(k) \\
[L]: \quad & \frac{\partial \pi}{\partial L} = 0 \Leftrightarrow F_L = f(k) - kf'(k) = w \\
& \frac{\partial F(K, L)}{\partial L} = \frac{\partial L F(K/L, 1)}{\partial L} = f(k) - kf'(k)
\end{aligned}$$

General equilibrium results

- Capital market clearing condition

  - The net rate of return from a unit of capital for a HH is $R-\delta$ given its depreciation within a period. No arbitrage condition requires that $R-\delta=r$ or $r=f'(k)-\delta$ and $W=K$.

- Given that $W=K$, $F_K=R$, and $F_L=w$ the $\dot{W}=rW+wL-C$ reduces to the resource constraint $Y=C+\dot{K}+\delta K$.

Steady-state/Balanced growth path

- Steady-state of the economy - where all variables grow at constant rates (balanced growth path - BGP)

Denote

$$\frac{\dot{Z}}{Z}=g_Z.$$

From the law of motion of capital

$$\dot{K}=sY-\delta K$$

follows that

$$g_K=\frac{\dot{K}}{K}=s\frac{Y}{K}-\delta.$$

- Given that $s,\delta=const$ it must be the case that the aggregate output Y and the total capital stock K grow at the same rate when the growth rate of total capital stock g_K is constant, i.e., $g_K=g_Y$

- From the resource constraint $C=(1-s)\,Y$. Therefore, the aggregate consumption and output grow at the same rate, i.e., $g_C=g_Y=g_K$

- $\frac{Y}{K}=F\left(1,\frac{L}{K}\right)$, therefore, since $\frac{Y}{K}=const$ the ratio $\frac{L}{K}$ should be constant. This implies that $g_Y=g_K=g_C=n$ or, equivalently, $g_c=g_k=g_y=g_K-n=0$.

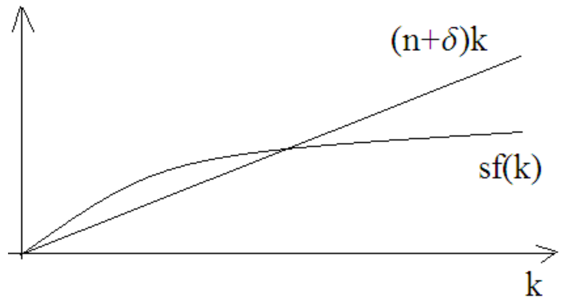
Neoclassical production function implies

- Uniqueness of the steady-state, i.e.,

$$\dot{k}=0\Rightarrow sf\left(k^*\right)=(n+\delta)\,k^*\Rightarrow k^*=k^*\left(s,n,\delta\right) \tag{1}$$

  The steady-state is unique since $\varphi\left(k\right):=sf\left(k\right)-(n+\delta)k$ has the following properties: $\varphi\left(0\right)=0,\lim_{k\to 0}\varphi'\left(k\right)=+\infty,\lim_{k\to+\infty}\varphi'\left(k\right)=-\left(n+\delta\right)$ and $\varphi''\left(k\right)<0$.

– On a graph the steady-state value of capital is determined from the intersection of $sf(k^*)$ curve and $(n+\delta)k^*$ line, i.e.,

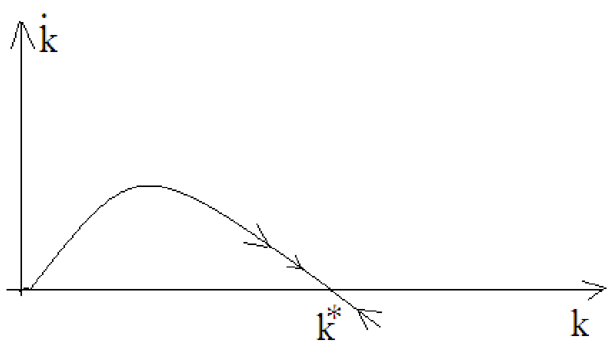


- From (1) it follows that $\frac{\partial k^*}{\partial s} > 0$: $d\varphi(k^*, s) = \frac{\partial \varphi(k^*,s)}{\partial k^*} dk^* + \frac{\partial \varphi}{\partial s} ds = 0$; $\frac{\partial \varphi(k^*,s)}{\partial k^*} = sf'(k^*) - (n+\delta) < 0$, $\frac{\partial \varphi(k^*,s)}{\partial s} = f(k^*)$ and $\frac{\partial k^*}{\partial s} = -\frac{\frac{\partial \varphi(k^*,s)}{\partial s}}{\frac{\partial \varphi(k^*,s)}{\partial k^*}} > 0$.

- During the transition the model gradually converges to this unique steady-state. The transition can be easily derived from the law of motion of capital written in per-capita terms.

$$\frac{d\dot{k}}{dk} = sf'(k) - (n+\delta)$$

In the neighborhood of steady-state, given that $f'' < 0$ and $s\frac{f(k^*)}{k^*} = (n+\delta)$, $\frac{d\dot{k}}{dk} < 0$. Therefore, the transition drawn in $\left(\dot{k}, k\right)$ is given by the following phase diagram



- For example when $Y = K^{\alpha}L^{1-\alpha}$ the steady state per-capita capital stock and output are $y = f(k) = k^{\alpha} \Rightarrow k^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}} \Rightarrow y^* = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$

- No growth in per-capita terms in the steady-state (in the absence of exogenous technological progress)

- Per-capita endogenous growth takes place only due to capital accumulation during the transition to the steady-state.

- Convergence

$$g_k = s\frac{f(k)}{k} - (n+\delta), \tag{2}$$

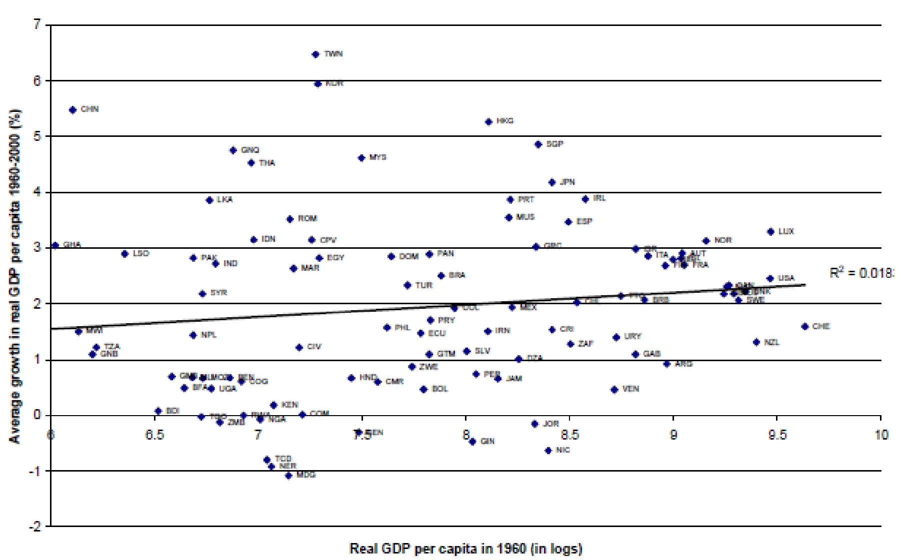
$$\frac{\partial g_k}{\partial k} = s\frac{f'(k)k - f(k)}{k^2} < 0. \tag{3}$$

  - Absolute convergence means that countries with lower capital per worker would exhibit higher growth, irrespective of any other characteristics of the economies.
  - Conditional convergence means that each country converges to its own steady-state. In other words, there is convergence of capital/ output per capita for a set of countries with similar savings, depreciation and population growth rates.

Solow model predicts conditional convergence

- The data do not support "absolute convergence", i.e., the data suggest that the poor do not grow faster than the rich and gradually catch up in terms of welfare independent of the specific characteristics of the countries

All countries in PWT 6.2.



- The data provide some support of "conditional convergence", i.e. the data suggest that every country grows faster the further away it is from its own steady-state. The latter is a function of the special characteristics of each country

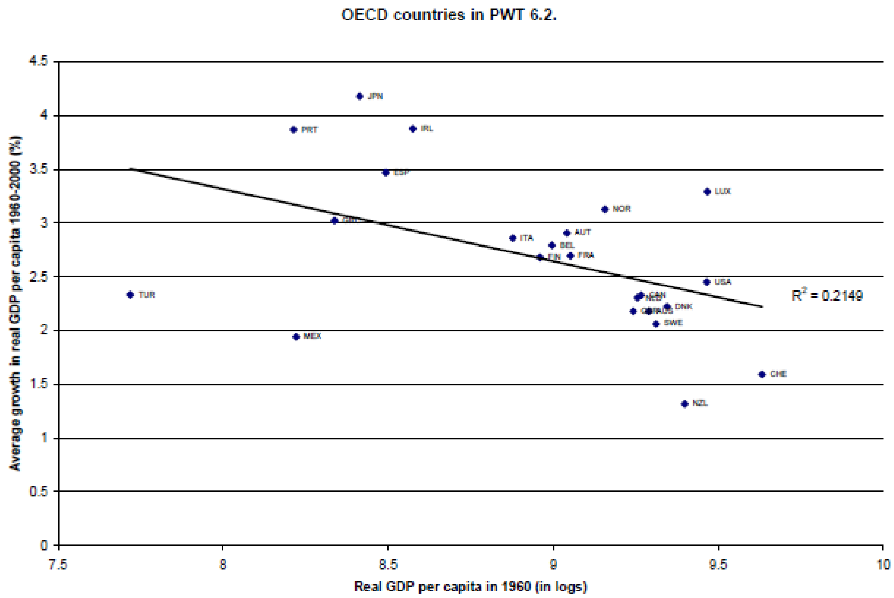


Illustration of conditional convergence

- From the law of motion of capital written in per-capita terms it follows that

$$\begin{aligned} s &= (n+\delta)\frac{k^*}{f(k^*)}, \\ g_k &= (n+\delta)\left[\frac{f(k)/k}{f(k^*)/k^*} - 1\right]. \end{aligned} \tag{4}$$

- Given the diminishing returns to capital, we know that $f(k)/k$ decreases as k increases. As a result, as k increases towards its steady-state value k^*, g_k falls to zero (its steady-state value).

- The model only predicts that we grow faster the further away we are from the steady-state.

Exogenous technological change and per-capita growth

- Redefine L in the production function $F(.)$ in the following way

$$L = A\tilde{L},$$

  where $\tilde{L}$ is the labor and A is the technology it uses for producing the output. Therefore, L is the effective labor.

- Let $\tilde{L}$ grow at rate n and A grow at rate g_A.

  – Denote the aggregates in effective labor units as

$$\tilde{y} = \frac{Y}{A\tilde{L}}, \tilde{k} = \frac{Y}{A\tilde{L}}, \tilde{c} = \frac{Y}{A\tilde{L}}.$$

- Repeating the analysis above for these variables it is straight forward to show that $g_{\tilde{y}} = g_{\tilde{k}} = g_{\tilde{c}} = g_K - g_{A\tilde{L}} = 0$. Therefore, $g_y = g_k = g_c = g_K - n = g_A$.
  - This implies that there can be per-capita growth in Solow-Swan model in case there is exogenous technological progress, $g_A > 0$.

Solow model and Kaldor stylized facts

1. Per capita output grows over time and its growth rate does not tend to decline.
2. Physical capital per worker grows over time.
   - *In this model both 1 and 2 require exogenous labor augmenting technology which grows at constant rate* g_A, *so that* $g_k = g_y = g_A$
3. Rate of return on capital is nearly constant:
   - In this model, *in the steady-state* $r = f'(k^*) - \delta$
4. The physical capital to output ratio is nearly constant
   - In steady-state this model predicts that $g_K = g_Y \Rightarrow \frac{Y}{K} = const$
5. The shares of capital and labor in national income are nearly constant:
   - $\frac{r^* k^*}{f(k^*)} = \frac{f'(k^*)k^*}{f(k^*)} = const; \frac{w^*}{f(k^*)} = 1 - \frac{f'(k^*)k^*}{f(k^*)} = const.$
6. The growth rate of output per worker within a country may exhibit short periods of fast growth: *Explained through transition dynamics.*

Across countries the growth rates of output per worker differ: *Solow model fails given the assumption on exogenous technological change.*

The (other) weaknesses of Solow model

- The savings rate is exogenous
- It cannot explain differences in y in terms of s or n
- It cannot explain differences in y in terms of differences in k since the income levels differ more than capital levels
  - Differences in y need to be explained by differences in (labor augmenting) technology used which is exogenous

- Endogenously generated per-capita growth rates can differ only along the transition path. Persistent differences cannot be explained. It can only explain short episodes of growth.

Solution: Different production function, i.e., drop the decreasing returns to scale and/or the Inada conditions

Golden rule level of capital and inter-generational issues In Solow-Swan model the golden rule level of capital is the amount of capital that maximizes consumption in the steady state.

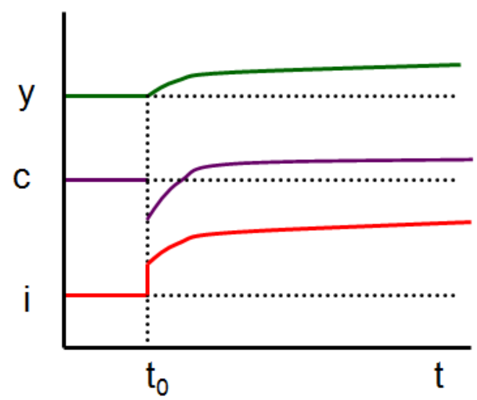
- In order to find it use the steady state conditions and write

$$c^* = (1-s)\, f(k^*) = f(k^*) - (n+\delta)\, k^*$$

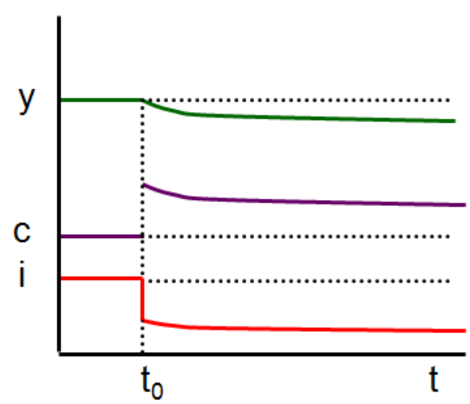
$$\Rightarrow$$

$$f'(k^*_G) = n+\delta; s_G = (n+\delta)\frac{k^*_G}{f(k^*_G)}.$$

- If $k^* < k^*_G$ since $\frac{\partial k^*}{\partial s} > 0$ the saving rate should be increased in order to arrive at k^*_G. The dynamics of the system will be the following in such a case



- If $k^* > k^*_G$ since $\frac{\partial k^*}{\partial s} > 0$ the saving rate should be decreased in order to arrive at k^*_G. The dynamics of the system will be the following in such case



In case we have overlapping generations increasing the savings rate s can imply lower consumption for the current generation and higher consumption for upcoming generations.

Additional: Harrod (1939)-Domar (1946) model

Secular stagnation was very popular about a century ago (it seems to get some attention now too). (Harrod, 1939; Domar, 1946) present a model and offer a related discussion.

Suppose that production function takes Leontief form

$$Y = F(K, L) = \min(AK, BL),$$

where $A, B > 0$. (Notice that this is the limiting case of constant elasticity of substitution production function were elasticity of substitution is 0, i.e., inputs are complements.) Such a production function implies that

$$F(K, L) = \begin{cases} AK & if \quad K \leq \frac{B}{A}L, \\ BL & if \quad K > \frac{B}{A}L. \end{cases}$$

Therefore, if $K \leq \frac{B}{A}L$ some part of labor force is idle and there is unemployment. In such a circumstance the unemployment rate is

$$\frac{L - \frac{A}{B}K}{L}.$$

In turn, when $K > \frac{B}{A}L$ some part of the capital is idle. In percentage terms, idle capital is

$$\frac{K - \frac{B}{A}L}{K}.$$

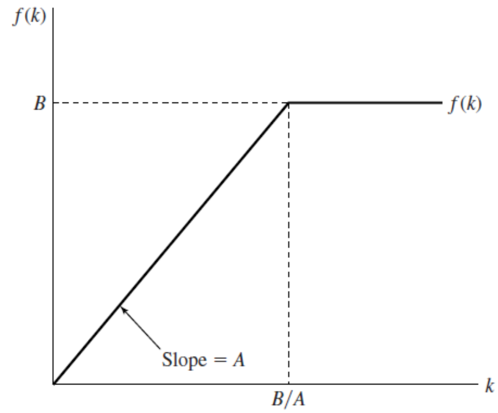
Consider now production function in per-capita terms. It is given by

$$f(k) = \min(Ak, B).$$

Therefore,

$$f(k) = \begin{cases} Ak & if \quad k \leq \frac{B}{A}, \\ B & if \quad k > \frac{B}{A}. \end{cases}$$

This following figure offers a plot of this function



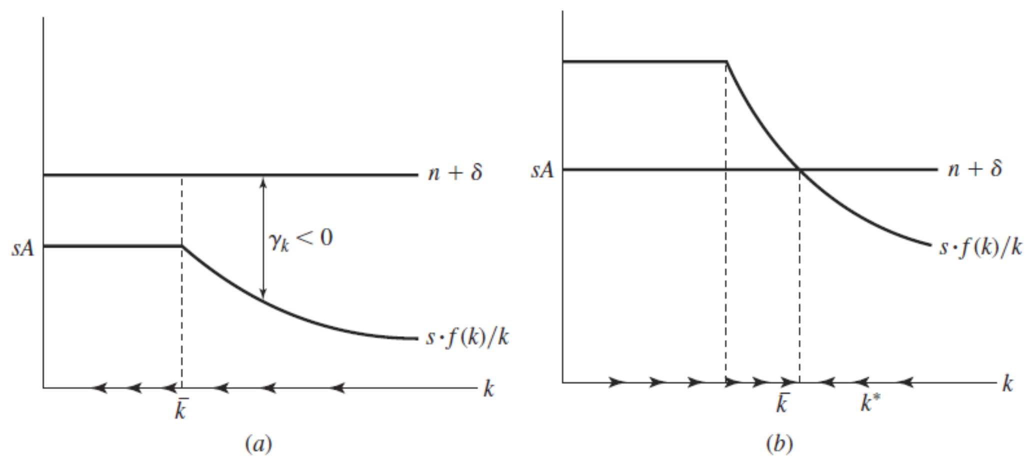
Notice also that the ratio of output and capital have the following form

$$\frac{f(k)}{k} = \begin{cases} A & if \quad k \leq \frac{B}{A}, \\ \frac{B}{k} & if \quad k > \frac{B}{A}. \end{cases}$$

Therefore, the fundamental equation of Solow-Swan model is

$$\frac{\dot{k}}{k} = \begin{cases} sA - (n+\delta) & if \quad k \leq \frac{B}{A}, \\ s\frac{B}{k} - (n+\delta) & if \quad k > \frac{B}{A}. \end{cases}$$

This following figure plots this relationship for $sA < n + \delta$ and $sA > n + \delta$.



Notice that the rate of growth of capital is negative when $sA < n + \delta$ in both cases (1) $k \leq \frac{B}{A}$ and (2) $k > \frac{B}{A}$ since $s\frac{B}{k} < sA$. Therefore, in such a case over time per capita capital, output, and consumption decline to 0. Moreover, the steady-state features permanently growing pool of unemployed people. In turn, when $sA > n + \delta$ there exists steady-state level of per-capita capital k^*,

$$k^* = \frac{sB}{n+\delta}.$$

Apparently, $k^* > \frac{B}{A}$. The economy converges to this level of capital if it starts above or below it. At the steady-state there is a growing pool of idle capital given by

$$\frac{K - \frac{B}{A}L}{K} = \frac{k^* - \frac{B}{A}}{k^*}.$$

The only way to have a full employment of capital and labor in this model

$$AK = BL.$$

is for the parameter values $sA = n + \delta$, which is unlikely to hold.

There are couple of problems, however, with this model. First, the average product of capital K in this model would usually depend on K and adjust to satisfy equality

$$s\frac{f(k)}{k} = n + \delta$$

in the steady-state as in Solow-Swan model. Second, the rate of savings could adjust to satisfy this condition. In particular, when agents maximize their discounted life-time utility selecting the amount of their savings they would not save (at constant rate) when the marginal product of capital is zero.

Additional: Poverty traps

One popular notion in development economics concerns *poverty traps.*

Definition 1 *An economy is in a poverty trap if there are multiple stable equilibria and the economy appears to be in an equilibrium which does not deliver the highest level of income (wealth.)*

In order to generate a poverty trap in a model economy consider the following setup. Let an economy have an access to two types of technologies: primitive (A) and modern (B). Primitive and modern goods are produced according to

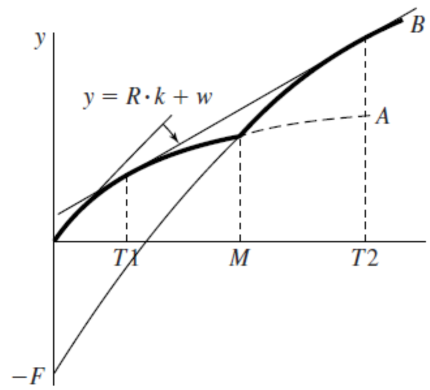
$$\begin{aligned} Y_A &= AK^{\frac{1}{2}}L^{\frac{1}{2}}, \\ Y_B &= BK^{\frac{1}{2}}L^{\frac{1}{2}}, \end{aligned}$$

where $A < B$. In order to use this better technology, the country has to pay managerial cost in each and every point in time which is proportional to labor force bL, where $b > 0$. This cost needs to be financed by the government which levies a tax on households to finance its expenditures.

In per-capita terms, net of these costs, production functions are given by

$$\begin{aligned} y_A &= Ak^{\frac{1}{2}}, \\ y_B &= Bk^{\frac{1}{2}} - b, \end{aligned}$$

and are drawn together below



If the government pays managerial costs than the firms use the modern technology. Meanwhile, they use the primitive technology if the government does not pay managerial costs. It is sensible to pay the costs if net benefits are positive, i.e., $y_B - b > y_A$ or $k > \left(\frac{b}{B-A}\right)^2$. Denote $\bar{k} = \left(\frac{b}{B-A}\right)^2$ and notice that at this point $y_B = y_A$. Assume that the government pays the costs if $k > \bar{k}$ and does not do so for lower values of k.

The economy is still governed by the fundamental equation of the Solow-Swan model with a slight modification

$$\dot{k} = \begin{cases} sAk^{\frac{1}{2}} - (n+\delta)k & if \quad k \leq \bar{k}, \\ sBk^{\frac{1}{2}} - b - (n+\delta)k & if \quad k > \bar{k}, \end{cases}$$

since at $\bar{k}$ it has to be that $y_B = y_A$.

Consider the case when there exist k_A^* and k_B^* such that $sAk^{\frac{1}{2}} - (n+\delta)k$ and $sBk^{\frac{1}{2}} - b - (n+\delta)k$ are zero. There are then 3 steady-states in this economy

$$\begin{aligned} k_A^* &= \left(\frac{sA}{n+\delta}\right)^2, \\ k_{B;1}^* &= \left[\frac{sB - \sqrt{(sB)^2 - 4(n+\delta)b}}{2(n+\delta)}\right]^2, \\ k_{B;2}^* &= \left[\frac{sB + \sqrt{(sB)^2 - 4(n+\delta)b}}{2(n+\delta)}\right]^2. \end{aligned}$$

It can be easily shown that $k_{B;1}^*$ is not stable in the sense that it is not an attractor. Whereas, k_A^* and $k_{B;2}^*$ are stable and k_A^* is a poverty trap.

Some proofs and additional results

That $\frac{\partial g_k}{\partial k} < 0$ in (3) follows from $f''(x) < 0$. Intuitively, if $f'' < 0$ then f function grows at a lower rate at each point than a linear function that is tangent to it at that point. To prove it use

$$f(tx_1 + (1-t)x_2) > tf(x_1) + (1-t)f(x_2)$$

which holds for $\forall t \in [0,1]$. Rearranging one gets

$$\frac{f\left(tx_1 + (1-t)\,x_2\right) - f\left(x_2\right)}{t} > f\left(x_1\right) - f\left(x_2\right)$$

The limit of the left-hand side when $t \to 0$ is equal to

$$\left(x_1 - x_2\right) \lim_{t\to 0} \frac{f\left(x_2 + t\left(x_1 - x_2\right)\right) - f\left(x_2\right)}{t\left(x_1 - x_2\right)} = \left(x_1 - x_2\right) f'\left(x_2\right).$$

Therefore, $\left(x_1 - x_2\right) f'\left(x_2\right) > f\left(x_1\right) - f\left(x_2\right)$. Now substitute $x_1 = 0$ and $x_2 = k \Rightarrow kf'(k) < f(k)$.

Meanwhile, in case of Cobb-Douglas production function it is possible to obtain the closed form solution for $k(t)$. From (2) or the fundamental equation. Denote $v = k^{1-\alpha}$ then

$$\frac{1}{1-\alpha}\dot{v} + (n+\delta)v = s.$$

The solution of this differential equation is

$$v(t) = \frac{s}{n+\delta} + v(0)\exp\left[-(1-\alpha)(n+\delta)\,t\right].$$

Therefore,

$$k^{1-\alpha}(t) = \frac{s}{n+\delta} + k^{1-\alpha}(0)\exp\left[-(1-\alpha)(n+\delta)\,t\right].$$

Continuous time optimal control - the basics and applications

The general continuous time constrained optimal control problem can be written in the following form

$$\max_{\{u_t|x_t\}_{t=t_0}^{t_1}} \left\{ F = \int_{t_0}^{t_1} f(t, x_t, u_t)\, dt \right\} \tag{5}$$

$$s.t.$$

$$\frac{dx_t}{dt} =: \dot{x}_t = g(t, x_t, u_t) \tag{6}$$

$$x(t_0) = x_0, x(t_1) \text{ is free.} \tag{7}$$

For simplicity in the reminder of the text assume that f and g are continuously differentiable functions of time and u is piecewise continuous function of time. The constraint (6) is the law of motion of the state variable x_t which is predetermined at the beginning of each "period" (e.g., capital). Meanwhile u_t is the control variable (e.g., the amount of consumption) which, given the value of x_t and its law of motion (6), we choose in order to maximize F (e.g., the lifetime utility of household). The solution of this problem is the optimal path of state and control variables, (x_t^*, u_t^*). This path should be feasible. In other words, it should satisfy the dynamic constraint (6), i.e., the law of motion, and the initial condition (7). Moreover, given the definition of control and state variables $u_t^* = u_t^*(x_t)$. The values of $x(t_1)$ and $u(t_1)$ satisfy maximization problem (i.e., these values are a choice).

Digression: *When we consider a household's intertemporal problem we - usually - have*

$$f(t, x_t, u_t) = \tilde{f}(u_t(x_t)) \exp(-\rho t),$$

where $e^{-\rho t}$ is the discounting function and ρ is the discount rate, $\tilde{f}$ is the instantaneous (one period) utility from consumption $c_t (\equiv u_t)$ - erroneously we use the letter u for $\tilde{f}$. The optimal consumption c_t^*, in turn, is function of capital $k_t (\equiv x_t)$. Meanwhile, the constraint (6) represents the accumulation rule of assets/capital - in such case we basically solve the optimal consumption and saving paths, where the latter determines the optimal path of capital.

The rigorous approach to solving the problem is through Lagrangian. Let q_t be the Lagrange multiplier of the constraint (6). The optimal problem written in terms of Lagrangian is the following.

$$\max_{\{u_t|x_t\}_{s=t_0}^{t_1}} \left\{ L = \int_{t_0}^{t_1} \{ f(t, x_t, u_t) + q_t [g(t, x_t, u_t) - \dot{x}_t] \}\, dt \right\}.$$

Integrate the last term by parts

$$-\int_{t_0}^{t_1} q_t \dot{x}_t dt = -\int_{t_0}^{t_1} q_t dx_t = -q_{t_1} x_{t_1} + q_{t_0} x_{t_0} + \int_{t_0}^{t_1} x_t \dot{q}_t dt$$

and rewrite the L

$$\max_{\{u_t|x_t\}_{s=t_0}^{t_1}} \left\{ L = \int_{t_0}^{t_1} [f(t,x_t,u_t) + q_t g(t,x_t,u_t) + \dot{q}_t x_t]\, dt - q(t_1)\, x(t_1) + q(t_0)\, x(t_0) \right\}.$$

Necessary conditions

Let u_t^* be the optimal control function. Construct a family of "comparison" controls $u_t^* + \alpha h_t$, where h_t is some function and α is a real number. Denote $y(t,\alpha)$ the path of the state variable generated by the control $u_t^* + \alpha h_t$. Assume that $y(t,\alpha)$ is differentiable in arguments and $y(t_0,\alpha) = x(t_0)$ for any α [i.e., the optimal path x_t^* and $y(t,\alpha)$ start from the same point]. Notice that $y(t,0) = x_t^*$.

With this comparison controls the value of the Lagrangian L is

$$\begin{aligned} L(\alpha) &= \int_{t_0}^{t_1} [f(t, y(t,\alpha), u_t^* + \alpha h_t) + q_t g(t, y(t,\alpha), u_t^* + \alpha h_t) + \dot{q}_t y(t,\alpha)]\, dt \\ &\quad - q(t_1)\, y(t_1,\alpha) + q(t_0)\, x_0. \end{aligned}$$

Further, for simplicity let $L(\alpha)$ have one and interior maximum and let it be differentiable. Consider the following first order condition with slight abuse of previous notation

$$0 = \left.\frac{dL(\alpha)}{d\alpha}\right|_{\alpha=0} := L'_\alpha(0) \tag{8}$$

$$\begin{aligned} L'_\alpha(0) &= \int_{t_0}^{t_1} \left(f'_x y'_\alpha + q_t g'_x y'_\alpha + \dot{q}_t y'_\alpha + f'_u h_t + q_t g'_u h_t \right) dt \\ &\quad - q(t_1)\, y'_\alpha(t_1, 0). \end{aligned}$$

Apparently, the exact value of the RHS of this expression depends on q_t. It depends also on h_t and the way h_t influences the path of the state variable $y(t,\alpha)$. Meanwhile, the condition (8) should hold for any h_t (thus any y'_α). Therefore, one should select q_t (and $\dot{q}_t$) so that it eliminates the influence of h_t - note that we are basically deriving the envelope condition which states that the gradient of the maximand at the optimal point is orthogonal. Select

$$\dot{q}_t = -\left[f'_x(t, x^*, u^*) + q_t g'_x(t, x^*, u^*) \right] \tag{9}$$

$$q(t_1) = 0. \tag{10}$$

Under such a choice,

$$\begin{aligned} 0 &= \tilde{L}'(0) \\ &= \int_{t_0}^{t_1} \left[f'_u(t, x^*, u^*) + q_t g'_u(t, x^*, u^*) \right] h_t dt, \end{aligned} \tag{11}$$

which should hold for any h_t. Therefore, it should hold also for $h_t = f'_u(t, x^*, u^*) + q_t g'_u(t, x^*, u^*)$, which means that

$$\int_{t_0}^{t_1} \left[f'_u(t, x^*, u^*) + q_t g'_u(t, x^*, u^*)\right]^2 dt = 0. \tag{12}$$

This in turn implies that

$$f'_u(t, x^*, u^*) + q_t g'_u(t, x^*, u^*) = 0. \tag{13}$$

The equations (9), (10), and (13) are the necessary conditions for optimality. Together with (6) and (7) they determine the optimal path of control and state variables (x_t^*, u_t^*).

A simple way for deriving the necessary conditions

Form a Hamiltonian

$$H(t, x_t, u_t, q_t) \equiv f(t, x_t, u_t) + q_t g(t, x_t, u_t),$$

where q_t is the costate variable and is part of the solution to the optimal problem. The necessary conditions are obtained as:

$$\frac{\partial H}{\partial u} = 0, \tag{14}$$

$$-\frac{\partial H}{\partial x} = \dot{q}, \tag{15}$$

$$\frac{\partial H}{\partial q} = \dot{x}. \tag{16}$$

Notice that (14) is the same as (13), (15) is the same as (9), and (16) is (6). In addition, one gets an obvious condition $x(t_0) = x_0$ and $q(t_1) = 0$. The latter plays the role of transversality condition (TVC) in terms of finite time problem.

Digression: *The TVC requires that in a dynamically optimal path the choices are made in a way that ensures that at the end of the time horizon the state variable (e.g., capital) has no value and therefore the constraint is not binding. In terms of economics, one wants the value of capital in terms of utility to be zero at the planning horizon. If its value is positive then at the end of the time the choice leaves a positive value of capital that gives no utility, which is against the optimality.*

In terms of economics, the costate variable measures the shadow value of the associated state variable. Hence, it captures the gains (value) in the optimal control problem that stem from marginally increasing the state variable.

Sufficient conditions

In order the necessary conditions to be also sufficient we need further conditions.

- the functions f and g are concave in both arguments
- the optimal trajectories of x, u, and q satisfy the necessary conditions

- x_t and q_t are continuous functions with $q_t \geq 0$ for all t and if g is nonlinear in x or u, or both.

In order to prove the sufficiency define $f^* := f(t, x^*, u^*)$ and $g^* := g(t, x^*, u^*)$ and

$$D := \int_{t_0}^{t_1} (f^* - f)\, dt.$$

Given that we are solving for a maximum we need to show that

$$D \geq 0.$$

Since f is concave

$$f^* - f \geq f_x^{*\prime}(x^* - x) + f_u^{*\prime}(u^* - u).$$

Therefore,

$$\begin{aligned} D &\geq \int_{t_0}^{t_1} \left[f_x^{*\prime}(x^* - x) + f_u^{*\prime}(u^* - u)\right] dt \qquad (17) \\ &= \int_{t_0}^{t_1} \left[(x^* - x)(-qg_x^* - \dot{q}) + (u^* - u)(-qg_u^*)\right] dt. \end{aligned}$$

Notice that

$$\begin{aligned} \int_{t_0}^{t_1} -\dot{q}(x^* - x)\, dt &= -\int_{t_0}^{t_1} (x^* - x)\, dq = -(x^* - x)\, q\big|_{t_0}^{t_1} + \int_{t_0}^{t_1} (g^* - g)\, q dt \\ &= \int_{t_0}^{t_1} (g^* - g)\, q dt \end{aligned}$$

since $x^*(t_0) = x(t_0)$ and $q(t_1) = 0$. Therefore, (17) can be written as

$$D \geq \int_{t_0}^{t_1} \left[(g^* - g) - g_x^*(x^* - x) - g_u^*(u^* - u)\right] q dt \geq 0.$$

The latter integral is greater or equal to zero since $q \geq 0$ and g is a concave function of x and u. This shows that the necessary conditions together with concavity of f and g and non-negativity of q are also sufficient conditions.

Infinite horizon discounted problem

A usual economic problem is written as

$$\max_{\{u_t|x_t\}_{t=0}^{+\infty}} \left\{ U = \int_0^{+\infty} \tilde{f}(x_t, u_t) \exp(-\rho t)\, dt \right\} \tag{18}$$

$s.t.$

$$\dot{x}_t = g(t, x_t, u_t) \tag{19}$$

$$x(0) = x_0 > 0 \tag{20}$$

Notice that while $\tilde{f}$ - the instantaneous utility - is at time t the costate involves the value of changing the state from x_t incrementally over time, i.e., to $t + dt$. Thus the costate (and the Hamiltonian) has to take this into account. The present value Hamiltonian (discount factor $= e^{-\rho t}$) is

$$H^P = \tilde{f}(x_t, u_t) e^{-\rho t} + q_t^P g(t, x_t, u_t).$$

While, the current value Hamiltonian (discount factor $= 1$) is

$$\begin{aligned} H^C &= e^{\rho t} H^P = \tilde{f}(x_t, u_t) + q_t^C g(t, x_t, u_t), \\ q_t^P &= q_t^C \exp(-\rho t). \end{aligned}$$

The necessary conditions for optimality for present value Hamiltonian are

$$\begin{aligned} & H_u^P = \tilde{f}_u(x_t, u_t) e^{-\rho t} + q_t^P g_u(t, x_t, u_t) = 0, \\ & \dot{q}_t^P = -H_x^P = -\left[\tilde{f}_x(x_t, u_t) e^{-\rho t} + q_t^P g_x(t, x_t, u_t) \right], \\ & \lim_{t \to +\infty} q_t^P x_t = 0. \end{aligned}$$

For current value Hamiltonian using the the definition of q_t^C ($\dot{q}_t^P = \dot{q}_t^C e^{-\rho t} - \rho q_t^C e^{-\rho t}$) the necessary conditions are

$$\begin{aligned} & H_u^C = \tilde{f}_u(x_t, u_t) + q_t^C g_u(t, x_t, u_t) = 0, \\ & \dot{q}_t^C = \rho q_t^C - H_x^C = \rho q_t^C - \left[\tilde{f}_x(x_t, u_t) + q_t^C g_x(t, x_t, u_t) \right], \\ & \lim_{t \to +\infty} q_t^C x_t \exp(-\rho t) = 0. \end{aligned}$$

The last conditions are the TVCs for infinite horizon optimal problem. They states that the value of state variable in terms of utility should be zero in the limit when t approaches $+\infty$.

Many states and controls

There could be many state and control variables - the numbers do not need to coincide. For more than one state simply one adds extra costate variables (multiplying the RHS of the dynamic constraints) to the Hamiltonian. For more than one control, one needs to derive one optimal condition for each control variable.

Continuous time Bellman equation (Hamilton-Jacobi-Bellman equation)

This section is for those who are familiar with recursive dynamic programming in discrete time. It illustrates the analogy between continuous time necessary conditions and the conditions derived for discrete time. Here I consider only the discounted problem, though all the logic can be applied for the more general case.

With a slight abuse of notation define the maximized value of the objective function as a function of the initial state x_t and initial time t [it's sufficient since $u_t = u(x_t)$].

$$V(t, x_t) = \max_{\{u_s : \dot{x}_s = g(x_s, u_s) | x_s\}_{s=t}^{+\infty}} \left\{ \int_t^{+\infty} \tilde{f}(x_s, u_s) \exp[-\rho(s-t)]\, ds \right\}.$$

This can be rewritten in recursive form in the following way:

$$V(t, x_t) =$$
$$\max_{\{u_s : \dot{x}_s = g(x_s, u_s) | x_s\}_{s=t}^{t+\Delta t}} \left\{ \int_t^{t+\Delta t} \tilde{f}(x_s, u_s) \exp[-\rho(s-t)]\, ds + V(t+\Delta t, x_{t+\Delta t}) \exp(-\rho \Delta t) \right\},$$

for any Δt.

Subtract from both sides $V(t, x_t)$ and divide by Δt.

$$\begin{aligned} 0 \quad = \quad & \max_{\{u_s : \dot{x}_s = g(x_s, u_s) | x_s\}_{s=t}^{t+\Delta t}} \left\{ \frac{1}{\Delta t} \int_t^{t+\Delta t} \tilde{f}(x_s, u_s) \exp[-\rho(s-t)]\, ds \right. \\ & \left. + \frac{V(t+\Delta t, x_{t+\Delta t}) \exp(-\rho \Delta t) - V(t, x_t)}{\Delta t} \right\}. \end{aligned}$$

Take the limit $\Delta t \to 0$ (i.e., continuous time). To evaluate the first term in cuerly barckets use the L'Hopital's rule:

$$\begin{aligned} & \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} \tilde{f}(x_s, u_s) \exp[-\rho(s-t)]\, ds \\ = \quad & \lim_{\Delta t \to 0} \frac{\frac{\partial(t+\Delta t)}{\partial \Delta t} \tilde{f}(x_{t+\Delta t}, u_{t+\Delta t}) \exp(-\rho \Delta t)}{\frac{\partial \Delta t}{\partial \Delta t}} = \tilde{f}(x_t, u_t). \end{aligned}$$

Meanwhile, apply the definition of differential in order to get

$$\begin{aligned}
& \lim_{\Delta t \to 0} \frac{V\left(t+\Delta t, x_{t+\Delta t}\right) \exp\left(-\rho \Delta t\right) - V\left(t, x_t\right)}{\Delta t} \\
= & \lim_{\Delta t \to 0} \left\{ \frac{\left[\exp\left(-\rho \Delta t\right) - 1\right] V\left(t+\Delta t, x_{t+\Delta t}\right)}{\Delta t} + \frac{V\left(t+\Delta t, x_{t+\Delta t}\right) - V\left(t, x_{t+\Delta t}\right)}{\Delta t} \right. \\
& \left. + \frac{V\left(t, x_{t+\Delta t}\right) - V\left(t, x_t\right)}{\Delta t} \right\} \\
= & -\rho V\left(t, x_t\right) + \dot{V}\left(t, x_t\right) + V_x'\left(t, x_t\right) \dot{x}_t.
\end{aligned}$$

In sum this means that

$$\rho V\left(t, x_t\right) = \max_{u_t | x_t} \left\{ \tilde{f}\left(x_t, u_t\right) + V_x'\left(t, x_t\right) g\left(x_t, u_t\right) + \dot{V}\left(t, x_t\right) \right\}, \tag{21}$$

which is the Hamilton-Jacobi-Bellman equation. The second term in RHS captures the value gains from marginal change in the state variable, while the third term stands for the gains over time. The maximization gives the FOC:

$$\tilde{f}_u' + V_x' g_u' = 0,$$

which is the necessary condition for optimality, $H_u^C = 0$, where $V_x' = q_t^C$. This shows how the costate captures the effect of the change of the state on the objective function in current value terms. It also shows that q_t^C depends on dynamic decisions.

The envelope condition is

$$\rho V_x' = \tilde{f}_x' + V_x' g_x + V_{xx}'' g + \dot{V}_x'.$$

This is the necessary condition which describes the dynamics of the costate variable $\dot{q}_t^C$ given that $\dot{x}_t = g\left(x_t, u_t^*\right)$.

The Ramsey-Cass-Koopmans model - (Ramsey, 1928; Cass, 1965; Koopmans, 1965)

The Solow-Swan model assumes exogenous and constant savings rate, when the savings are the source of capital accumulation and are a decision variable for the savers (households). The Ramsey-Cass-Koopmans (in short Ramsey model) model endogenizes the savings rate.

We will see that in the steady-state the saving rate in the Ramsey model is constant, similar to Solow-Swan model. Therefore, basically we will simply re-examine the results of the Solow-Swan model, while relaxing the assumption of exogeneity of the savings.

Main assumptions

- Neoclassical production function: $Y = F(K, AL)$, where A is labor augmenting technology
- One-sector model of growth
  - i.e., both capital and consumption goods are produced with the same technology
- From the consumption-side
  - A continuum of infinitely lived and identical households (HHs) of mass L
  - The representative household (HH) is endowed with a unit of labor and chooses its consumption c, labor supply (and the evolution of assets $\dot{\varpi}$) to maximize the lifetime utility U, where
$$U = \int_0^{+\infty} u(c_t) L \exp(-\rho t)\, dt$$
    * $u(c)$ is the instantaneous utility from consumption of amount c of final good in per-capita terms. The instantaneous utility function is increasing and concave in c [i.e., $u' > 0$, $u'' < 0$] and satisfies the Inada conditions [i.e., $\lim_{c \to 0} u'(c) = +\infty$, $\lim_{c \to +\infty} u'(c) = 0$]. The concavity implies that HH prefers to smooth consumption over time.
    * The pure rate of time preference is $\rho > 0$
    * The budget constraint of HH is $\dot{\varpi} = (r - n)\varpi + w - c$
    * Since the utility here should be percieved in cardinal sense, the HH maximizes simply its utility multiplied by the size of the representative HH.

Further assumptions

- Technology grows at exogenous rate $\frac{\dot{A}}{A} = g_A$, $A(0) > 0$ - given
- Population grows at exogenous rate $\frac{\dot{L}}{L} = n$, $L(0) > 0$ - given

Market equilibrium

The firm side is similar to Solow-Swan model. Formally, setting the final goods as numeraire the representative firm's optimization problem is

$$\pi = \max_{K,L} \{F(K, AL) - RK - wL\}.$$

Therefore, the first order conditions (optimal rules) are

$$[K] \quad : \quad \frac{\partial \pi}{\partial K} = 0 \Leftrightarrow F_K = R, \tag{22}$$

$$[L] \quad : \quad \frac{\partial \pi}{\partial L} = 0 \Leftrightarrow F_L = w. \tag{23}$$

The representative HH chooses consumption path to maximize its lifetime utility. Its means of savings is accumulation of capital. Formally, the HH's problem is

$$\begin{aligned}
&\max_c \int_0^{+\infty} u(c_t) \exp\left[-(\rho - n)\, t\right] dt, \\
&s.t. \\
&\dot{\varpi} = (r - n)\varpi + w - c, \\
&\varpi(0) > 0 \text{ - given.}
\end{aligned}$$

If written in terms of current value Hamiltonian the HH's problem is

$$\begin{aligned}
&\max_c \{u(c) + q\left[(r - n)\varpi + w - c\right]\}, \\
&s.t. \\
&\varpi(0) > 0 \text{ - given,}
\end{aligned}$$

where the q is the shadow price of a unit of assets. Denote $H = u(c) + q\left[(r - n)\varpi + w - c\right]$. Therefore, the optimal rules are

$$[c] \quad : \quad \frac{\partial H}{\partial c} = 0 \Leftrightarrow u'(c) = q, \tag{24}$$

$$[\varpi] \quad : \quad \dot{q} = q\,(\rho - n) - \frac{\partial H}{\partial \varpi} = q\,(\rho - r)\,, \tag{25}$$

$$[TVC] \quad : \quad \lim_{\tau \to +\infty} \varpi(\tau) q(\tau) \exp\left[-(\rho - n)\,\tau\right] = 0.$$

From the first optimal rule it follows that

$$\dot{q} = \dot{c} u''(c).$$

Therefore,

$$r - \rho = -\frac{\dot{q}}{q} = -\frac{du'/dt}{u'} = -\frac{\dot{c}}{c}\frac{cu''(c)}{u'(c)},$$

or the optimal consumption path is

$$\frac{\dot{c}}{c}=-\frac{u'(c)}{u''(c)c}\left(r-\rho\right).$$

The TVC states that the value of the current asset holdings in infinity is zero. Formally, this is part of the open boundary problem given by the maximization of H.

Note the following

- $\frac{\dot{c}}{c}>0$ if $r-\rho>0$
- The sensitivity of the growth of consumption to $r-\rho$ is higher, the lower is $-\frac{u'(c)}{u''(c)c}$, which is the intertemporal elasticity of substitution. This elasticity is a measure of the responsiveness of consumption to changes in the marginal utility, i.e., it measures the willingness to deviate from consumption smoothing.
  - In a special case of constant intertemporal elasticity of substitution (CIES) utility function

    $$u(c)=\frac{c^{1-\theta}-1}{1-\theta};\theta>0$$

    the growth rate of consumption is given by

    $$\frac{\dot{c}}{c}=\frac{1}{\theta}\left(r-\rho\right)$$

    since

    $$-\frac{u'(c)}{u''(c)c}=-c^{-\theta-1}\frac{1}{-\theta c^{-\theta-1}}=\frac{1}{\theta}.$$

The CIES assumption is preserved in what follows.

The equilibrium in the asset market delivers again

$$\begin{aligned}R &= r+\delta,\\ W &= K.\end{aligned}$$

This gives the law of motion for capital $\dot{K}=Y-C-\delta K$, given that $Y=F\left(K,AL\right)=RK+wL$. The last equation is implied by the homogeneity of degree one assumption and states that the final good producer earns zero profit (note that there is perfect competition in final good market).

Balanced growth path

All variables of the model need to grow at constant rates (BGP).

- On a BGP $g_c=\frac{\dot{c}}{c}=const$. We have CIES utility function and $\rho=const$, therefore,

$$g_c=\frac{1}{\theta}\left(r-\rho\right).$$

On the BGP, therefore the interest rate should be constant, $r = const.$ In our setup, constant interest rate then will imply that savings rate is constant. The intuition behind is that on the BGP there should be no shifts in the shares of aggregates (notice that $C + S = Y$).

- Use the constant returns to scale assumption and write

$$r + \delta = \frac{\partial F(K, AL)}{\partial K} = \frac{\partial F\left(\frac{K}{AL}, 1\right)}{\partial \frac{K}{AL}} = f'\left(\frac{K}{AL}\right). \tag{26}$$

- Given that $f'' < 0$ the ratio $\frac{K}{AL}$ should be constant on the BGP in order to have $r = const.$

- Given that the ratio $\frac{K}{AL}$ is constant from the constant returns to scale assumption it follows that

$$\frac{Y}{K} = F\left(1, \frac{AL}{K}\right) = const.$$

  - Given that $\frac{Y}{K}$ is constant on the balanced growth path from the law of motion of capital it follows that the ratio $\frac{C}{K}$ also should be constant,

$$\frac{C}{K} = \frac{Y}{K} - \delta - g_K.$$

    Therefore, $g_K = g_Y = g_C$. Moreover, from $\frac{Y}{K} = F(1, \frac{AL}{K})$ it follows that on the balanced growth path $g_K = g_Y = g_C = g_L + g_A = n + g_A$.

In order to derive the steady-state and to characterize the transition dynamics, redefine the model in units of effective labor, i.e., AL. Let $\tilde{y} := \frac{Y}{AL}$, $\tilde{k} := \frac{K}{AL}$, and $\tilde{c} := \frac{C}{AL}$. Also, for $Y = AL \times F\left(\frac{K}{AL}, 1\right) := AL \times f(\tilde{k})$.

From these definitions it follows that $R = F_K = f'(\tilde{k})$, $w/A = F_L/A = f(\tilde{k}) - f'(\tilde{k})\tilde{k}$. In the steady-state $g_{\tilde{y}} = g_{\tilde{k}} = g_{\tilde{c}} = 0$. The steady-state and the transition dynamics of the model can be summarized by the following system of equations

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{1}{\theta}\left[f'\left(\tilde{k}\right) - \delta - \rho - \theta g_A\right], \tag{27}$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{f\left(\tilde{k}\right)}{\tilde{k}} - \frac{\tilde{c}}{\tilde{k}} - (\delta + n + g_A). \tag{28}$$

$$\tilde{k}(0) > 0 - \text{given}$$

$$\lim_{\tau \to +\infty} \tilde{k}(\tau)\, q(\tau) \exp\left[-(\rho - n - g_A)\tau\right] = 0. \tag{29}$$

The first equation follows from the optimal path of consumption given that

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\dot{c}}{c} - g_A,$$

and

$$r = f'\left(\tilde{k}\right) - \delta.$$

The second equation follows from the law of motion of capital given that

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{k}}{k} - g_A = \frac{\dot{K}}{K} - (n + g_A).$$

In the steady-state $\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{\tilde{c}}}{\tilde{c}} = 0 \Rightarrow$ we can solve for the steady-state values of $\tilde{c}$ and $\tilde{k}$ from (27) and (28). Let $F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$.

$$\begin{aligned}
& f'\left(\tilde{k}^*\right) = \delta + \rho + \theta g_A \\
\Rightarrow \quad & \tilde{k}^* = \left(\frac{\alpha}{\delta + \rho + \theta g_A}\right)^{\frac{1}{1-\alpha}} \Rightarrow f\left(\tilde{k}^*\right) = \left(\frac{\alpha}{\delta + \rho + \theta g_A}\right)^{\frac{\alpha}{1-\alpha}}, \qquad (30) \\
& \tilde{c}^* = f\left(\tilde{k}^*\right) - (\delta + n + g_A)\tilde{k}^* \\
\Rightarrow \quad & \tilde{c}^* = \left(\frac{\alpha}{\delta + \rho + \theta g_A}\right)^{\frac{\alpha}{1-\alpha}} \left[1 - \frac{\alpha(\delta + n + g_A)}{\delta + \rho + \theta g_A}\right]. \qquad (31)
\end{aligned}$$

Transition dynamics

The transition dynamics of the model in $\left(\tilde{k}, \tilde{c}\right)$ space is characterized by the Jacobian of the system of equations (28) and (27) evaluated in the neighborhood of the steady-state

$$\begin{aligned}
J &= \begin{pmatrix} \frac{\partial \dot{\tilde{k}}}{\partial \tilde{k}} & \frac{\partial \dot{\tilde{k}}}{\partial \tilde{c}} \\ \frac{\partial \dot{\tilde{c}}}{\partial \tilde{k}} & \frac{\partial \dot{\tilde{c}}}{\partial \tilde{c}} \end{pmatrix} \\
&= \begin{pmatrix} f'\left(\tilde{k}\right) - (\delta + n + g_A) & -1 \\ \frac{1}{\theta} f''(\tilde{k})\tilde{c} & \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g_A\right] \end{pmatrix}.
\end{aligned}$$

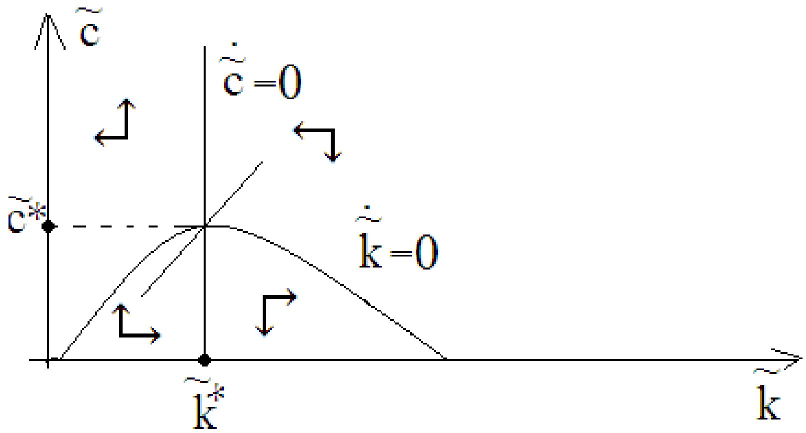
Notice that at the steady-state $\dot{\tilde{c}} = 0$, therefore,

$$J_{SS} = \begin{pmatrix} f'\left(\tilde{k}_{SS}\right) - (\delta + n + g_A) & -1 \\ \frac{1}{\theta} f''(\tilde{k}_{SS})\tilde{c}_{SS} & 0 \end{pmatrix} = \frac{1}{\theta} f''(\tilde{k}_{SS})\tilde{c}_{SS} < 0$$

Since $\det J = \mu_1 * \mu_2$, where $\mu_{1,2}$ are the eigenvalues of the matrix J, we have that μ_1 and μ_2 have different signs. This means that we have saddle path with one stable arm and one unstable arm. The stable arm corresponds to negative eigenvalue, while the unstable arm corresponds to the positive eigenvalue.[2]

The phase diagram of the system is as follows.

[2] For 2x2 matrices eigenvalues $\mu_{1,2}$ can be found from $\det M = \mu_1 * \mu_2$ and $trace(M) = \mu_1 + \mu_2$.



Starting at any level of capital $\tilde{k}(0) > 0$ the economy makes a discrete jump to the saddle-path which passes through the intersection of $\frac{\dot{\tilde{k}}}{\tilde{k}} = 0$ and $\frac{\dot{\tilde{c}}}{\tilde{c}} = 0$ and moves along that path toward the steady-state. (This saddle-path is illustrated in the above figure.) In other words, given $\tilde{k}(0) > 0$ consumers select $\tilde{c}(0)$ so that the economy is on the saddle-path. Any other $\tilde{c}(0)$ violates either Euler equation or the TVC.

- If $\tilde{c}(0)$ is higher, then the economy transits toward $\tilde{k} = 0$ and hits the vertical axis in finite time. At the point where $\tilde{k} = 0$ consumption $\tilde{c}$ jumps to 0 which violates Euler equation (Notice also that at that point if $\theta \geq 1$ the instantaneous utility u is $-\infty$).

- If $\tilde{c}(0)$ is lower, then the economy transits toward $\tilde{c} = 0$ and hits the horizontal axis in finite time. In order to see what happens during this transition rewrite the transversality condition (29) using (25) and (26) in the following manner

$$q(\tau) = q(0)\exp\left\{\int_0^\tau \left[\rho + \delta - f'\left(\tilde{k}\right)\right] dv\right\}$$
$$\lim_{\tau\to+\infty} \tilde{k}(\tau)\exp\left\{-\int_0^\tau \left[f'\left(\tilde{k}\right) - \delta - n - g_A\right] dv\right\} = 0.$$

At the steady-state in order TVC to hold it has to be the case that $f'\left(\tilde{k}\right) - (\delta + n + g_A) > 0$. At the same time at the steady-state $f'(\tilde{k}) - (\delta + \rho + \theta g_A) = 0$. Therefore, $\tilde{k}$ that satisfies $f'\left(\tilde{k}\right) = \delta + n + g_A$ (denote it by $\tilde{k}^{**}$) is higher than steady-state level of $\tilde{k}$ since $f'' < 0$. At $\tilde{k}^{**}$ (for fixed $\tilde{k}$) consumption is $\tilde{c} = \tilde{k}^{**}\left[\frac{f(\tilde{k}^{**})}{\tilde{k}^{**}} - (\delta + n + g_A)\right] > 0$. Moreover, at $\tilde{k}^{**}$ consumption declines and $\tilde{k}$ increases. However, as $\tilde{k}$ increases f' declines therefore $f'\left(\tilde{k}\right) - (\delta + n + g_A)$ declines below zero and TVC does not hold.

The behavior of the savings rate

The savings rate is given by $s = \frac{f(\tilde{k}) - \tilde{c}}{f(\tilde{k})} = 1 - \frac{\tilde{c}}{f(\tilde{k})}$. It endogenously changes along the transition path. It can increase or decrease during transition. During the transition there are two forces that affect the saving rate: income effect and substitution effect. If the the economy starts with low level of capital then as capital increases the interest rate declines. The inter-temporal substitution effect then reduces willingness to save. However, higher income tends to increase willingness to save.

Consider the simple case of Cobb-Douglas production function, $\tilde{y} = \tilde{k}^{\alpha}$. In such a case in the steady-state the saving rate can be pinned down from the steady-state values of capital and consumption per (effective) labor, (28) and (27),

$$s^* = 1 - \frac{\tilde{c}^*}{\left(\tilde{k}^*\right)^{\alpha}} = \frac{\alpha\left(\delta + n + g_A\right)}{\delta + \rho + \theta g_A}.$$

Note that the TVC condition (29) implies that

$$\rho - r - (\rho - n - g_A) < 0.$$

Therefore, at the steady-state from (28) it follows that

$$n + (1 - \theta)\, g_A < \rho.$$

This parameter restriction allows to have bounded utility and well-defined optimal problem. It implies that $s^* < 1$.

In order to assess the behavior of the savings rate on the transition path consider the average propensity to consume $\tilde{x} = \frac{\tilde{c}}{\tilde{y}} = 1 - s$ and assume that . The growth rate of $\tilde{x}$ is

$$\frac{\dot{\tilde{x}}}{\tilde{x}} = \frac{\dot{\tilde{c}}}{\tilde{c}} - \frac{\dot{\tilde{y}}}{\tilde{y}} = \frac{\dot{\tilde{c}}}{\tilde{c}} - \alpha\frac{\dot{\tilde{k}}}{\tilde{k}}.$$

Meanwhile, the transition dynamics of $\tilde{x}$ can be described by the expressions (27) and (28),

$$\begin{aligned}
\frac{\dot{\tilde{x}}}{\tilde{x}} &= \left\{\frac{1}{\theta}\left[\alpha\tilde{k}^{\alpha-1} - (\delta + \rho + \theta g_A)\right] - \alpha\left[(1 - \tilde{x})\,\tilde{k}^{\alpha-1} - (\delta + n + g_A)\right]\right\}, \\
&= \left[\alpha\left(\delta + n + g_A\right) - \frac{1}{\theta}\left(\delta + \rho + \theta g_A\right)\right] - \left(1 - \frac{1}{\theta} - \tilde{x}\right)\alpha\tilde{k}^{\alpha-1}. && (32) \\
\frac{\dot{\tilde{k}}}{\tilde{k}} &= (1 - \tilde{x})\,\tilde{k}^{\alpha-1} - (\delta + n + g_A)\,, && (33)
\end{aligned}$$

The locus of the first equation is then

$$\begin{aligned}
\tilde{x} &= \left(1 - \frac{1}{\theta}\right) + \psi\frac{\tilde{k}^{1-\alpha}}{\alpha}, \\
\psi &= \left[\frac{1}{\theta}\left(\delta + \rho + \theta g_A\right) - \alpha\left(\delta + n + g_A\right)\right].
\end{aligned}$$

Depending on the sign of ψ it is either increasing or decreasing with $\tilde{k}$. If $\psi = 0$ then $\tilde{x} = \left(1 - \frac{1}{\theta}\right)$.

Setting $\dot{\tilde{k}} = 0$ the locus of the second equation is

$$\tilde{x} = 1 - (\delta + n + g_A)\, \tilde{k}^{1-\alpha},$$

which is decreasing in $\tilde{k}$.

The Jacobian of the system is then

$$J = \begin{pmatrix} \alpha \tilde{k}^{\alpha-1} & \left(1 - \frac{1}{\theta} - \tilde{x}\right) \alpha (1-\alpha)\, \tilde{k}^{\alpha-2} \\ -\tilde{k}^{\alpha-1} & -(1-\tilde{x})(1-\alpha)\, \tilde{k}^{\alpha-2} \end{pmatrix}.$$

The determinant of J is

$$\begin{aligned} \det J &= -\alpha \tilde{k}^{\alpha-1} \left[(1-\tilde{x})(1-\alpha)\, \tilde{k}^{\alpha-2} \right] + \left(1 - \frac{1}{\theta} - \tilde{x}\right) \alpha (1-\alpha)\, \tilde{k}^{\alpha-2} \tilde{k}^{\alpha-1} \\ &= -\frac{1}{\theta} \alpha (1-\alpha)\, \tilde{k}^{2\alpha-3} < 0 \end{aligned}$$

This means that $\tilde{x}$ and the saving rate $(1-\tilde{x})$ are saddle-path stable since the eigenvalues have alternating signs. Therefore, depending on ψ, $\tilde{x}$ and $1-\tilde{x}$ either increase or decrease to their steady-state levels. In the special case when $\psi = 0$

$$1 - \tilde{x} = \frac{1}{\theta}.$$

The constant savings rate assumed in the Solow-Swan model is a special case of the Ramsey model, which is true under these specific parameter values.

- Note that when $\theta = 1$ (logarithmic utility) the inter-temporal substitution and income effects from changes in interest rate exactly cancel each other. When $\theta < 1$ consumers tolerate large deviations from smooth consumption profile and substitution effect is dominant. In particular the propensity to consume declines with interest rate. Whereas when $\theta > 1$ consumers do not tolerate large deviations from smooth consumption profile and income effect is dominant. In such a case propensity to consume increases with interest rate.

Evaluation of the model

- Predictions of the Solow-Swan model remain under this more general framework
- With exogenous technological change, the model performs well in accounting for facts 1-5. However, it cannot explain the facts 1 and 2
- Its explanatory power is not improved for fact 6

Extensions

Government in the Ramsey model

Assume that there is a government in the economy described by the Ramsey model. The government consumes amount G of final goods. For now, suppose that these purchases have neither utility nor productive effects. (We will allow later government purchases to positively affect firms' output.) The government also makes amount V lump-sum transfers to households. The government runs a balanced budget through taxes that it collects. The taxes are proportional and time-invariant levies on wage inome, τ_w, asset income, τ_a, consumption, τ_c, and firms' earnings $(\tilde{\pi})$, τ_f . Therefore, the government's budget constraint is

$$G + V = \tau_w wL + \tau_a rW + \tau_c C + \tau_f \tilde{\pi}.$$

These taxes modify the representative household's budget constraint. It is now

$$\dot{\varpi} = [(1-\tau_a) r - n] \varpi + (1-\tau_w) w - (1+\tau_c) c + v.$$

Assuming CIES utility function, this implies that the household's written in terms of current value Hamiltonian is

$$\max_c H = \frac{c^{1-\theta}-1}{1-\theta} + q \{[(1-\tau_a) r - n] \varpi + (1-\tau_w) w - (1+\tau_c) c + v\},$$
$s.t.$

$\varpi(0) > 0$ - given,

Therefore, the optimal rules are

$$\begin{aligned}
[c] &: \quad c^{-\theta} = (1+\tau_c) q, \\
[\varpi] &: \quad -\dot{q} = q [(1-\tau_a) r - \rho], \\
[TVC] &: \quad \lim_{\tau \to +\infty} \varpi(\tau) q(\tau) \exp[-(\rho - n) t] = 0.
\end{aligned}$$

From the first two conditions it follows that now familiar Euler equation is

$$\frac{\dot{c}}{c} = \frac{1}{\theta} [(1-\tau_a) r - \rho].$$

Notice that τ_c does not affect consumption path, though τ_a does so. This happens because τ_c is time-invariant and reduces equally both present and future consumption. Meanwhile, although τ_a is also time invariant it affects capital accumulation and, therefore, income and consumption in the future.

Suppose that the firms' taxable earnings are equal to output less wage payments and depreciation

$$\tilde{\pi} = F(K, L) - wL - \delta K.$$

Therefore, firms' profits after tax are

$$\pi = (1 - \tau_f) [F(K, L) - wL - \delta K] - rK.$$

notice that $R = r + \delta$.

This implies that firms' optimal rules are

$$\begin{aligned} [K] &: \ r = (1 - \tau_f) \left[\frac{\partial F(K, L)}{\partial K} - \delta \right], \\ [L] &: \ w = \frac{\partial F(K, L)}{\partial L}, \end{aligned}$$

which imply that firms make zero profits. These conditions in per capita terms are

$$\begin{aligned} r &= (1 - \tau_f) \left[f'(k) - \delta \right], \\ w &= f(k) - kf'(k). \end{aligned}$$

From the budget constraints of the government and household it follows that

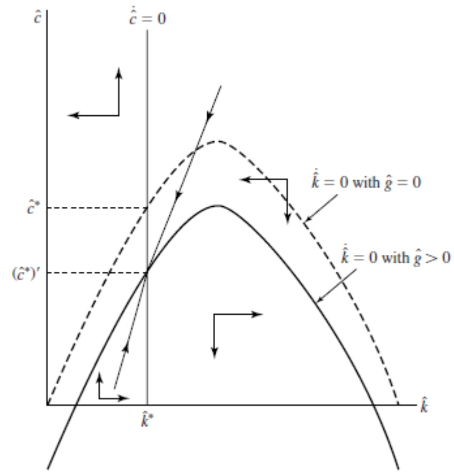
$$\dot{\varpi} = (r - n) \varpi + w - c + \tau_f [f(k) - w - \delta k] - \hat{g}.$$

In equilibrium assets and capital holdings are the same since there is no debt. Therefore, using the conditions above and Euler equation we have

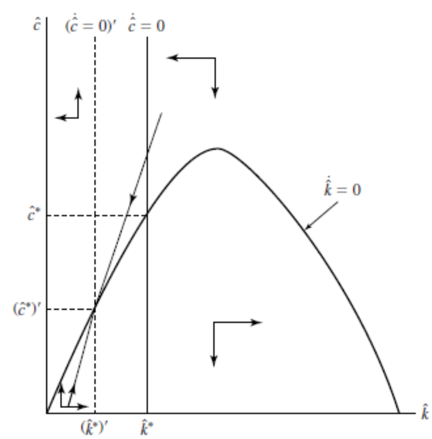
$$\begin{aligned} \dot{k} &= f(k) - (n + \delta) k - c - \hat{g}, \\ \frac{\dot{c}}{c} &= \frac{1}{\theta} \left\{ (1 - \tau_a) (1 - \tau_f) \left[f'(k) - \delta \right] - \rho \right\}. \end{aligned}$$

Notice that τ_w does not show up in these modified equilibrium rules. This happens because the household supplies its labor inelastically.

Assume now that $\hat{g} = \tau_a = \tau_f = 0$. Clearly, in such a case this system of differential equations is equivalent to (27), (28), and (29) with $g_A = 0$ and phase diagram of the system is the one offered above. Consider an unanticipated increase in $\hat{g}$ keeping $\tau_a = \tau_f = 0$. This increase is financed with a combination of τ_c and τ_w. In such a case $\dot{k}$ shifts downward, which implies a new and lower level of steady-state capital and consumption. After this unanticipated increase in $\hat{g}$ the model economy following TVC makes a discrete jump to the stable-arm and converges gradually to the new steady-state. The phase diagram of this transition is offered below (dashed lines are for the system where $\hat{g} = 0$!)



In turn, consider an unanticipated increase in any of the tax rates τ_a or τ_f. Such an increase shifts $\frac{\dot{c}}{c}$ to the right. Therefore, it implies lower steady-state level of capital and consumption. Similarly, after this unanticipated increase in a tax rate the model economy following TVC makes a discrete jump to the stable-arm and converges gradually to the new steady-state. The phase diagram of this transition is offered below (solid lines are for the system where $\tau_a = \tau_f = 0$!)



Productive public expenditures in the Ramsey model

Consider the set up offered above. However, suppose instead that government purchases are dedicated for creating productive public services for each firm. The production function of the firms takes a form of

$$Y = F\left(K, L; \frac{G}{Y}\right),$$

where G is total public expenditure, $\frac{G}{Y}$ is public services per output of a firm, $F\left(K, L; \frac{G}{Y}\right)$ is homogenous of degree one in K and L, and proportional to $\left(\frac{G}{Y}\right)^{\gamma}$ where $\gamma \in (0, 1)$, i.e.,

$$Y = F(K, L)\left(\frac{G}{Y}\right)^{\gamma},$$

Public services are productive means that $\frac{\partial Y}{\partial (G/Y)} > 0$, $\frac{\partial^2 Y}{\partial K \partial (G/Y)} > 0$, and $\frac{\partial^2 Y}{\partial L \partial (G/Y)} > 0$, which holds since $\gamma > 0$.

Suppose, further that (1) firms take $\frac{G}{Y}$ as a parameter, (2) assets are capital K, (3) depreciation is paid by the household, (4) public expenditure G is financed through a proportional and time-invariant tax on households' gross income $\tau = \tau_a = \tau_w$, $\tau \in (0, 1)$,

$$G = \tau \left(wL + RK \right),$$

and (5) transfers are $V = 0$.

This implies that the budget constraint of the households is given by

$$\dot{K} = (1 - \tau) \left(RK + wL \right) - \delta K - C.$$

Therefore, the representative household's budget constraint is

$$\dot{k} = (1 - \tau) \left(Rk + w \right) - (n + \delta) k - c$$

Assuming CIES utility function, this implies that the household's written in terms of current value Hamiltonian is

$$\begin{aligned}
&\max_c H = \frac{c^{1-\theta} - 1}{1 - \theta} + q \left[(1 - \tau) \left(Rk + w \right) - (n + \delta) k - c \right], \\
&s.t. \\
&k(0) > 0 \text{ - given,}
\end{aligned}$$

Therefore, the optimal rules are

$$\begin{aligned}
[c] &: \quad c^{-\theta} = q, \\
[\varpi] &: \quad -\dot{q} = q \left[(1 - \tau) R - \delta - \rho \right], \\
[TVC] &: \quad \lim_{\tau \to +\infty} \varpi(\tau) q(\tau) \exp \left[- (\rho - n) t \right] = 0.
\end{aligned}$$

From the first two conditions it follows that now familiar Euler equation is

$$\frac{\dot{c}}{c} = \frac{1}{\theta} \left[(1 - \tau) R - \delta - \rho \right].$$

In turn, firms' optimal rules are

$$\begin{aligned}
[K] &: \quad R = \frac{\partial F \left(K, L; \frac{G}{Y} \right)}{\partial K}, \\
[L] &: \quad w = \frac{\partial F \left(K, L; \frac{G}{Y} \right)}{\partial L},
\end{aligned}$$

which implies that

$$G = \tau Y.$$

Given that $G = \tau Y$ these conditions in per capita terms are

$$\begin{aligned} R &= f'(k)\,\tau^{\gamma}, \\ w &= f(k)\,\tau^{\gamma} - kf'(k)\,\tau^{\gamma}. \end{aligned}$$

This implies that

$$\begin{aligned} \dot{k} &= (1-\tau)\,\tau^{\gamma} f(k) - (n+\delta)\,k - c, \\ \frac{\dot{c}}{c} &= \frac{1}{\theta}\left[(1-\tau)\,\tau^{\gamma} f'(k) - \delta - \rho\right]. \end{aligned}$$

Suppose that the government sets tax rate τ in order to maximize $\frac{\dot{c}}{c}$. Given that $\gamma \in (0,1)$ function $\phi(\tau) = (1-\tau)\,\tau^{\gamma}$ is strictly concave in τ, i.e., there is a laffer curve. Therefore, the maximum of $\frac{\dot{c}}{c}$ is attained at

$$\tau^{*} = \frac{\gamma}{\gamma+1}.$$

It interesting alsowhat is the optimal tax rate for the Social Planner. In order to derive the optimal tax rate solve the Social Planner's problem:

$$\begin{aligned} &\max_{c,\tau} H = \frac{c^{1-\theta}-1}{1-\theta} + q\left[(1-\tau)\,\tau^{\gamma} f(k) - (n+\delta)\,k - c\right], \\ &s.t. \\ &k(0) > 0 \text{ - given,} \end{aligned}$$

The resulting optimal rules for consumption and savings are

$$\begin{aligned} [c] &: \ c^{-\theta} = q, \\ [k] &: \ \dot{q} = q(\rho - n) - \left[(1-\tau)\,\tau^{\gamma} f'(k) - (n+\delta)\right], \end{aligned}$$

which imply that consumption and capital follow

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{1}{\theta}\left[(1-\tau)\,\tau^{\gamma} f'(k) - \delta - \rho\right], \\ \dot{k} &= (1-\tau)\,\tau^{\gamma} f(k) - (n+\delta)\,k - c. \end{aligned}$$

This system of equations is apparently the same as the system of equations that governs the behaviour of the economy in decentralized equilibrium. Maximizing H with respect to τ, in turn, yields

$$\tau^{*} = \frac{\gamma}{\gamma+1}.$$

Therefore, with $\tau = \frac{\gamma}{\gamma+1}$ the decentralized equilibrium yields the first-best outcome.

The AK model - Spillovers *à la* Romer (1986)

We started from the standard Solow-Swan growth model. This model, as well as the Ramsey model, has neoclassical production function for the final good. With neoclassical production function these models cannot endogenously generate long run growth since the returns on capital decline with accumulation of capital.

The model presented below assumes a neoclassical production function - at the "individual level." In addition, it assumes that the labor augmenting technology is a function of average per-capita capital stock. While doing so, it has in mind some "learning by doing" effects/spillovers as in Arrow (1962), i.e., the workers learn/become more productive while working with desks, computers, etc.

Main assumptions

- The level of technology/ efficiency that augments the labor input in the production is a function of the average capital-labor ratio in the economy. The motivation for this is that investment of a firm brings productivity gains from its use of the labor. The firm builds up the knowledge (technical expertise) of how to efficiently use the capital by accumulating it, i.e., there is "learning-by-doing" (e.g., production lines). This learning-by-doing effect has an aggregate impact, when any individual firm's technical efficiency is public knowledge, so that all firms can benefit from the technological advance for the use of capital in the production. This gives the link between "A_i"(i.e., some *ith* firm's efficiency) and the capital-labor ratio in the economy.
  - The level of technology is assumed to be $A \equiv \lambda k$, where k is the average of per-capita capital stock and $\lambda > 0$ measures the efficiency of use of the capital.
  - There are continuum of (ex ante) identical firms of mass one.
  - The production function in a firm takes the form: $Y_i = F(K_i, AL_i) = L_i F(k_i, \lambda k) \equiv L_i \lambda k f\left(\frac{k_i}{\lambda k}\right)$
    - Thus, there are decreasing returns to capital at the firm-level since the firm is so small that it does not take into account its impact on A. However, there are constant returns to capital in symmetric equilibrium since $k_i = k$.
- One-sector model of growth: $\dot{K} = Y - C - \delta K$
- From the consumption-side, the representative household (HH) chooses its consumption and next period assets to maximize its lifetime utility $U = \int_0^{+\infty} u(c) \exp\left[-(\rho - n)t\right] dt$, subject to its budget constraint: $\dot{\varpi} = (r-n)\varpi + w - c$, where $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$.

Further assumptions

- Population grows at exogenous rate n

Market equilibrium

For any individual producer the level of efficiency $A(\lambda k)$ is taken as given, along with the prices of inputs, when choosing capital and labor to maximize its per period profits π_i, i.e.,

$$\max_{K_i, L_i} \pi_i = F(K_i, AL_i) - RK_i - wL_i.$$

Therefore the optimal rules are

$$\begin{aligned} [K] &: \frac{\partial \pi_i}{\partial K_i} = 0 \Leftrightarrow \\ R &= F_{K_i}(K_i, AL_i) = \frac{\partial L_i \lambda k f\left(\frac{k_i}{\lambda k}\right)}{\partial K_i} = f'\left(\frac{k_i}{\lambda k}\right), \end{aligned}$$

and

$$\begin{aligned} [L] &: \frac{\partial \pi_i}{\partial L_i} = 0 \Leftrightarrow \\ w &= F_{L_i}(K_i, AL_i) = AF_{AL_i}(K_i, AL_i) \\ &= \frac{\partial L_i \lambda k f\left(\frac{k_i}{\lambda k}\right)}{\partial L_i} = \lambda k f\left(\frac{k_i}{\lambda k}\right) - k_i f'\left(\frac{k_i}{\lambda k}\right). \end{aligned}$$

Since all producers are identical, the equilibrium must be symmetric, i.e., $k_i = k$ for $\forall i$. Therefore, from the asset market equilibrium condition ($\varpi = k$ and $r = R - \delta$) it follows that the net rate of returns on assets in the economy is

$$r = f'\left(\frac{1}{\lambda}\right) - \delta.$$

The standard optimal consumption path that comes from the intertemporal maximization problem of the HH is

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho).$$

As a result, the equilibrium is characterized by two dynamic equations (i.e., the law of motion of capital in per-capita terms and the optimal consumption path)

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right], \\ \frac{\dot{k}}{k} &= \lambda f\left(\frac{1}{\lambda}\right) - \delta - n - \frac{c}{k}. \end{aligned}$$

Meanwhile, the standard TVC applies, that the value of the assets (capital) is equal to zero in the end of the planning horizon

$$\lim_{t \to +\infty} k(t)\, q(t) \exp\left[-(\rho - n)t\right] = 0.$$

Balanced growth path

- Given that in equilibrium the marginal product of capital is independent of the level of per-capita capital, it is always constant, i.e.,

$$r = f'\left(\frac{1}{\lambda}\right) - \delta.$$

  - The constant returns to capital ensure that there can exist long-run growth driven by capital accumulation in this model.

- Consumption growth is constant in any equilibrium path, i.e.,

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right].$$

  - Increase of the per capita consumption over time requires that the externalities in the capital stock (as measured by λ) be large enough, to increase the net marginal product of capital, $f'\left(\frac{1}{\lambda}\right) - \delta$, above the time preference rate, i.e.,

$$f'\left(\frac{1}{\lambda}\right) - \delta > \rho.$$

  - Given an initial level (choice) of consumption per capita, $c(0)$, the economy is always along a BGP. In other words, there is no transition dynamics in this model and the economy immediately jumps to a balanced growth path. The proof is offered below.
    - From the first dynamic equation (optimal path of consumption) follows that

$$\begin{aligned} \int \frac{\dot{c}}{c} dt &= \int \frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right] dt \Rightarrow \\ \int \frac{1}{c}\frac{dc}{dt} dt &= \frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right] t + m_0 \Rightarrow \end{aligned}$$

    where m_0 is some constant,

$$\begin{aligned} \int \frac{1}{c} dc &= \frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right] t + m_0 \Rightarrow \\ \ln c &= \frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right] t + m_0 \Rightarrow \\ c &= e^{m_0} e^{\frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right] t}. \end{aligned}$$

    The e^{m_0} is then the initial value of the consumption since $c(0) = e^{m_0}$. From the law of motion of per-capita capital it follows that

$$\dot{k} = \left(\lambda f\left(\frac{1}{\lambda}\right) - \delta - n\right) k - c\left(0\right) e^{\frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right] t}.$$

The solution to this differential equation is given by the (linear combination) sum of the general solution of the homogenous differential equation and a particular solution of the non-homogenous differential equation. In other words, solve first

$$\dot{k} = \left(\lambda f\left(\frac{1}{\lambda}\right) - \delta - n\right) k \Rightarrow k^{h,g} = k^{h,g}\left(0\right) e^{\left(\lambda f\left(\frac{1}{\lambda}\right) - \delta - n\right) t},$$

then find a solution to the general equation. A solution is

$$k^{nh,p} = m_1 c\left(0\right) e^{\frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right] t},$$

where m_1 is found by plugging the $k^{nh,p}$ to the law of motion of per-capita capital, i.e.,

$$\begin{aligned}
& m_1 c\left(0\right) \frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right] \exp\left\{\frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right] t\right\} = \\
& \left[\lambda f\left(\frac{1}{\lambda}\right) - \delta - n\right] m_1 c\left(0\right) \exp\left\{\frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right] t\right\} \\
& -c\left(0\right) \exp\left\{\frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right] t\right\} \\
\Rightarrow & \\
& m_1 = 1/\left\{\left[\lambda f\left(\frac{1}{\lambda}\right) - \delta - n\right] - \frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right]\right\}.
\end{aligned}$$

Thus the solution of the differential equation is

$$k = k^{h,g}\left(0\right) \exp\left\{\left[\lambda f\left(\frac{1}{\lambda}\right) - \delta - n\right] t\right\} + m_1 c\left(0\right) \exp\left\{\frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right] t\right\}.$$

Given that $f'' < 0$ (and $\rho > n$ to have bounded U) the following inequalities hold

$$\lambda f\left(\frac{1}{\lambda}\right) - \delta - n > f'\left(\frac{1}{\lambda}\right) - \delta - \rho > 0.$$

From household's optimization problem, in turn, follows that the rate of return on capital accumulation in terms of utility is

$$\begin{aligned}
\frac{\dot{q}}{q} &= \rho - r \\
\lim_{t \to +\infty} k(t) q(t) \exp\left[-\left(\rho - n\right) t\right] &= 0.
\end{aligned}$$

Therefore, the transversality condition requires that $k^{h,g}(0) = 0$ and

$$k = m_1 c(0) \exp\left\{\frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right] t\right\}$$
$$m_1 c,$$

which means that per-capita capital grows at the same (constant) rate as the consumption.

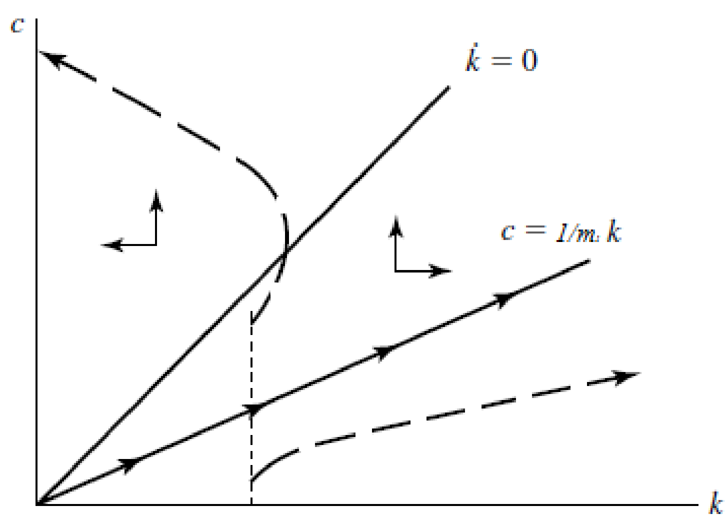
- There is another and more intuitive way to show that the growth rate of per-capita capital is always constant and equal to the growth rate of consumption.

  - Capital per head will grow at a constant rate only if the consumption-to-capital ratio remains constant over time. In order to examine the properties of the BGP, examine the behavior of the $\frac{c}{k}$ ratio, i.e.,

$$\frac{\dot{c/k}}{c/k} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k} = \frac{c}{k} + \frac{f'\left(\frac{1}{\lambda}\right) - \theta\lambda f\left(\frac{1}{\lambda}\right) - (1-\theta)\delta - \theta n - \rho}{\theta}.$$

  Note that the above dynamic equation is unstable in $\frac{c}{k}$. Moreover, unless $\frac{\dot{c}}{c} = \frac{\dot{k}}{k}$, the growth rate of k either should cease or should increase to infinity. Both cases would violate TVC. Therefore, it must be that the BGP is characterized by constant $\frac{c}{k}^* = \frac{(1-\theta)\delta + \theta\lambda f\left(\frac{1}{\lambda}\right) - f'\left(\frac{1}{\lambda}\right) + \rho - \theta n}{\theta}$. In the event of a structural change, consumption in this model makes a "discrete" shift to ensure that $\frac{c}{k}(t) = \frac{c}{k}^*$ and the economy is set again on a BGP.

- In terms of phase diagram this corresponds to having



- If consumption is lower than $c = (1/m_1)\,k$ then capital grows at a higher rate than c and TVC is violated. If consumption is higher than $c = (1/m_1)\,k$ then in finite time the economy converges to vertical axis where $k = 0$. At that point consumption has to make a discrete jump to 0 and Euler equation is violated.

- The steady-state growth rate of per capita consumption and capital is

$$g = \frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{1}{\theta}\left[f'\left(\frac{1}{\lambda}\right) - \delta - \rho\right].$$

- The consumption to capital ratio and savings rate in this economy are

$$\begin{aligned}
\frac{c}{k} &= \frac{1}{m_1} = \frac{(1-\theta)\left[\delta - \lambda f\left(\frac{1}{\lambda}\right)\right] - \theta n + \rho + \lambda f\left(\frac{1}{\lambda}\right) - f'\left(\frac{1}{\lambda}\right)}{\theta}, \\
s &= \frac{Y - C}{Y} = 1 - \frac{c}{k}\frac{1}{\lambda f\left(\frac{1}{\lambda}\right)} = \frac{f'\left(\frac{1}{\lambda}\right) + (\theta - 1)\delta + \theta n - \rho}{\theta \lambda f\left(\frac{1}{\lambda}\right)}.
\end{aligned}$$

Comparative statics

- Increase in λ increases g and has ambiguous effect on savings rate s
  - Higher λ implies higher growth rate g since it increases the effectiveness of capital per head in increasing the labor productivity
- Increase in θ or ρ decrease both g and s
  - The higher θ and ρ imply lower saving rate s since the first one increases the consumption smoothing and the second one induces higher consumption at current period. In turn, lower savings rate implies lower growth g through lower capital accumulation and, thus, learning-by-doing spillovers.

Social Planner's problem

Note that the welfare theorems will not work in this model since knowledge externalities are assumed, which are inherent to the accumulated capital stock and are not internalized in competitive equilibrium. Due to these externalities the competitive equilibrium outcome may not be the first best (the socially optimal one).

The Social Planner (SP) internalizes these externalities and therefore he identifies that $k_i = k$, when considering the marginal product of capital in the production. The SP selects the paths of quantities that deliver maximum utility (social welfare). The SP's problem is

$$
\begin{aligned}
&\max_{c} \int_{0}^{+\infty} \frac{c_t^{1-\theta}-1}{1-\theta} \exp\left[-\left(\rho-n\right)t\right]dt, \\
&s.t. \\
&\dot{k} = \lambda k f\left(\frac{1}{\lambda}\right) - c - \left(n+\delta\right)k, \\
&k\left(0\right) > 0 - \text{given}, \\
&\lim_{t\to+\infty} k\left(t\right)q\left(t\right)\exp\left[-\left(\rho-n\right)t\right] = 0.
\end{aligned}
$$

This problem in terms of current value Hamiltonian is

$$
\max_{c} H_{SP} = \frac{c^{1-\theta}-1}{1-\theta} + q\left[\lambda k f\left(\frac{1}{\lambda}\right) - c - \left(n+\delta\right)k\right].
$$

The first order conditions (optimal rules) are

$$
\begin{aligned}
\left[c\right] &: \quad c^{-\theta} = q, \\
\left[k\right] &: \quad \dot{q} = q\left(\rho-n\right) - q\left[\lambda f\left(\frac{1}{\lambda}\right) - \left(n+\delta\right)\right].
\end{aligned}
$$

Therefore,

$$
\frac{\dot{c}}{c} = \frac{1}{\theta}\left(\lambda f\left(\frac{1}{\lambda}\right) - \delta - \rho\right).
$$

The same way as in competitive equilibrium it can be argued that the capital per head grows at the same (constant) rate as consumption and there is no transition. Moreover, TVC holds as long as

$$
\left(1-\theta\right)\left(\lambda f\left(\frac{1}{\lambda}\right) - \delta - \rho\right) - \theta\left(\rho-n\right) < 0
$$

which ensures that U is bounded.

Thus,

$$
g^{SP} \equiv \frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{1}{\theta}\left[\lambda f\left(\frac{1}{\lambda}\right) - \delta - \rho\right],
$$

and

$$
\left(\frac{c}{k}\right)^{SP} \equiv \frac{c}{k} = \frac{\left(1-\theta\right)\left[\delta - \lambda f\left(\frac{1}{\lambda}\right)\right] - \theta n + \rho}{\theta}.
$$

Given that f is concave function

$$
\frac{f\left(\frac{1}{\lambda}\right)/\left(1/\lambda\right)}{f'\left(\frac{1}{\lambda}\right)} > 1.
$$

Therefore, SP achieves higher growth rate

$$
g^{SP} > g.
$$

Moreover, in percentage terms SP saves more than agents in decentralized equilibrium

$$\frac{c}{k} = \left(\frac{c}{k}\right)^{SP} + \frac{\lambda f\left(\frac{1}{\lambda}\right) - f'\left(\frac{1}{\lambda}\right)}{\theta}.$$

- This is why SP achieves higher long-run growth.
- A growth promoting policy would try to eliminate the difference between the $\frac{c}{k} = \left(\frac{c}{k}\right)^{SP}$. Since $\frac{c}{k} < \left(\frac{c}{k}\right)^{SP}$ this policy would motivate higher savings in decentralized equilibrium. Therefore, a policy could be a simple subsidy to the production that equates the private return to capital to the socially optimal one and finances the subsidy through lump sum tax imposed on HH.

Human capital accumulation: The Lucas (1988) model

Preliminary remarks

In contrast to the models presented so far, Lucas (1988) assumes that there are two types of assets endogenously accumulated in the economy, physical capital and human capital. The idea is very simple and it is that in addition to producing, for instance, more infrastructure we also "produce" better (or more) educated workers. The better educated workers, then, produce more, while using the same amount of labor. Therefore, the labor productivity increases, and this, together with capital accumulation, may enable long run growth.[3]

It is worth emphasizing that the biggest difference between Romer (1986) and Lucas (1988) models is that the latter endogenizes the process of labor productivity growth through human capital accumulation, when the former thinks of spillover effects (learning-by doing).[4]

If presented in one sector form, the final good production side and the asset accumulation processes of Lucas (1988) model can be written as

$$Y = AK^{\alpha}H^{1-\alpha} = C + I_K + I_H,$$

where H is the human capital input, I_K and I_H are the investments for physical capital and human capital accumulation, i.e.,

$$\begin{aligned} I_K &= \dot{K} + \delta_K K, \\ I_H &= \dot{H} + \delta_H H, \end{aligned}$$

where δ_K is the depreciation rate physical capital and δ_H is the depreciation rate of human capital. Given that we consider an equilibrium where both assets are accumulated, the returns to both assets should be equal. For the current exercise let $\delta_K = \delta_H \equiv \delta$. This would imply that,

$$\begin{aligned} \frac{\partial Y}{\partial K} - \delta &= \frac{\partial Y}{\partial H} - \delta \Rightarrow \\ \alpha \frac{Y}{K} &= (1-\alpha)\frac{Y}{H} \Rightarrow \\ H &= \frac{1-\alpha}{\alpha} K \Rightarrow \\ Y &= \left[A \left(\frac{1-\alpha}{\alpha} \right)^{1-\alpha} \right] K. \end{aligned}$$

Thus, in terms of algebra, the ideas behind the Romer (1986) and Lucas (1988) models are quite

[3] This model is much motivated by Becker (1962).

[4] Lucas offers also a learning-by-doing model in the same paper, Lucas (1988).

similar. Both end up having an aggregate production function which is linear in capital.[5,6]

This one sector model is a simple outline of the model that Lucas (1988) offers. The model with corresponding assumptions is the following.

Main assumptions

- This is a two-sector model of growth, where the physical capital is still produced with the same technology as the consumption good, but human capital is produced with a different technology.

- Human capital is the essential input for the production of new human capital. The motivation for this is that the human capital of one generation is an important factor in affecting the formation of human capital of the later generations. If the production of human capital is within a household, that would be the human capital "embodied" in the parents. If its production is through formal education, then that would be the human capital of the teachers with their methodologies (e.g., books).

  - The accumulation (production) of human capital H follows a law of motion

    $$\dot{H} = BH_H - \delta_H H,$$

    where H_H is the human capital used for its own production. Every unit of human capital produces $B > 0$ new units of human capital. This stock depreciates at a rate $\delta_H > 0$ (e.g., due to "aging").

    Note: There are no diminishing returns to the production of human capital with this type of production function for the human capital. The non-decreasing returns to the production of human capital will be the engine of long-run growth in this model. The increasing stock of human capital drives the accumulation of physical capital and the economy grows indefinitely. If, instead, the production of human capital had decreasing returns to its input, this model would have the same predictions as the Solow-Swan model and it would not be able explain growth in the long-run.

- The production of final output combines physical capital stock and human capital H_Y, i.e., $Y = AK^{\alpha}H_Y^{1-\alpha}$,where H_Y is the human capital used in production of final good. Standard neoclassical assumptions apply.

- From the consumption-side, the representative HH chooses its consumption path, the assets (physical and human capital) in the next period and the allocation of its human capital input between final good and human capital production, in order to maximize its lifetime utility

[5] The models that will be presented throughout the course, again, will be quite similar to Romer (1986) in terms of the aggregate production function.

[6] This analysis can be extended to any neoclassical production function $F(K, H)$.

$U=\int_0^{+\infty}u(C)\exp\left(-\rho t\right)dt$, subject to standard budget constraint and the law of motion of human capital, where $u\left(C\right)=\frac{C^{1-\theta}-1}{1-\theta}$.

Market equilibrium

Define the fraction of human capital used in the production of final output as: $u\equiv\frac{H_Y}{H}$. There are no externalities involved in the input and output markets. By the first welfare theorem it is known that the competitive equilibrium will achieve the first-best allocations. The second welfare theorem implies that one can directly solve for the optimal allocations, as there are prices that will support the competitive equilibrium that achieves such intratemporal and intertemporal allocations.

The intertemporal allocation problem has two controls, consumption and allocation of human capital in the two sectors of production that compete for it. There are two state variables, human and physical capital. Physical capital accumulation requires the saving of output (consumption), while the human capital accumulation requires investments in terms of real resources (human capital needs to be driven out of the production of final output).

Having in mind the welfare theorems, the representative households problem is

$$\begin{aligned}
&\max_{u,C}U=\int_0^{+\infty}\frac{C_t^{1-\theta}-1}{1-\theta}\exp\left(-\rho t\right)dt,\\
&s.t.\\
&\dot{K}=AK^{\alpha}(uH)^{1-\alpha}-\delta_K K-C, &(34)\\
&\dot{H}=B\left(1-u\right)H-\delta_H H &(35)\\
&K\left(0\right),H\left(0\right)>0-given.
\end{aligned}$$

Let q_K and q_H be the shadow prices for the physical and human capital, respectively. The problem, if written in terms of current value Hamiltonian, is given by

$$\max_{u,C}H_{LC}=\frac{C^{1-\theta}-1}{1-\theta}+q_K\left[AK^{\alpha}(uH)^{1-\alpha}-\delta_K K-C\right]+q_H\left[B(1-u)H-\delta_H H\right].$$

The optimal rules are

$$\begin{aligned}
[C] &: C^{-\theta}=q_K, &(36)\\
[u] &: q_K(1-\alpha)\frac{Y}{u}=q_H BH, &(37)\\
[K] &: \dot{q}_K=q_K\rho-q_K\left(\alpha\frac{Y}{K}-\delta_K\right) &(38)\\
&= -q_K\left(\alpha\frac{Y}{K}-\delta_K-\rho\right),\\
[H] &: \dot{q}_H=q_H\rho-\left\{q_K(1-\alpha)\frac{Y}{H}+q_H\left[B(1-u)-\delta_H\right]\right\}. &(39)
\end{aligned}$$

The standard TVCs apply, one for each of the state variables, i.e.,

$$\begin{aligned}\lim_{t\to+\infty} q_K(t)\,K(t)\exp(-\rho t) &= 0,\\ \lim_{t\to+\infty} q_H(t)\,H(t)\exp(-\rho t) &= 0.\end{aligned}$$

From (36) and (38) follows that

$$\frac{\dot{C}}{C} = \frac{1}{\theta}\left(\alpha\frac{Y}{K} - \delta_K - \rho\right). \tag{40}$$

From (37) and (39) follows that

$$\begin{aligned}\dot{q}_H &= q_H\rho - \left\{q_H\frac{BH}{(1-\alpha)\frac{Y}{u}}(1-\alpha)\frac{Y}{H} + q_H\left[B(1-u)-\delta_H\right]\right\}\\ &= q_H\rho - \{q_H Bu + q_H\left[B(1-u)-\delta_H\right]\} \Rightarrow\\ \dot{q}_H &= -q_H\left(B - \delta_H - \rho\right). \qquad (41)\end{aligned}$$

Balanced growth path

From the optimal consumption path (40) and that the growth rate of consumption at steady-state should be constant, follows that the aggregate output Y and capital stock K grow at the same rate, i.e., $g_K = g_Y$. From the resource constraint (or the law of motion of capital) $\frac{\dot{K}}{K} = \frac{AK^{\alpha}(uH)^{1-\alpha}}{K} - \delta_K - \frac{C}{K} = \frac{Y}{K} - \delta_K - \frac{C}{K}$ follows that in steady-state the consumption and capital grow at the same rate, i.e., $g_C = g_K = g_Y$. From the production of human capital, given that $B, \delta_H = const$ and in steady-state $\frac{\dot{H}}{H} = const$, follows that the share of human capital in production of final good is constant, i.e.,

$$\begin{aligned}\dot{H} &= B(1-u)H - \delta_H H \Rightarrow\\ u &= 1 - \frac{g_H + \delta_H}{B} = const. \qquad (42)\end{aligned}$$

From the production of final good $Y = AK^{\alpha}(uH)^{1-\alpha}$ follows that

$$\frac{Y}{K} = \frac{AK^{\alpha}(uH)^{1-\alpha}}{K} = A\left(u\frac{H}{K}\right)^{1-\alpha}.$$

Given that $g_K = g_Y$ and $A, u = const$ the growth rates of physical and human capital are equal, i.e., $g_K = g_H = g_C = g_Y \equiv g$. From (37) and given that $g_H = g_Y$ and $u, \alpha, B = const$ follows that

$$\frac{\dot{q}_H}{q_H} = \frac{\dot{q}_K}{q_K}.$$

This result should not be a surprising result. Given that both human and physical capital should be accumulated in balanced growth path equilibrium, the rates of return on their accumulation $\frac{\dot{q}_i}{q_i}$ $(i = H, K)$ are equal.

This equality implies then that

$$-\frac{\dot{q}_H}{q_H} = (B - \delta_H - \rho) = \left(\alpha \frac{Y}{K} - \delta_K - \rho\right) = -\frac{\dot{q}_K}{q_K}.$$

Therefore, from the optimal consumption path (40) it follows that

$$g = \frac{1}{\theta}\left(\alpha \frac{Y}{K} - \delta_K - \rho\right) = \frac{1}{\theta}\left(B - \delta_H - \rho\right). \tag{43}$$

Thus, given that $g_H = g_C = g$, from (42) it follows that

$$\begin{aligned} u^* &= 1 - \frac{(B - \delta_H - \rho) + \theta \delta_H}{\theta B} \\ &= \frac{(\theta - 1)(B - \delta_H) + \rho}{\theta B}. \end{aligned} \tag{44}$$

In order to show that $u^* > 0$, consider, for instance, the TVC for human capital

$$\lim_{t \to +\infty} q_H(t) H(t) \exp(-\rho t) = 0.$$

Given that in steady-state $\frac{\dot{q}_H}{q_H} = -(B - \delta_H - \rho)$ and $g_H = \frac{1}{\theta}(B - \delta_H - \rho) \Rightarrow$ in order the TVC to hold

$$\begin{aligned} -(B - \delta_H - \rho) - \rho + \frac{1}{\theta}(B - \delta_H - \rho) &< 0 \Rightarrow \\ \frac{1}{\theta}(B - \delta_H - \rho) &< (B - \delta_H) \Rightarrow \\ (\theta - 1)(B - \delta_H) + \rho &> 0 \Rightarrow \\ u^* &> 0. \end{aligned}$$

Meanwhile, from (43) follows that in steady-state

$$\left(\frac{Y}{K}\right)^* = \frac{B - \delta_H + \delta_K}{\alpha}. \tag{45}$$

From the law of motion of capital follows that in steady-state

$$\begin{aligned} \left(\frac{C}{K}\right)^* &= \left(\frac{Y}{K}\right)^* - \delta_K - \frac{1}{\theta}\left[\alpha\left(\frac{Y}{K}\right)^* - \delta_K - \rho\right] \\ &= \frac{1}{\theta}\left[(\theta - \alpha)\frac{B - \delta_H + \delta_K}{\alpha} - (\theta - 1)\delta_K + \rho\right]. \end{aligned}$$

Therefore the savings rate is

$$s^* = \left(\frac{Y - C}{Y}\right)^* = \frac{\left(\frac{Y}{K}\right)^* - \left(\frac{C}{K}\right)^*}{\left(\frac{Y}{K}\right)^*} = \frac{\alpha(B - \delta_H - \rho) + \alpha\theta\delta_K}{\theta(B - \delta_H + \delta_K)}.$$

Comparative statics

- Increase in B increases g^*. Ambiguous effects on $s^*$and u^* (for $\frac{1}{\theta} \geq 1$, s^* increases and u^* decreases)
- Increase in θ (or ρ) decreases both g^* and s^*, while it increases u^*
- Increase in α increases s^* but has no effect on g^* and u^*

Transition dynamics

In order to describe the transition dynamics of this model consider variables that do not grow in the steady-state

$$\begin{aligned} \frac{C}{K} &:= \chi, \\ \frac{K}{H} &:= \omega, \end{aligned}$$

and u. χ is control like variable. Given the level of K it can change within period. In contrast, ω is a state like variable. It cannot change within a period.

Rewrite the model in terms of these variables. From the definitions of χ and ω and (34) and (40) it follows that

$$\begin{aligned} \frac{\dot{\chi}}{\chi} &= g_C - g_K = \frac{1}{\theta}\left[\alpha A u^{1-\alpha}\omega^{-(1-\alpha)} - \delta_K - \rho\right] - \left[A u^{1-\alpha}\omega^{-(1-\alpha)} - \delta_K - \chi\right] \\ &= \frac{\alpha-\theta}{\theta} A u^{1-\alpha}\omega^{-(1-\alpha)} + \chi - \frac{1}{\theta}\left[\rho + (1-\theta)\,\delta_K\right] \end{aligned} \tag{46}$$

Further, from the definition of χ and ω and (34) and (35) it follows that

$$\begin{aligned} \frac{\dot{\omega}}{\omega} &= g_K - g_H \\ &= A u^{1-\alpha}\omega^{-(1-\alpha)} - \chi - B\,(1-u) + (\delta_H - \delta_K) \end{aligned} \tag{47}$$

In turn, from (37), (38), and (41) it follows that

$$\begin{aligned} u &= \frac{1-\alpha}{B}\frac{q_K}{q_H}\frac{Y}{H}, \\ \frac{\dot{q}_K}{q_K} &= -\left(\alpha\frac{Y}{K} - \delta_K - \rho\right), \\ \frac{\dot{q}_H}{q_H} &= -\left(B - \delta_H - \rho\right). \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\dot{u}}{u} &= \frac{\dot{q}_K}{q_K} - \frac{\dot{q}_H}{q_H} + g_{\frac{Y}{K}} + g_{\frac{K}{H}} \\ &= -\left(\alpha\frac{Y}{K} - \delta_K - \rho\right) + (B - \delta_H - \rho) + (1-\alpha)\frac{\dot{u}}{u} - (1-\alpha)\frac{\dot{\omega}}{\omega} + \frac{\dot{\omega}}{\omega}.\end{aligned}$$

Using (47) it can be rewritten as

$$\frac{\dot{u}}{u} = \frac{1-\alpha}{\alpha}B + \frac{1-\alpha}{\alpha}(\delta_K - \delta_H) + Bu - \chi. \tag{48}$$

Equations (46), (47), and (48) give 3 differential equations that can be solved for 3 unknown functions χ, ω, and u. It is further convenient to define and use

$$z = u^{1-\alpha}\omega^{-(1-\alpha)}. \tag{49}$$

The growth rate of z is

$$\begin{aligned}\frac{\dot{z}}{z} &= (1-\alpha)\left(\frac{\dot{u}}{u} - \frac{\dot{\omega}}{\omega}\right) \\ &= \frac{1-\alpha}{\alpha}\left[B + (\delta_K - \delta_H)\right] - (1-\alpha)Az.\end{aligned}$$

Dropping ω and using z instead of it gives the following system of differential equations

$$\frac{\dot{z}}{z} = \frac{1-\alpha}{\alpha}\left[B + (\delta_K - \delta_H)\right] - (1-\alpha)Az, \tag{50}$$

$$\frac{\dot{\chi}}{\chi} = -\frac{1}{\theta}\left[\rho + (1-\theta)\delta_K\right] - \frac{\theta-\alpha}{\theta}Az + \chi, \tag{51}$$

$$\frac{\dot{u}}{u} = \frac{1-\alpha}{\alpha}\left[B + (\delta_K - \delta_H)\right] + Bu - \chi. \tag{52}$$

The first differential equation depends on z only. Appendix in the end of this section show that integrating it gives

$$z(t) = \frac{a\frac{z(0)}{a-bz(0)}}{\exp(-at) + b\frac{z(0)}{a-bz(0)}},$$

which implies that

$$\lim_{t\to+\infty} z(t) = \frac{a}{b} = \frac{1}{\alpha}\frac{B + (\delta_K - \delta_H)}{A}.$$

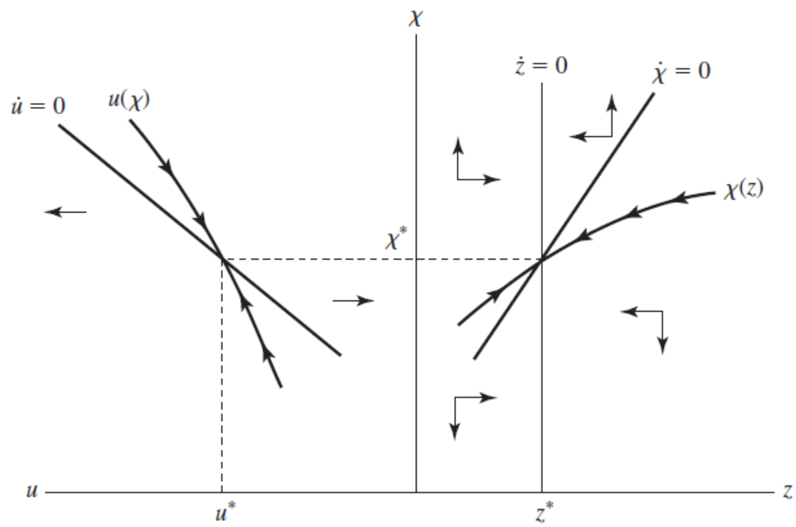
In general one could attempt to solve also for $\chi(t)$ and $u(t)$. For the current exercise, however, it is sufficient to analyze their dynamics. In order to do that notice that $\frac{\dot{z}}{z}$ and $\frac{\dot{\chi}}{\chi}$ depend on z and χ only. Therefore, the dynamics of z and χ are characterized by Jacobian

$$J_1 = \begin{pmatrix} -(1-\alpha)A & 0 \\ -\frac{\theta-\alpha}{\theta}A & 1 \end{pmatrix}.$$

The determinant of J_1 is negative. Therefore, we have saddle-path stability. Along this path $z(t)$ and $\chi(t)$ increase (or decrease) to their steady-state values.

Next, notice that $\frac{\partial}{\partial u}\frac{\dot{u}}{u} > 0$. Therefore, in order to have stability it has to be the case (i.e., parameters have to be such) that when χ is greater/less than its steady-state value $\frac{\dot{u}}{u} < 0/\frac{\dot{u}}{u} > 0$. In such a circumstance, u and χ decline/increase to their steady-state values.

For any of the variables using $*$ to denote the steady-state value of, in terms of phase diagram this corresponds to



Returning to ω

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{u}}{u} + A\left(z - z^*\right),$$

where $z^* = \frac{1}{\alpha}\frac{B+(\delta_K-\delta_H)}{A}$. In turn using (51) the growth rate of u (52) can be rewritten as

$$\begin{aligned}
\frac{\dot{u}}{u} &= \frac{1-\alpha}{\alpha}\left[B + (\delta_K - \delta_H)\right] + Bu - \frac{1}{\theta}\left[\rho + (1-\theta)\,\delta_K\right] - \frac{\theta-\alpha}{\theta}\alpha A z - \frac{\dot{\chi}}{\chi} \\
&= \frac{\theta-\alpha}{\theta}\alpha\left\{\frac{1}{\alpha}\left[B + (\delta_K - \delta_H)\right] - Az\right\} + B\left(u - u^*\right) - \frac{\dot{\chi}}{\chi} \\
&= -\frac{\theta-\alpha}{\theta}\alpha A\left(z - z^*\right) + B\left(u - u^*\right) - \frac{\dot{\chi}}{\chi}, \\
Bu^* &= \left[\frac{\theta-\alpha}{\theta}\left[B + (\delta_K - \delta_H)\right] + \frac{1}{\theta}\left[\rho + (1-\theta)\,\delta_K\right]\right] - \frac{1-\alpha}{\alpha}\left[B + (\delta_K - \delta_H)\right].
\end{aligned}$$

Therefore

$$\frac{\dot{\omega}}{\omega} = \frac{\alpha}{\theta}A\left(z - z^*\right) + B\left(u - u^*\right) - \frac{\dot{\chi}}{\chi}.$$

Now, if $z(0) < z^*$ then z increases over-time to its steady-state value. Assuming that $\theta > \alpha$, this implies from (51) that χ also increases toward its steady-state value (i.e., $\dot{\chi} > 0$). Therefore, u increases toward its steady-state value according to (52). This implies that $\frac{\dot{\omega}}{\omega} < 0$ and the system can be on stable path only if $\omega(0) > \omega^*$. If, however, $z(0) > z^*$ then $\dot{\chi} < 0$ and $u > u^*$. Therefore,

$\frac{\dot{\omega}}{\omega} > 0$ and $\omega(0) < \omega^*$.

In terms of the original variables in the model we have that

$$\begin{aligned} g_C &= \frac{1}{\theta}\left(\alpha A z - \delta_K - \rho\right), \\ g_K &= A z - \chi - \delta_K, \\ g_H &= \frac{1}{\theta}\left[\alpha A z^* - (\rho + \delta_K)\right] - B\left(u - u^*\right). \end{aligned}$$

This implies that g_C increases/decreases together with z, which is negatively correlated with ω. Therefore, if $\omega = \frac{K}{H}$ is higher/lower than its steady-state value then g_C increases/decreases over time. Meanwhile, if u is higher/lower than its steady-state value then g_H declines/increases over time. The analysis of the behavior of g_K is not so straight-forward, however.

This analysis applies to the close neighborhood of steady-state [i.e., $u \neq u^*$ but $u \in (0,1)$]. In case the economy in terms of $\frac{K}{H}$ ratio is very far away from the steady-state then during some part of the transition only one of the types of capital will be accumulated.

Kaldor stylized facts and first models of endogenous growth

Assume that the aggregate human capital is uniformly distributed across the population: $H = hL$ and there is no population growth. This suggests that the production function may be thought as one with capital and labor. Human capital plays the role of labor-augmenting technological progress that is endogenously generated by savings from the final output production, i.e. $Y = AK^{\alpha}(uhL)^{1-\alpha}$. In equilibrium: $g^* = \frac{\dot{H}}{H} = \frac{\dot{h}}{h}$.

- $\frac{Y}{L} = Au^{1-\alpha}\left(\frac{K}{hL}\right)^{\alpha} h$ increases at a rate g
- $\frac{K}{L}$ also increases at a rate g
- $\frac{Y}{K}$ is constant
- The real interest rate $r = B - \delta_H$ is constant
- The wage rate $w = \frac{\partial Y}{\partial(uL)} = (1-\alpha)\frac{Y}{H}h$ increases at rate g
- Growth rates across countries differ in the long-run due to technology and preference parameters. Initial conditions (initial levels of human and physical capital stock) have a permanent effect on the level of welfare. When economies start with different endowments, the model predicts no convergence in levels of GDP per capita, even if countries have the same long-run growth rate. Growth rates can vary also along the transition path.

Further comments

Lucas motivated the importance of human capital accumulation for long-run economic growth, by forming two different (yet complementary) models. The first model allows for human capital accumulation out of the market (e.g., education sector) that would imply that there is a trade-off between current consumption and future one, since human capital needs to be driven out of current production sector. Furthermore, in his original specification he allowed for both internal and external returns to human capital in the final-good production [spillovers *à la* Romer, $Y = (AH^{\gamma})\, K^{\alpha} H_Y^{1-\alpha}; \gamma > 0$].

The second model allows on-the-job accumulation of human capital, i.e., another form of "learning-by-doing". He assumed multiple goods, with different rates of human capital accumulation as by-product of their production. The trade-off in this case is that human capital accumulation takes a form of a less desirable mix of current consumption goods. Growth promoting policies implied by either model are very different (education subsidies *vs.* industrial policy).

The research challenge that Lucas acknowledges himself is that human capital is not a measurable factor, and in particular its potential external effects. He proposes that a good example of the importance of external effects of human capital is the formation of cities.

Overall, the model lacks a good justification of the non-decreasing returns to the human capital accumulation. The accumulation of human capital differs importantly from the accumulation of knowledge and therefore this model is not a model of technological progress. The important difference between them is that human capital is rival and excludable while knowledge is not rival, though can be excludable.

Empirically, human capital growth cannot explain cross-country growth differences. There is only some limited support that human capital matters as an input to R&D. The latter comes out an important factor in driving aggregate productivity and explaining the cross-country variation in the growth and levels of GDP per capita.

Externalities in the production of goods and human capital

There may be important positive externalities steaming from the availability of well educated personnel both in final goods production and in human capital accumulation - the employees in both "sectors" may get more productive if the rest are well educated. Due to such externalities the private returns from human capital will be biased downwards. Therefore, the competitive equilibrium would not deliver socially optimal allocations.

One way to model such externalities, that would deliver tractable results and balanced growth path, is as follows. Let the final goods production be

$$Y = \left(AH_Y^{\gamma_1}\right) K^{\alpha} H_Y^{1-\alpha}, \tag{53}$$

where $\gamma_1 > 0, A = const$, and firms do not take into account the term in the brackets. Meanwhile, the accumulation of human capital is

$$\dot{H} = \left(BH_H^{\gamma_2}\right) H_H^{1-\gamma_2} - \delta_H H, \tag{54}$$

where $\gamma_2 > 0$ and the household does not take into account the term in the brackets. This means that the amount of human capital in a sector has a positive effect on the productivity in the sector. An alternative specification would be that the amount of human capital in the economy (H) has positive effect on the productivity in both production and education sectors. The latter case could be motivated for example with economy wide better environment due to higher tuition level(s). In contrast, the former setup would be more appropriate when peer effects play a role (e.g., there is learning from the peers). The preference is given to the former setup since the latter does not yield tractable (analytic) results for a wide range of parameter values. Basically this is because (54) is used to determine the share of human capital in human capital accumulation, $\frac{H_H}{H}$. In case externalities stem from the economy wide amount of human capital (54) implies non-linear and not so straight forward to solve expression for $\frac{H_H}{H}$.

In the model with (53), in decentralized equilibrium from the production side it will follow that

$$u := \frac{H_Y}{H}, \tag{55}$$

$$Y = AK^{\alpha} (uH)^{1+\gamma_1-\alpha}, \tag{56}$$

$$w = (1-\alpha) \frac{Y}{uH}, \tag{57}$$

$$r = \alpha \frac{Y}{K} - \delta_K. \tag{58}$$

It is straight forward to notice that there is a distortion: the wage rate of human capital is not equal to its marginal product,

$$w = (1-\alpha) \frac{Y}{uH} \neq MP_H = (1+\gamma_1-\alpha) \frac{Y}{uH}.$$

In turn, with slight abuse of notation (i.e., K instead of W), under (54) the representative household's problem is

$$\begin{aligned}
&\max_{u,C} U = \int_0^{+\infty} \frac{C_t^{1-\theta}-1}{1-\theta} \exp(-\rho t)\, dt, \\
&s.t. \\
&\dot{K} = (1+\tau_K) rK + \left(1+\tau_H^1\right) wuH - C - T, \\
&\dot{H} = \tilde{B} \left[(1-u) H\right]^{1-\gamma_2} - \delta_H H, \\
&K(0), H(0) > 0 - given,
\end{aligned}$$

where T is a lump-sum tax and τ_K, τ_H^1 are proportional taxes/subsidies and

$$\tilde{B} = B \left[(1-u) H\right]^{\gamma_2}$$

in equilibrium.

Let q_K and q_H be the shadow prices of the physical and human capital, respectively. The optimal problem of HH if written in terms of current value Hamiltonian is

$$\begin{aligned} \max_{u,C} H_{LC} &= \frac{C^{1-\theta}-1}{1-\theta} + q_K \left[(1+\tau_K) rK + \left(1+\tau_H^1\right) wuH - C - T \right] \\ &+ q_H \left(1+\tau_H^2\right) \left\{ \tilde{B} \left[(1-u) H \right]^{1-\gamma_2} - \delta_H H \right\}, \end{aligned} \tag{59}$$

where τ_H^2 is a proportional tax/subsidy that allows the government to alter the accumulation of human capital. For each additional unit of human capital the government compensates the household with τ_H^2 units of human capital.

In equilibrium T is such that the government runs balanced budget, i.e.,

$$T = \tau_K rK + \tau_H^1 wuH + \frac{q_H}{q_K} \tau_H^2 \left[B (1-u) H - \delta_H H \right].$$

It is worth noticing that since q_H and q_K are the shadow prices of H and K (in terms of utility) $\frac{q_H}{q_K}$ is simply the price of H relative to K.

The optimal rules which follow from (59) are

$$\begin{aligned} [C] &: C^{-\theta} = q_K, \\ [u] &: q_K \left(1+\tau_H^1\right) w = q_H \left(1+\tau_H^2\right) (1-\gamma_2) B, \\ [K] &: \dot{q}_K = \rho q_K - q_K (1+\tau_K) r, \\ [H] &: \dot{q}_H = \rho q_H - \left\{ q_K \left(1+\tau_H^1\right) wu + q_H \left(1+\tau_H^2\right) \left[B (1-\gamma_2)(1-u) - \delta_H \right] \right\}, \\ [TVC_K] &: \lim_{t \to +\infty} q_K K \exp(-\rho t) = 0, \\ [TVC_H] &: \lim_{t \to +\infty} q_H H \exp(-\rho t) = 0. \end{aligned}$$

Focusing on balanced growth path and fixing the share u and the proportional taxes $\tau_K, \tau_H^1, \tau_H^2$, after some algebra it can be shown that the rate of growth of consumption is

$$\begin{aligned} \frac{\dot{C}}{C} &= \frac{1}{\theta} \left[(1+\tau_K) r - \rho \right] \\ &= \frac{1}{\theta} \left\{ \left(1+\tau_H^2\right) \left[B (1-\gamma_2) - \delta_H \right] - \rho + g_w \right\}. \end{aligned} \tag{60}$$

From (60) and (58) it follows that

$$g_K = g_Y.$$

In turn, from (56) and (57) it follows that

$$g_Y = (1+\gamma_1-\alpha) g_H + \alpha g_K, \tag{61}$$

$$g_w = g_Y - g_H. \tag{62}$$

From the human capital accumulation rule and law of motion of capital it follows that

$$\begin{aligned} g_H &= B(1-u) - \delta_H, \\ g &\equiv g_K = g_Y = g_C. \end{aligned}$$

Therefore, from (60)-(62) it follows that

$$g = \frac{1+\gamma_1-\alpha}{1-\alpha} g_H, \tag{63}$$

$$g_w = \frac{\gamma_1}{1+\gamma_1-\alpha} g,$$

$$g = \frac{1}{\theta} \frac{\left(1+\tau_H^2\right)\left[B(1-\gamma_2) - \delta_H\right] - \rho}{1 - \frac{1}{\theta}\frac{\gamma_1}{1+\gamma_1-\alpha}}. \tag{64}$$

A sufficient condition for having positive denominator is

$$\theta \geq 1.$$

Meanwhile, the nominator is non-negative if, for example,

$$(1-\gamma_2) B - \delta_H - \rho \geq 0,$$

which is slightly stronger than the condition for non-negative growth rate in the model without externalities $B - \delta_H - \rho \geq 0$. Hereafter these conditions are assumed to hold.

The share u can be found from (63),

$$\begin{aligned} 1-u &= \frac{g_H + \delta_H}{B} \\ &= \frac{1}{B}\left(\frac{1-\alpha}{1+\gamma_1-\alpha} g + \delta_H\right). \end{aligned} \tag{65}$$

From (64) and (65) it is evident that u increases with γ_2. This is because when γ_2 increases the private returns on human capital accumulation decline. From (64) and (65) it can be shown also that u increases with γ_1. This, in turn, is because when γ_1 increases human capital becomes more productive in the production sector.

The relative quantities of aggregates on balanced growth path, therefore, are

$$\begin{aligned} \frac{Y}{K} &= \frac{1}{\alpha}\left(\frac{\theta g + \rho}{1+\tau_K} + \delta_K\right), \\ \frac{C}{K} &= \frac{Y}{K} - g - \tau_H^2 g_H \frac{q_H}{q_K}\frac{H}{K}, \\ \frac{q_H}{q_K w} &= \frac{1}{(1-\alpha) u}\frac{K}{Y}\frac{q_H}{q_K}\frac{H}{K} = \frac{1+\tau_H^1}{\left(1+\tau_H^2\right)(1-\gamma_2) B}. \end{aligned}$$

In social optimum it is easy to show that

$$g^{SP} = \frac{1}{\theta} \frac{B - \delta_H - \rho}{1 - \frac{1}{\theta}\frac{\gamma_1}{1+\gamma_1-\alpha}}.$$

This implies that in decentralized equilibrium the government needs to subsidize the accumulation of human capital and set

$$\begin{aligned} \tau_H^2 &: \quad g = g^{SP} \\ \tau_H^2 &= \frac{B - \delta_H}{B(1-\gamma_2) - \delta_H} - 1 > 0 \end{aligned}$$

in order to attain the first best outcome, $g = g^{SP}$. According to (63) this would immediately correct the share of human capital allocated to human capital accumulation, u. Therefore, it will correct the share of human capital in the production sector. Given that τ_H^2 corrects the growth rate, it corrects also the return on physical capital in case $\tau_K = 0$. Therefore, in such case it corrects the ratio $\frac{Y}{K}$.

Meanwhile, τ_H^1 can be used in order to correct the ratio $\frac{q_H}{q_K w}$ and set it equal to $\left(\frac{q_H}{q_K MP_H}\right)^{SP}$,

$$\tau_H^1 = \frac{1+\gamma_1-\alpha}{1-\alpha}\left(1+\tau_H^2\right)(1-\gamma_2) - 1.$$

It can be easily noticed, however, that such policy won't correct the ratio $\frac{C}{K}$. Therefore, there will be still dynamic inefficiency.

In order to check these arguments set-up and solve the Social Planner's problem.

$$\begin{aligned} &\max_{u,C} U = \int_0^{+\infty} \frac{C_t^{1-\theta} - 1}{1-\theta} \exp(-\rho t)\, dt, \\ &s.t. \\ &\dot{K} = AK^{\alpha}(uH)^{1+\gamma_1-\alpha} - \delta_K K - C, \\ &\dot{H} = B(1-u)H - \delta_H H, \\ &K(0), H(0) > 0 - given, \end{aligned}$$

In terms of current value Hamiltonian,

$$\begin{aligned} \max_{u,C} H_{LC}^{SP} &= \frac{C^{1-\theta}-1}{1-\theta} + q_K\left[AK^{\alpha}(uH)^{1+\gamma_1-\alpha} - \delta_K K - C\right] \\ &\quad + q_H\left[B(1-u)H - \delta_H H\right]. \end{aligned}$$

The optimal rules which follow are

$$\begin{aligned}
&[C] : C^{-\theta} = q_K, \\
&[u] : q_K \left(1+\gamma_1-\alpha\right) \frac{Y}{uH} = q_H B, \\
&[K] : \dot{q}_K = \rho q_K - q_K \left(\alpha \frac{Y}{K} - \delta_K\right), \\
&[H] : \dot{q}_H = \rho q_H - \left\{ q_K \left(1+\gamma_1-\alpha\right) \frac{Y}{uH} u + q_H \left[B\left(1-u\right) - \delta_H\right] \right\}, \\
&[TVC_K] : \lim_{t\to+\infty} q_K K \exp\left(-\rho t\right) = 0, \\
&[TVC_H] : \lim_{t\to+\infty} q_H H \exp\left(-\rho t\right) = 0.
\end{aligned}$$

After some algebra, on balanced growth path

$$\begin{aligned}
\frac{\dot{C}}{C} &= \frac{1}{\theta}\left(\alpha\frac{Y}{K} - \delta_K - \rho\right), \\
&= \frac{1}{\theta}\left[B - \delta_H - \rho + \left(g_Y - g_H\right)\right].
\end{aligned}$$

From the first expression and the resource constraint it follows that

$$g^{SP} \equiv g_Y = g_K = g_C.$$

Therefore,

$$\begin{aligned}
g^{SP} &= \frac{1+\gamma_1-\alpha}{1-\alpha} g_H, \\
g_Y - g_H &= \frac{\gamma_1}{1+\gamma_1-\alpha} g^{SP},
\end{aligned}$$

and

$$g^{SP} = \frac{1}{\theta}\frac{B-\delta_H-\rho}{1-\frac{1}{\theta}\frac{\gamma_1}{1+\gamma_1-\alpha}}.$$

From this expression, it follows that

$$1 - u^{SP} = \frac{g_H + \delta_H}{B}.$$

This means that in decentralized equilibrium without policies both the growth rate and the allocation of human capital to human capital accumulation are lower than in social optimum. The suggested policy then corrects both.

Appendix

Integrating

$$\frac{\dot{z}}{z} = \frac{1-\alpha}{\alpha}\left[B + \left(\delta_K - \delta_H\right)\right] - \left(1-\alpha\right) A z$$

gives

$$\int_0^t \frac{dz}{(1-\alpha)\left[\frac{1}{\alpha}B+\frac{1}{\alpha}(\delta_K-\delta_H)\right]z-(1-\alpha)Az^2}=t.$$

In order to find the value of the first integral denote

$$a:=\frac{1-\alpha}{\alpha}\left[B+(\delta_K-\delta_H)\right], b:=(1-\alpha)A,$$

and rewrite

$$\int_0^t \frac{dz}{z(az-bz)}=t.$$

The ratio in the integral can be rewritten as

$$\frac{1}{z(a-bz)}=\frac{D_1}{z}+\frac{D_2}{(a-bz)}$$

where, for convenience,

$$D_1=\frac{1}{a}, D_2=\frac{b}{a}.$$

Therefore, the integral above is

$$\begin{aligned}
\int_0^t \frac{dz}{z(az-bz)} &= \int_0^t \left[\frac{D_1}{z}+\frac{D_2}{(a-bz)}\right]dz \\
&= D_1\left[\ln z(t)-\ln z(0)\right]-\frac{D_2}{b}\int_0^t \frac{1}{(a-bz)}d(a-bz) \\
&= D_1\left(\left[\ln z(t)-\ln z(0)\right]-\left\{\ln\left[a-bz(t)\right]-\ln z\left[a-bz(0)\right]\right\}\right)
\end{aligned}$$

Finally taking exponents from both sides of the integral equation gives

$$\frac{z(t)}{a-bz(t)}=\frac{z(0)}{a-bz(0)}\exp(at),$$

$$z(t)=\frac{a\frac{z(0)}{a-bz(0)}}{\exp(-at)+b\frac{z(0)}{a-bz(0)}}.$$

R&D based models of growth: The Romer (1990) model - horizontal product innovation

The main ideas behind Romer (1990) model are

1. technology is an important factor in production
2. technological progress is market outcome, i.e., it is endogenously generated
3. technology is a *good* with special features

   - technology is not rival, i.e., if it is created it can be used at zero cost any time after by everyone
   - technology is at least partly excludable, i.e., one can restrict the access of others to its technology, to some extent (thus s/he can earn returns)

Main assumptions

- This is a multi-sector model of R&D-based endogenous growth

  The sectors are

  1. R&D sector - produces "blueprints" of new types of capital goods $\dot{A}$. R&D production uses L_A share of total labor L. The existent set of capital goods types A increases the productivity of the R&D sector, i.e., positive knowledge externalities operate in the production of new blueprints, creating increasing returns in this sector

     $$\dot{A} = BAL_A, B > 0, \tag{66}$$

     where B is the efficiency of "blueprint" creation.

     – An example could be the creation of wireless telephone while using the knowledge of transmitting information via radio waves and voice encoding.

  2. Capital goods sector - The sector uses the blueprints and produces intermediate capital goods for final goods production. It is characterized by monopolistic competition. There is free-entry in the market of new blueprints. Entrepreneurs compete for patent that provides them with infinite-horizon property rights on a new blueprint. The acquisition of a patent allows an entrepreneur to employ exclusively the new blueprint and produce a distinct type of capital good thereafter. The production of capital goods/varieties requires investment in terms of the (foregone) final goods - producing 1 unit of a capital good requires 1 unit of final goods.

     – Each and every capital goods producing firm first buys (invests) the blueprint of capital good. It then enters to capital goods market and stays there forever. In capital goods market the firm has to have strictly positive profit streams in order to recover the entry cost. Since it has to have positive profits it "has" to be a price setter. Moreover, in free entry equilibrium the investment cost equals to the value derived in the market, i.e., the present value of discounted profit streams.

3. Final goods sector - Final goods producers employ L_Y share of total labor L and variety (a set) of capital goods $x(i), i \in [0, A]$ in the production

$$Y = L_Y^{1-\alpha} \int_0^A x^{\alpha}(i)\, di.$$

   - Integral over capital goods is appropriate since we have continuous arrival of A in (66). It is, however, a result of simplification and we will treat it as a sum.
   - Notice that the elasticity of substitution between capital goods is

$$\varepsilon = -\frac{d \ln \left(\frac{x(i)}{x(j)}\right)}{d \ln \left(\frac{Y'_{x(i)}}{Y'_{x(j)}}\right)} = \frac{1}{1-\alpha}.$$

   - In a symmetric equilibrium in this model the aggregate capital services are summarized as

$$K = \int_0^A x di = Ax.$$

   Aggregate output then can be written as $Y = (AL_Y)^{1-\alpha} K^{\alpha}$.

   Final good producers are fully competitive in input and output markets. The final good is the numeraire good which may be either consumed or invested.

4. From the consumption-side, the representative HH chooses its consumption and next period assets to maximize its lifetime utility $U = \int_0^{+\infty} u(C_t) \exp(-\rho t)\, dt$, subject to the standard budget constraint.

Further assumptions

- All capital varieties depreciate fully within one period in final goods production (i.e., capital goods x can be thought to represent capital services)

- No growth in population, i.e., $L = const$

Market equilibrium

Final goods sector

The final goods producers maximize their profits taking the price of their inputs, labor (w) and capital goods/varieties ($p_x(i)$, $\forall i$) as given. The problem of the representative final goods producer is

$$\max_{\{x(i)\}_{i\in[0,A]},L_Y} L_Y^{1-\alpha}\int_0^A x(i)^\alpha di - \int_0^A p_{x(i)}x(i)di - wL_Y$$

The optimal rules are

$$[L_Y] \quad : \quad Y_L' = \frac{\partial Y}{\partial L_Y} = (1-\alpha)\frac{Y}{L_Y} = w, \tag{67}$$

$$[x(i)] \quad : \quad Y_{x(i)}' = \frac{\partial Y}{\partial x(i)} = \alpha L_Y^{1-\alpha} x\,(i)^{\alpha-1} = p_x(i); \forall i, \tag{68}$$

where the first expression describes the demand for labor and the second describes the demand for a capital good.[7]

Capital goods sector

Each capital good producer i, within every period maximizes its profits $\pi_x(i)$, by selecting the price $p_x\,(i)$ and the quantity of production $x\,(i)$. For every unit of capital that it produces it needs to invest one unit of final good that it "borrows" from HH at the current price of final output (which is set to one, i.e., the final good is the numeraire), i.e., $\pi_x(i) = p_x(i)x(i) - x(i)$. The firm takes as given the price of the output it uses in the production and the demand that it's good is facing from the final good producers. Since the firm does not have dynamic constraints it's problem is[8]

$$\begin{aligned}
&\max_{p_x(i),x(i)} \pi_x\,(i) = p_x\,(i)\,x\,(i) - x\,(i) \\
&s.t. \\
&p_x\,(i) = \alpha L_Y^{1-\alpha} x^{\alpha-1}\,(i)\,.
\end{aligned}$$

The optimal rule(s) are derived by plugging the inverse demand function of capital good to the profit function and taking the derivative with respect $x(i)$, i.e., solve the following problem

$$\max_{x(i)}\{\alpha L_Y^{1-\alpha} x\,(i)^\alpha - x\,(i)\}$$

$\Rightarrow$

$$[x(i)] : 1 = \alpha^2 L_Y^{1-\alpha} x(i)^{\alpha-1}.$$

This implies that

$$x := x\,(i) = \alpha^{\frac{2}{1-\alpha}} L_Y. \tag{69}$$

[7] Expression (68) holds since we treat the integral as a sum, otherwise the change of any of $x(i)$-s should not have had a measurable effect on the integral.

[8] It is worth to note that the objective function is (70) and in order to solve the optimal problem we could use Hamiltonian. However, since there are no dynamic constraints such solution is tantamount to the proposed one.

From (68) and (69) follows that

$$p_x := p_x(i) = \alpha L_Y^{1-\alpha} x^{\alpha-1} = \frac{1}{\alpha} \Rightarrow$$
$$\pi_x := \pi_x(i) = (p_x - 1)\, x = \frac{1-\alpha}{\alpha} \alpha^{\frac{2}{1-\alpha}} L_Y > 0.$$

Note that the imperfect competition in the market equilibrium implies that the price of capital good is a constant mark-up ($\frac{1-\alpha}{\alpha}$) above its marginal cost, and the quantity of supplied capital good will be lower than the one selected in a perfectly competitive market. Note also that in equilibrium the final output is linear in technology since $Y = L_Y^{1-\alpha} A x^{\alpha} = \alpha^{\frac{2\alpha}{1-\alpha}} L_Y A$, and the economy can experience long-run growth driven by technological progress, which here takes the form of expanding variety of capital goods.

Firm entry into capital goods industry

The potential capital good producer (an enterprenuer) in order to establish its firm competes with other potential producers in bidding for a new blueprint, where the blueprint is produced in fully competitive market. It makes an up-front (prior to entry) payment for the blueprint. In free entry equilibrium this payment (cost of entry) is equal to the value derived by the firm in the capital market,

$$V_x(t) = \int_t^{+\infty} \pi_x(\tau) \exp\left[-\int_t^{\tau} r(s)ds\right] d\tau, \tag{70}$$

where t is the entry date and $r(s)$ is the instantaneous real interest rate that the representative HH earns on its asset holdings.

From (70) it follows that

$$\begin{aligned} \dot{V}_x(t) &= -\pi_x(t) \exp\left[-\int_t^t r(s)\,ds\right] + \int_t^{+\infty} \pi_x(\tau) \frac{\partial}{\partial t} \exp\left[-\int_t^{\tau} r(s)\,ds\right] d\tau \\ &= -\pi_x(t) + \int_t^{+\infty} \pi_x(\tau) \left\{ \frac{\partial}{\partial t}\left[-\int_t^{\tau} r(s)\,ds\right] \frac{\partial}{\partial\left[-\int_t^{\tau} r(s)\,ds\right]} \exp\left[-\int_t^{\tau} r(s)\,ds\right] \right\} d\tau \\ &= -\pi_x(t) + r(t)\, V_x(t), \end{aligned}$$

which is standard Hamilton-Jacobi-Bellman equation. It can be rewritten as

$$rV_x = \pi_x + \dot{V}_x.$$

If $V_x(t)$ is constant in equilibrium over time, then

$$V_x = \frac{\pi_x(t)}{r(t)}.$$

This condition implies that at every point in time, the instantaneous excess of revenue over marginal cost must be just sufficient to cover the interest rate cost on the initial investment on a new blueprint.

Another way of thinking this is that a HH lends V_x to the entrepreneur for him to "buy" a blueprint and establish a firm and then receives in every period the "dividends" that equal to the per period profits.

Under the free-entry the value generated by the entry of a firm $V_x\dot{A}$ is equal to the cost of generating the blueprint wL_A,

$$V_x\dot{A} = wL_A. \tag{71}$$

R&D sector

Any "blueprint" is owned by a capital good producer which has a value $V_x(t)$. Thus, the price/value of a "blueprint" is $V_x(t)$ and the problem of a "blueprint" producer is

$$\max_{L_A} \{V_x BAL_A - wL_A\}.$$

Assuming fully competitive market, the optimal rule resembles (71),

$$w = V_x BA.$$

Labor market

The labor market equilibrium needs to guarantee that the value of the marginal product of labor is equated in two sectors which use it, i.e., the final good and R&D sectors. Therefore, the wage rate needs to be equal in both sectors, i.e.,

$$w = V_x BA = (1-\alpha)\frac{Y}{L_Y} = \alpha^{\frac{2\alpha}{1-\alpha}} A(1-\alpha). \tag{72}$$

Thus,

$$V_x = \frac{\alpha^{\frac{2\alpha}{1-\alpha}}(1-\alpha)}{B}.$$

Therefore, indeed $\dot{V}_x = 0$ and $V_x = \frac{\pi_x}{r}$.

From (72) it also follows that

$$\begin{aligned} w &= V_x(t)BA = \frac{\pi_x}{r}BA \\ &= \frac{1-\alpha}{\alpha}\alpha^{\frac{2}{1-\alpha}}L_Y\frac{1}{r}BA\frac{Y}{Y} \\ &= \alpha(1-\alpha)\frac{1}{r}BY. \end{aligned} \tag{73}$$

From $w = (1-\alpha)\frac{Y}{L_Y}$ and (73) it follows that

$$L_Y = \frac{r}{\alpha B}. \tag{74}$$

The household side

From the standard HH intertemporal maximization problem, it follows that its consumption over time follows the path

$$\frac{\dot{C}}{C} = \frac{1}{\theta}(r - \rho). \tag{75}$$

The standard TVC ensures that the value of the asset holdings of the HH is equal to zero in the limit, i.e., the growth of the assets does not exceed the real interest rate.

Balanced growth path

All variables of the model need to grow at constant rates (BGP).

- For $g_A =: \frac{\dot{A}}{A} = B(L - L_Y)$, to be constant in equilibrium, it must be that the allocation of labor in the final goods sector is constant over time, i.e., $\dot{L}_Y = 0$. From (75) it follows that the interest rate should be constant on BGP and $g_C =: \frac{\dot{C}}{C} = \frac{1}{\theta}(r - \rho)$.
- From the FOCs of the capital goods producers, it follows that x is time invariant, implying that aggregate capital stock $K = Ax$ grows at rate $g_K = g_A$. The economy grows due to capital-deepening (or specialtization of labor) which is entirely driven by R&D which expands the number of capital goods types.
- Time invariance of x implies also that π_x is time invariant.
- The production function of final goods $Y = \alpha^{\frac{2\alpha}{1-\alpha}} L_Y A$ implies that along the BGP $g_Y = g_A$.
- From households' budget constraint

  $$\dot{W} = rW + wL - C$$

  and that in this model assets are firms (i.e., $W = V_x A$) it follows that

  $$\dot{W} = \frac{\pi_x}{V_x} V_x A + (1 - \alpha)\frac{Y}{L_Y} L - C$$

  therefore, since $r = \frac{\pi_x}{V_x}$

  $$V_x g_A = \pi_x + (1 - \alpha)\frac{L}{L_Y}\frac{Y}{A} - \frac{C}{A}.$$

  Given that $g_{\pi_x} = g_{V_x} = 0$ and $g_Y = g_A \Rightarrow$ along the BGP $g_C = g_A = g_Y = g_K =: g$.

Note that in equilibrium, there is a positive relation between growth and the real interest rate, as implied by the consumption-side, but a negative one implied by the technology/ production-side, as

$$g_A = B(L - L_Y) = B(L - \frac{r}{\alpha B}). \tag{76}$$

The latter is due to the fact that higher interest rate reduces the present discounted value of any capital variety firm. By doing so, it reduces the market incentives to direct (labor) resources away from the final good production into the production of new assets/savings instruments. Along the unique BGP, these two forces are equated implying a unique real interest rate and labor allocation. These two can be derived by equating the (75) and (76)

$$\frac{1}{\theta}(r-\rho) = B\left(L - \frac{r}{\alpha B}\right) \Rightarrow$$
$$r = \frac{\alpha}{\alpha+\theta}(\theta BL + \rho)$$

$\Rightarrow$

$$L_Y = \frac{\theta BL + \rho}{(\alpha+\theta)B},$$

and

$$g = \frac{r-\rho}{\theta} = \frac{\alpha BL - \rho}{\alpha+\theta}.$$

The condition for positive long-run growth sets a minimum bound on the scale of the economy: $L > \frac{\rho}{B\theta}$. Note that the TVC is always satisfied under this condition, i.e., $r > g_A$.

Comparative statics

- Increase in B and L increase g, by reducing L_Y
- Increase in θ or ρ decreases g, by increasing L_Y
- Increase in α also increases g, by reducing L_Y

Kaldor stylized facts

- $Y/L = Ax^{\alpha}$ increases at a rate g
- $K/L = A\frac{x}{L}$ also increases at a rate g
- Y/K is constant
- The real interest rate r is constant
- The wage rate $w = (1-\alpha)\frac{Y}{L_Y} = (1-\alpha)\left(\frac{x}{L_Y}\right)^{\alpha} A$ increases at rate g
- Growth rates across countries differ in the long-run due to technology and preference parameters. Long-run growth takes place due to the endogenous R&D which expands the variety of capital goods. Short-episodes of fast growth are due to changes in the underlying parameters and/or the scale of the economy.

The model predicts that there is no convergence in terms of GDP per capita.

The Social Planner's problem

In the decentralized market equilibrium there are two sources of inefficiency which drive the equilibrium growth outcome away from the first-best:

1. Monopoly rights/patents

2. Positive knowledge externalities in the production of new blueprints of capital goods

The social planner faces the following optimal control problem, where the state of the economy is summarized by A

$$\begin{aligned}
&\max_{\{x(i)\},L_Y,C} \int_0^{+\infty} \frac{C_t^{1-\theta}-1}{1-\theta} \exp\left(-\rho t\right) dt, \\
&s.t. \\
&L_Y^{1-\alpha} \int_0^A x^{\alpha}\left(i\right) di = C + \int_0^A x\left(i\right) di, \\
&\dot{A} = B(L-L_Y)A, \\
&A\left(0\right) > 0 - given.
\end{aligned}$$

where the second equation is the resource constraint. The third term in resource constraint is the available capital (services) stock $K = \int_0^A x\left(i\right) di$. In this model the capital depreciates in a period; thus, investments equal to the stock of capital.

Let q_K and q_A denote the shadow prices of "capital" and "knowledge" respectively. Optimal rules imply the following conditions that govern equilibrium in every point in time

$$\begin{aligned}
[C]: &\quad C^{-\theta} = q_K, \\
[x(i)]: &\quad x\left(i\right) = \alpha^{\frac{1}{1-\alpha}} L_Y, \forall i, &(77) \\
\Rightarrow & \\
[L_Y]: &\quad q_K(1-\alpha)\left(\frac{x}{L_Y}\right)^{\alpha} = Bq_A, &(78) \\
[A]: &\quad \dot{q}_A = q_A\rho - q_K\left(L_Y^{1-\alpha}x^{\alpha} - x\right) - q_A B(L-L_Y). &(79)
\end{aligned}$$

Equation (77) shows that the Social Planner would choose to produce to the point that the marginal product of each capital goods type equates its marginal cost (one unit of output), hence $x^{SP} > x$. Equation (78) equates the value of the marginal product of labor in the final good and R&D production. It implies that over time, the rates of returns are equated, i.e., $-\frac{\dot{q}_K}{q_K} = -\frac{\dot{q}_A}{q_A}$. From the optimal rule for consumption then follows that

$$\frac{\dot{C}}{C} = -\frac{1}{\theta}\frac{\dot{q}_K}{q_K} = -\frac{1}{\theta}\frac{\dot{q}_A}{q_A},$$

where $\frac{\dot{q}_A}{q_A}$ can be derived from (77), (78), and (79)

$$\begin{aligned}
\frac{\dot{q}_A}{q_A} &= \rho - \frac{q_K}{q_A}\left[L_Y^{1-\alpha}\left(\alpha^{\frac{1}{1-\alpha}}L_Y\right)^{\alpha} - \alpha^{\frac{1}{1-\alpha}}L_Y\right] - B(L-L_Y) \\
&= \rho - \frac{q_K}{q_A}\left(L_Y\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}L_Y\right) - B(L-L_Y) \\
&= \rho - B\left[(1-\alpha)\left(\frac{x}{L_Y}\right)^{\alpha}\right]^{-1} L_Y\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha) - B(L-L_Y) \\
&= \rho - B\left[\left(\frac{x}{L_Y}\right)^{\alpha}\right]^{-1} L_Y\alpha^{\frac{\alpha}{1-\alpha}} - B(L-L_Y) \\
&= \rho - B\alpha^{-\frac{\alpha}{1-\alpha}}L_Y\alpha^{\frac{\alpha}{1-\alpha}} - B(L-L_Y) \\
&= \rho - BL.
\end{aligned}$$

Balanced growth path

From the blueprint production follows that $L_Y = const$ on BGP. Given that $L_Y = const$ the x is constant. Therefore, from resource constraint follows that the macroeconomic aggregates grow at the same constant rate on balanced growth path. Denote that constant growth rate by g^{SP}.

The socially optimal long-run growth is given by

$$g^{SP} = \frac{BL-\rho}{\theta}.$$

Since $\alpha \in (0,1)$ the $g^{SP} > g$,i.e.,

$$\frac{BL-\rho}{\theta} > \frac{\alpha BL-\rho}{\alpha+\theta}.$$

Moreover, since $g^{SP} > g$ from the "blueprint" production follows that $L_Y^{SP} < L_Y$, i.e., the social planner would allocate more of the labor resources into the production of R&D. Therefore, growth promoting policies increase the incentives to innovate by subsidies to the production of R&D (e.g., subsidies to the employment of labor in R&D) that will make firms internalize knowledge externalities they generate by each new variety that they discover. The distortion of the imperfect competition may be alleviated by a subsidy to the purchases of the capital goods and/ or subsidies to the production of final output that would increase the demand for capital goods. These policies would finance those subsidies by lump-sum taxes on the household.

Additional - Erosion of monopoly power

Everything else the same as in the standard model assume that in each time interval dT the monopolists lose their monopoly power with a probability $p \times dT$. The parameter p is the arrival rate of monopoly power erosion (the probability of losing the monopoly power at time T given that the firm has maintained it till that time),

$$\begin{aligned} p &= \lim_{dT\to 0}\frac{\Pr\left(t\leq T\leq t+dT|T\geq t\right)}{dT} \\ &= \frac{1}{1-F\left(T\right)}\lim_{dT\to 0}\frac{\Pr\left(t\leq T\leq t+dT\right)}{dT}=\frac{f\left(T\right)}{1-F\left(T\right)}, \end{aligned}$$

where $F\left(T\right)$ is the probability of losing the monopoly power after T and $f\left(.\right)=F\prime\left(.\right)$. Therefore, the probability that the monopolist which has entered at time t does not lose its market power by time $\tau\geq t$ is

$$\begin{aligned} \int_t^\tau\frac{d\left[1-F\left(T\right)\right]}{\left[1-F\left(T\right)\right]} &= -\int_t^\tau pdT, \\ \ln\left[1-F\left(\tau\right)\right]-\ln\left[1-F\left(t\right)\right] &= -p\left(\tau-t\right), \\ 1-F\left(\tau\right) &= \exp\left[-p\left(\tau-t\right)\right]. \end{aligned}$$

The expected present value of the monopolist, therefore, is

$$\begin{aligned} V_x\left(t\right) &= \int_t^{+\infty}\left(\pi_x\left(t\right)\exp\left[-\left(\tilde{r}+p\right)\left(\tau-t\right)\right]+0*\left\{1-\exp\left[-p\left(\tau-t\right)\right]\right\}\right)d\tau \\ &= \int_t^{+\infty}\pi_x\left(t\right)\exp\left[-\left(\tilde{r}+p\right)\left(\tau-t\right)\right]d\tau, \\ \tilde{r} &= \frac{1}{\tau-t}\int_t^\tau r(s)ds. \end{aligned}$$

This implies that the Hamilton-Jacobi-Bellman equation is

$$\dot{V}_x=-\pi_x+\left(r+p\right)V_x$$

Given that everything else remained the same from the condition (72) it follows that

$$\begin{aligned} V_x &= const \\ &\Rightarrow \\ V_x &= \frac{\pi_x}{r+p}. \end{aligned}$$

This means that (74) transforms into

$$L_Y=\frac{r+p}{\alpha B}.$$

This expression, in turn, implies that (75) transforms into

$$g_A=B\left(L-L_Y\right)=B\left(L-\frac{r+p}{\alpha B}\right).$$

Combining this condition with the Euler Equation from the consumer's side results in

$$\frac{1}{\theta}(r-\rho)=B\left(L-\frac{r+p}{\alpha B}\right).$$

Therefore, the rate of return on investments, labor force allocation to final goods production, and the growth rate of the economy are

$$\begin{aligned}
r &= \frac{\alpha}{\alpha+\theta}\left[\theta\left(BL-\frac{p}{\alpha}\right)+\rho\right], \\
L_Y &= \frac{\theta BL+\rho+p}{(\alpha+\theta)B}, \\
g &= \frac{\alpha BL-\rho-p}{\alpha+\theta}.
\end{aligned}$$

This means that the rate of return and the growth rate decline with p, and L_Y increases with it. This is because when p increases the incentive to innovate declines, which reduces $r, 1-L_Y$, and g.

It is worth to notice that the non-zero probability of losing the monopoly power pushes the decentralized equilibrium even farther from the social optimum, where there is no monopoly power, therefore, no (non-zero) probability of losing it.

R&D based models of growth: The Jones (1995) model - horizontal product innovation, (semi-endogenous) growth without scale effects

Motivation

The dependence of the per-capita growth rate on the scale of the economy in Romer (1990) can seem to be problematic. This scale effect, perhaps, would not be very well justified, for example, for China and India. Romer points out that one should think about the scale of the economy in terms of the part of labor force that has the ability to conduct research (i.e., human capital) rather than the entire labor force. However, as Jones (1995) points out even if one does so the scale effect is still problematic. Jones shows that the allocation of labor in the R&D sector in the U.S. economy grew in the period 1950-1990 (measured as the number of scientists and engineers engaged in R&D production), along with a constant long-run TFP and output per capita growth [see the figure below - source: Jones (1995)].[9]

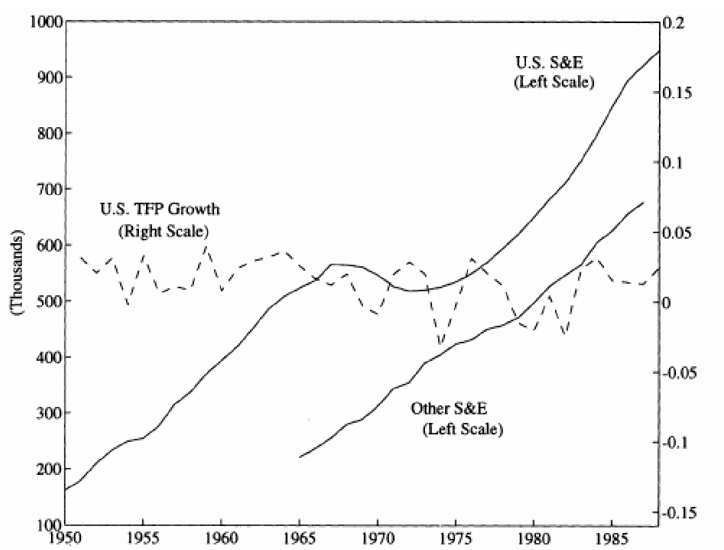


FIG. 1.—Scientists and engineers engaged in R & D and U.S. TFP growth. Source: The number of scientists and engineers engaged in R & D is taken from National Science Foundation (1989) and various issues of the *Statistical Abstract of the U.S. Economy*. TFP growth rates are calculated using the private business sector data in Bureau of Labor Statistics (1991). "Other S&E" is the sum of scientists and engineers engaged in R & D for France, West Germany, and Japan.

The scale effect in Romer (1990) (and many other R&D based models of endogenous growth) may be circumvented by an alternative R&D production function. The micro-foundation of the R&D production function used in the first generation of the R&D-driven endogenous growth models, $\dot{A} = \delta L_A$, suggests that a random process governs the arrival rate δ of new discoveries; the productivity of R&D is taken as given from the point of view of the individual researcher. In Romer (1990)

[9] Note that this pattern speaks against the balanced growth path analysis in Romer (1990) since in that paper the growth rate on BGP is positively related to the number of "scientists."

$\delta = BA$ which introduces pure positive spillover effects in the knowledge production, as the arrival rate increases linearly in the present stock of knowledge.

Jones (1995) employs richer formulation. He sets $\delta = BA^{\phi}m$. When (1) $\phi < 1$ the discovery of new ideas becomes increasingly more difficult; (2) $\phi = 1$ the formulation is essentially the same as in Romer (1990); and (3) $\phi > 1$ the discovery of new ideas becomes easier. Jones argues that due to overlap/duplication in research the effective labor force in R&D process is not L_A but rather L_A^{λ}, $\lambda \in (0,1]$. In particular, he assumes that $\delta = BA^{\phi}L_A^{\lambda-1}$ (i.e., $m = L_A^{\lambda-1}$). Thus R&D process is given by

$$\dot{A} = \delta L_A = \left(BA^{\phi}L_A^{\lambda-1}\right) L_A.$$

Under the assumption $\phi < 1$, this model delivers a BGP which is consistent with an increasing number of persons devoted to R&D. Growth of the economy along this path is function of the growth rate of population and the parameters governing the production of new knowledge, λ and ϕ, rather than the scale of the economy.

Main assumptions

- This is a multi-sector model of R&D-based endogenous growth

  The sectors are

  1. R&D sector - produces blueprints of new varieties/types of capital goods $\dot{A}$. R&D process employs labor L_A. The production of blueprints is given by

     $$\dot{A} = \delta L_A,$$

     where $\delta = BA^{\phi}L_A^{\lambda-1}$ is external for blueprint producers.
  2. Capital goods sector - The sector uses the blueprints and produces intermediate capital goods for the final goods production. It is characterized by monopolistic competition. There is free-entry in the market of new blueprints. Entrepreneurs compete for patent that provides them with infinite-horizon property rights on a new blueprint. The acquisition of a patent allows an entrepreneur to employ exclusively the new blueprint and produce a distinct capital good thereafter. The production of capital goods/varieties requires investment in terms of the (foregone) capital services - r units of final goods produce a unit of a capital good.
  3. Final goods sector - Final goods producers employ labor, L_Y, and variety/set of capital goods, $x(i)$ $i \in [0, A]$, in the production

     $$Y = L_Y^{1-\alpha} \int_0^A x(i)^{\alpha} di.$$

     These firms are fully competitive in input and output markets.

4. From the consumption-side, the representative HH chooses its consumption and savings to maximize its lifetime utility $U = \int_0^{+\infty} \frac{c_t^{1-\theta}-1}{1-\theta} \exp\left[-(\rho-n)t\right] dt$ subject to standard budget constraint.

Further assumptions

- All capital goods depreciate fully within one period.
- Population grows at constant rate n.

Preliminary remarks on balanced growth path

In equilibrium

$$\begin{aligned} \dot{A} &= BA^{\phi}L_A^{\lambda-1}L_A \Rightarrow \\ \frac{\dot{A}}{A} &= BA^{\phi-1}L_A^{\lambda}. \end{aligned} \tag{80}$$

On BGP $g_A = \frac{\dot{A}}{A} = const.$ Therefore,

$$0 = (\phi - 1) g_A + \lambda (g_s + n),$$

where $s = \frac{L}{L_A}$. Thus,

$$g_A = \frac{\lambda n}{1-\phi}$$

when $g_s = 0$. We will show that on BGP the aggregates (in per-capita terms) grow at the same rate as A and $g_s = 0$.

Market equilibrium

Final goods sector

Everything is similar to Romer (1990). The problem of the representative final goods producer is

$$\max_{\{x(i)\}_{i\in[0,A]}, L_Y} \left\{ L_Y^{1-\alpha} \int_0^A x(i)^{\alpha} di - \int_0^A p_{x(i)} x(i) di - wL_Y \right\}.$$

The optimal rules are

$$[L_Y] \quad : \quad w = (1-\alpha), \tag{81}$$

$$[x(i)] \quad : \quad p_{x(i)} = \alpha L_Y^{1-\alpha} x(i)^{\alpha-1} =; \forall i. \tag{82}$$

Capital goods sector

The problem of some ith capital good producer is

$$\begin{aligned}&\max_{p_x(i),x(i)} \left\{p_{x(i)} x\left(i\right) - rx\left(i\right)\right\} \\ &s.t. \\ &p_{x(i)} = \alpha L_Y^{1-\alpha} x\left(i\right)^{\alpha-1}.\end{aligned}$$

The optimal rules are derived by plugging the inverse demand for capital good to the profit function and taking the derivative with respect $x\left(i\right)$, i.e.,

$$\max_{x(i)} \{\alpha L_Y^{1-\alpha} x(i)^{\alpha} - rx(i)\}$$

$\Rightarrow$

$$[x(i)] : r = \alpha^2 L_Y^{1-\alpha} x(i)^{\alpha-1} \Rightarrow x(i) = \left(\frac{\alpha^2}{r}\right)^{\frac{1}{1-\alpha}} L_Y. \tag{83}$$

From (82) and (83) it follows that

$$p_x\left(i\right) = \alpha L_Y^{1-\alpha} x^{\alpha-1} = \frac{r}{\alpha} \Rightarrow \tag{84}$$

$$\pi_x\left(i\right) = \left(p_x - r\right) x = \frac{1-\alpha}{\alpha} rx. \tag{85}$$

Given the symmetry across the different types of capital goods in final goods production [i.e., (83) does not depend on index i], in equilibrium all capital goods producers will make the same price and quantity choices [i.e., $p_x(i) = p_x$ and $x(i) = x \ \forall i$].

Similar to Romer (1990), imperfect competition in market equilibrium implies that the price of capital goods is a constant mark-up ($\frac{1-\alpha}{\alpha}$) on marginal cost, and the quantity of supplied capital goods will be lower than the one selected in a perfectly competitive market.

In equilibrium final output is

$$Y = L_Y^{1-\alpha} A x^{\alpha} = Ax\left(\frac{L_Y}{x}\right)^{1-\alpha} \tag{86}$$

$$= A\left(\frac{\alpha^2}{r}\right)^{\frac{\alpha}{1-\alpha}} L_Y. \tag{87}$$

From (85), (86) and (83) it follows that

$$\frac{Y}{A} = \left(\frac{L_Y}{x}\right)^{1-\alpha} \frac{\alpha}{1-\alpha}\frac{\pi_x}{r} = \left[\left(\frac{\alpha^2}{r}\right)^{\frac{-1}{1-\alpha}}\right]^{1-\alpha} \frac{\alpha}{1-\alpha}\frac{\pi_x}{r} \tag{88}$$

$$= \frac{\pi_x}{\alpha\left(1-\alpha\right)}.$$

Therefore,

$$\pi_x = \alpha\left(1-\alpha\right)\frac{Y}{A}. \tag{89}$$

Denote

$$K = \int_0^A x(i)di.$$

In equilibrium

$$\begin{aligned} K &= Ax = A\frac{\alpha}{1-\alpha}\frac{\pi_x}{r} = \frac{1}{r}\alpha^2 Y \Rightarrow \\ r &= \alpha^2\frac{Y}{K} \end{aligned} \tag{90}$$

Firm entry into capital goods market

Everything is similar to Romer (1990), except as it is shown below $\dot{V}_x(t) \neq 0$, where $V_x(t)$ is the value of the capital firm which enters at time t,

$$V_x(t) = \int_t^\infty \pi_x(\tau)\exp\left[-\int_t^\tau r(s)ds\right]d\tau, \tag{91}$$

where r is the instantaneous real interest rate.

From (70) it follows that

$$V_x(t) = \frac{\pi_x(t)+\dot{V}_x(t)}{r(t)}. \tag{92}$$

Free entry condition

$$V_x(t)\dot{A} = wL_A.$$

This condition states that the value generated by the entry of a firm (or firms) $V_x\dot{A}$ is equal to the cost of generating the blueprint(s) wL_A.

R&D sector

The price of a blueprint is $V_x(t)$ and the problem of the representative blueprint producer is

$$\max_{L_A}\left\{V_x\delta L_A - wL_A\right\},$$

where $\delta = BA^\phi L_A^{\lambda-1}$ is not controlled by the firm. Assuming that in equilibrium there is technological progress, optimal rule is

$$w = V_x\delta.$$

Labor market equilibrium

Labor market equilibrium equates wages in both sectors,

$$\begin{aligned} w &= V_x\delta = (1-\alpha)\frac{Y}{L_Y} \\ &= (1-\alpha)A\left(\frac{\alpha^2}{r}\right)^{\frac{\alpha}{1-\alpha}}. \end{aligned} \tag{93}$$

Thus,

$$V_x = \frac{1-\alpha}{B}A^{1-\phi}L_A^{1-\lambda}\left(\frac{\alpha^2}{r}\right)^{\frac{\alpha}{1-\alpha}}. \tag{94}$$

The household side

Everything is standard and written in per-capita terms. The optimal consumption path is

$$g_c = \frac{1}{\theta}(r-\rho). \tag{95}$$

The standard TVC ensures that the value of the asset holdings of the HH is equal to zero in the limit, i.e., the growth of the value of the assets does not exceed the discount rate.

Balanced growth path

From the formula for R&D process (80) it follows that

$$g_A = \frac{\lambda}{1-\phi}(n+g_s). \tag{96}$$

From the relation between final output Y and profits of capital goods producers π_x (80) it follows that

$$g_y = g_A + g_{\pi_x} - n. \tag{97}$$

Given that r should be constant on the BGP from Hamilton-Jacobi-Bellman equation (92) it follows that

$$\begin{aligned} g_{V_x} &= g_{\pi_x}, \\ g_y &= g_A + g_{V_x} - n. \end{aligned} \tag{98}$$

From the supply of capital good x it follows that

$$g_{V_x} = g_{\pi_x} = n - g_s\frac{s}{1-s}. \tag{99}$$

Therefore,

$$g_y = \frac{\lambda}{1-\phi}(n+g_s) - g_s\frac{s}{1-s}.$$

Given that on BGP all variables should grow at constant rates it follows from this equation that $g_s = 0$.

In order to find the growth rate of per capita consumption consider the budget constraint,

$$\dot{W} = rW + wL - C.$$

In equilibrium the asset holdings are the rented capital K and the shares of capital good producing firms with total value $V_x A$. Therefore, the budget constraint can be written as

$$(K + \dot{V_x A}) = r(K + V_x A) + wL - C.$$

From (92) it follows that

$$\dot{K} = rK + A\pi_x - V_x\dot{A} + wL - C.$$

From (89) it follows that

$$\dot{K} = rK + \alpha(1-\alpha)Y - V_x\dot{A} + wL - C.$$

From definition of s and (81) it follows that

$$\begin{aligned} w &= (1-\alpha)\frac{Y}{L}\frac{L}{L_Y} = (1-\alpha)\frac{Y}{L}\frac{1}{1-s} \Rightarrow \\ \dot{K} &= rK + \alpha(1-\alpha)Y - V_x\dot{A} + (1-\alpha)\frac{1}{1-s}Y - C. \end{aligned}$$

From the free entry condition if follows that

$$V_x\dot{A} = wL_A = (1-\alpha)Y\frac{L_A}{L_Y} = (1-\alpha)Y\frac{s}{1-s}.$$

which implies that

$$\dot{K} = rK + \alpha(1-\alpha)Y + (1-\alpha)Y - C.$$

From (90) it follows that

$$rK = \alpha^2 Y$$

$$\Rightarrow$$

$$\dot{K} = Y - C.$$

Therefore, this expression together with (90) it follows that

$$\begin{aligned} g_C &= g_Y = g_K = g_A + n. \\ &\Rightarrow \\ g_c &= g_y = g_k = g_A = \frac{\lambda n}{1-\phi}. \end{aligned}$$

R&D based models of growth: The Smulders and van de Klundert (1995) model - vertical product innovation and market structures

Motivation

In Romer (1990) and Jones (1995) R&D process creates new types of goods (i.e., we have horizontal product expansion). An interpretation of such framework is that the firms engage in R&D (or pay for R&D) for developing their distinct types of goods. Meanwhile, they never upgrade their good or upgrade the way they produce it. A suitable example for such case would be that Intel develops the architecture of central processing units x86 and starts producing its (first) Intel 8_086, though never enhances that architecture and its tech process (for example, to any of the Intel Core i7 editions).

Smulders and van de Klundert (1995), in contrast, focuses on inter-firm R&D. Similar to Romer (1990) and Jones (1995) it considers large firms in the capital goods sector to the extent that these firms are infinitely lived and are price setters. In contrast, however, Smulders and van de Klundert allow the firms to continually upgrade their productivity (or equivalently the quality of their product) through innovation. They call this in-house R&D which within the current example would correspond to the case when Intel improves the architecture and the tech process of Intel 8_086.

In Smulders and van de Klundert (1995) the motivation behind R&D is the competition between firms. The outcome of R&D process allows to reduce marginal costs in production and, therefore, set lower price. With a lower price the innovator would capture a bigger market. Smulders and van de Klundert (1995) further strengthens this point and models the final goods production function (i.e., the demand for capital goods) so that the actions of any firm in the capital goods sector can have an effect on the demand of the rest of the firms (e.g., Intel and AMD). In order to do that they use a more general CES aggregator where capital goods are not additively separable as in Romer (1990). This allows them to have strategic interactions between capital goods producers and Smulders and van de Klundert focus on the mechanisms how the market structure of capital goods producing industry and competition type in capital goods market (Cournot and Bertrand) can affect growth and welfare in the economy. These strategic interactions are shown to matter for innovation.[10]

Main assumptions

- Multi-sector model of R&D-based endogenous growth

  The sectors are

  1. Ordinary goods sector

[10] The model presented in the next section assumes a simplified structure for the R&D process, which is similar to the one used in van de Klundert, T., and Smulders, S. (1997). Growth, competition and welfare. *Scandinavian Journal of Economics 99*, 99-118. This helps avoiding lots of unnecessary algebra and does not qualitatively change the discussion.

- Ordinary goods producers have a Ricardian technology and employ only labor, L_Y,

$$Y = L_Y.$$

These firms are in perfect competition in both input and output markets.

2. Capital goods producing sector

   - Each firm produces its unique type of capital good $x(j) \equiv x_j$, $j = \overline{1, N}$, with a technology

   $$x_j = \lambda_j L_{x_j},$$

   where the L_{x_j} is the share of total labor force L hired by the jth firm and λ_j is its productivity.
   - In order to operate the firm has to hire fixed number of managers f.
   - Each firm can hire labor force L_{r_j} (researchers) in order to improve its productivity, i.e., improve the knowledge how to produce or create better/more efficient production instructions. The researchers use the current knowledge of the firm, λ_j, in order to create the new/better one $\dot{\lambda}_j$. The productivity improvement process is given by

   $$\dot{\lambda}_j = \xi \Lambda \lambda_j^{1-\alpha} L_{r_j},$$
   $$\xi > 0, 0 < \alpha < 1,$$

   where ξ is an exogenous efficiency level, $1 - \alpha$ is the degree to which knowledge contributes to the productivity improvement process, and Λ is a spillover term. The spillovers originate from the knowledge for production of the remaining firms. Λ is such that in equilibrium new knowledge for production $\dot{\lambda}_j$ is linear in current knowledge λ_j - there is no population growth in this model, therefore, this will be a necessary condition for a balanced growth path.

     * For example,

     $$\Lambda = \left(\frac{1}{N} \sum_{j=1}^{N} \lambda_j \right)^{\alpha},$$

     An interpretation for this is that the knowledge for production of any firm becomes public; however, only some part of that knowledge helps the other firms to enhance their productivity. An example could be informal discussions amongst the scientists employed in different firms, through which the scientists share information how to improve the production process. Patents and their use for generation of other patents could be another example in case the buyers of patents have the power/right of making a take-it or leave-it offer (see for further discussion **?**).

   (a) There is free entry into capital goods producing sector at no cost. This means that the entrant gets its patent for the design of its capital good for free. However, once it enters at each period it incurs two types of fixed costs. First, the firm has to hire fixed amount of labor force f and, second, the firm can invest in productivity improvement.

  - The firm has to have strictly positive gross profits (or profit flows) in order to cover these costs. Thus, the firm has to be price setter. Moreover, it is assumed that the fixed cost associated with compensation of f is such that the net (of these fixed costs) profits of only finite number of firms will be non-negative. In other words, although there are no entry costs, the maximum number of firms in the market will be constant (denote by N), since the profits are zero and negatively related to the number of firms. Therefore, we consider an oligopolistic market, i.e., there are barriers to entry.

- From the consumption-side, the representative HH chooses its consumption and next period assets to maximize its lifetime utility $U = \int_0^{+\infty} \frac{C_t^{1-\theta}-1}{1-\theta} \exp(-\rho t)\, dt$ subject to standard budget constraint, where C is the consumption good which is a basket (Cobb-Douglas combination) of Y and X

$$C = Y^{\sigma} X^{1-\sigma}, 0 < \sigma < 1.$$

  In turn, X is a basket (CES aggregate with elasticity of substitution ε) of capital goods x_j,

$$X = \left(\sum_{j=1}^{N} x_j^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}, \varepsilon > 1.$$ [11]

Further assumptions

- All capital varieties depreciate fully within one period
- There is no population growth

Market equilibrium

The household side

The representative household's maximization problem can be divided into three steps. In the first step, the household selects the path of consumption C that delivers maximum utility. That path is standard, it follows from the lifetime utility maximization, and it is given by

$$\frac{\dot{C}}{C} = \frac{1}{\theta}(r - \rho). \tag{100}$$

[11] The assumption that $\varepsilon > 1$ is necessary condition for imperfect competition in the capital goods market.

In the second step household chooses the optimal combination of Y and X in the good consumed, C. In other words, the household solves

$$\max_{Y,X}\left\{C-P_YY-P_XX\right\},$$
$$s.t.$$
$$C=Y^{\sigma}X^{1-\sigma}, \tag{101}$$

where P_Y and P_X are the prices of Y and X, respectively. The optimal rules then are

$$[Y]:P_YY=\sigma C, \tag{102}$$
$$[X]:P_XX=(1-\sigma)\,C. \tag{103}$$

In the third step, the household chooses the optimal combination of capital goods in X. In other words, household solves

$$\max_{\{x_j\}_{j=1}^N}\left\{P_XX-\sum_{j=1}^N p_{x_j}x_j\right\}$$
$$s.t.$$
$$X=\left(\sum_{j=1}^N x_j^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{104}$$

where N is the number of capital goods. The optimal rules are

$$\left[x_j,\forall j=\overline{1,N}\right]:P_X\frac{\partial}{\partial x_j}\left(\sum_{j=1}^N x_j^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}=p_{x_j}\Leftrightarrow$$
$$x_j=\left(\frac{P_X}{p_{x_j}}\right)^{\varepsilon}X. \tag{105}$$

Plugging these demand functions back into X gives an expression for the price level

$$P_X=\left(\sum_{j=1}^N p_{x_j}^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}. \tag{106}$$

Ordinary goods production

The problem of the representative ordinary goods producer is

$$\max_{L_Y}\left\{P_YY-wL_Y\right\}$$
$$s.t.$$
$$Y=L_Y. \tag{107}$$

Therefore its demand for labor is

$$[L_Y]: P_Y = w. \tag{108}$$

Capital goods production

The revenues of the jth capital goods producing firm are from the sales of capital good x_j. The costs are the labor force compensation for the production of x_j, the labor force compensation for productivity improvement, and the fixed cost wf. Thus the profit function of the jth capital goods producing firm is

$$\begin{aligned}\pi_{x_j} &= p_{x_j} x_j - \left(wL_{x_j} + wL_{r_j} + wf\right) \\ &= p_{x_j} \lambda_j L_{x_j} - w\left(L_{x_j} + L_{r_j} + f\right),\end{aligned}$$

where p_{x_j} is the price of the capital good x_j. The capital goods producing firm maximizes its present value of future profit streams V_j. It discounts the profit streams with instantaneous interest rate r. The problem of the capital goods producing firm is

$$\begin{aligned}&\max_{L_{x_j}, L_{r_j}} \left\{ V_j = \int_0^{+\infty} \pi_{x_j} \exp\left[-\int_0^{\tau} r(t)\, dt\right] d\tau \right\}, \\ &s.t. \\ &\pi_{x_j} = p_{x_j} \lambda_j L_{x_j} - \left(wL_{x_j} + wL_{r_j} + wf\right), \\ &\dot{\lambda}_j = \xi \Lambda \lambda_j^{1-\alpha} L_{r_j}.\end{aligned} \tag{109}$$

The integral starts from $t = 0$ since there are no entry costs and the entry will be instantaneous at time 0 - so, basically, we have jumped a little bit ahead and used an equilibrium condition; This does not alter the generality.

Let q_λ be the shadow value of relaxing the constraint (109). The problem in terms of current value Hamiltonian is

$$\max_{L_{x_j}, L_{r_j}} H_F = \pi_{x_j} + q_{\lambda_j} \xi \Lambda \lambda_j^{1-\alpha} L_{r_j}.$$

The optimal rules are

$$
\begin{aligned}
[L_{x_j}]: \quad & \frac{\partial p_{x_j}\lambda_j L_{x_j}}{\partial L_{x_j}} - w = 0 \Leftrightarrow \\
& \frac{p_{x_j}x_j}{L_{x_j}} + x_j \frac{\partial p_{x_j}}{\partial L_{x_j}} - w = 0 \Leftrightarrow \\
& \frac{p_{x_j}x_j}{L_{x_j}} + p_{x_j} \frac{\partial x_j}{\partial L_{x_j}} \frac{x_j}{p_{x_j}} \frac{\partial p_{x_j}}{\partial x_j} - w = 0 \Leftrightarrow \\
& \frac{p_{x_j}x_j}{L_{x_j}} - \frac{p_{x_j}x_j}{L_{x_j}} \frac{1}{e_j} - w = 0 \quad \left(e_j \equiv -\frac{\partial x_j}{\partial p_{x_j}} \frac{p_{x_j}}{x_j} \right) \Leftrightarrow \\
& \frac{p_{x_j}x_j}{L_{x_j}} \left(1 - \frac{1}{e_j}\right) = w, \qquad (110) \\
[L_{r_j}]: \quad & q_{\lambda_j} \frac{\dot{\lambda}_j}{L_{r_j}} = w, \qquad (111) \\
[\lambda_j]: \quad & \dot{q}_{\lambda} = q_{\lambda_j} r - q_{\lambda_j} \frac{\partial H_F}{\partial \lambda_j} \Leftrightarrow \\
& \dot{q}_{\lambda_j} = q_{\lambda_j} r - q_{\lambda_j} \left[(1-\alpha) \frac{\dot{\lambda}_j}{\lambda_j} + \frac{\partial p_{x_j}\lambda_j L_{x_j}}{\partial \lambda_j} \right] \Leftrightarrow \\
& \dot{q}_{\lambda_j} = q_{\lambda_j} r - \left[q_{\lambda_j} (1-\alpha) \frac{\dot{\lambda}_j}{\lambda_j} + \frac{p_{x_j}x_j}{\lambda_j} \left(1 - \frac{1}{e_j}\right) \right], \qquad (112)
\end{aligned}
$$

where (110) is the supply of the capital good x_j, (111) is the demand for labor for productivity improvement, e_j is the elasticity of substitution between capital goods perceived by the jth capital good producing firm. Under Bertrand competition

$$e_j \equiv e_j^B = \varepsilon - \left[\frac{(\varepsilon - 1)\, p_{x_j}^{1-\varepsilon}}{\sum_{j=1}^{N} p_{x_j}^{1-\varepsilon}} \right], \qquad (113)$$

and under Cournot competition

$$e_j \equiv e_j^C = \frac{\varepsilon}{1 + \left[(\varepsilon - 1) \frac{x_j^{\frac{\varepsilon-1}{\varepsilon}}}{\sum_{j=1}^{N} x_j^{\frac{\varepsilon-1}{\varepsilon}}} \right]}, \qquad (114)$$

(see the derivations in the end of this chapter). The terms in square brackets measure the impact of other firms on the demand of the jth capital good producing firm. In other words, they measure the extent of strategic interactions between capital good producing firms. Moreover, these terms indicate the difference between the perceived elasticity of substitution (e) and the actual elasticity of substitution (ε). Therefore, they indicate some of the distortions in the economy which stem from monopolistic competition with finite number of firms. In a symmetric equilibrium when the number of firms increases these distortions tend to zero since the terms in square brackets tend to zero. Finally, (112) is the rate of return on productivity improvement.

Remaining equilibrium conditions

In equilibrium the value of the capital goods basket is equal to the sum of the values of capital goods, i.e.,

$$P_X X = \sum_{j=1}^{N} p_{x_j} x_j. \tag{115}$$

This means that there is no resource waste - zero profit.

Moreover, in equilibrium the labor market clears, i.e.,

$$L = L_Y + \sum_{j=1}^{N} \left(L_{x_j} + L_{r_j} + f \right). \tag{116}$$

Hereafter consider a symmetric equilibrium in capital goods market and a balanced growth path.

Balanced growth path and symmetric equilibrium

In symmetric equilibrium (drop index j) from (110), (111), and (112) it follows that

$$\frac{\dot{q}_\lambda}{q_\lambda} = r - \left[(1-\alpha)\, \xi L_r + \frac{L_x}{L_r} g_\lambda \right].$$

From (111) on balanced growth path it follows that

$$\frac{\dot{q}_\lambda}{q_\lambda} = g_w - g_\lambda + g_{L_r}.$$

From productivity improvement process (109) it follows that

$$\frac{\dot{q}_\lambda}{q_\lambda} = r - \left[(1-\alpha)\, g_\lambda + \frac{L_x}{L_r} g_\lambda \right],$$

and

$$g_{L_r} = 0.$$

Therefore,

$$\begin{aligned} \frac{\dot{q}_\lambda}{q_\lambda} &= r - \left[(1-\alpha)\, g_\lambda + \frac{L_x}{L_r} g_\lambda \right] \Leftrightarrow \\ L_r &= \frac{g_\lambda}{r - g_w + \alpha g_\lambda} L_x. \end{aligned} \tag{117}$$

The expression (117) shows the relation between L_r and L_x given the growth rates and the interest rate. It implies that

$$g_{L_x} = g_{L_r} = 0.$$

In turn, from the rules for optimal combination of the ordinary good and basket of capital goods in consumption good, (102) and (103), it follows that

$$P_Y Y = \frac{\sigma}{1-\sigma} P_X X. \tag{118}$$

Therefore, from the production function of the ordinary goods (107) and labor demand in that sector (108), the equilibrium rule for the basket of capital goods (115) and from the supply of capital goods (110) it follows that

$$\begin{aligned} P_Y Y &= w L_Y = \frac{\sigma}{1-\sigma} \frac{e}{e-1} w N L_x \Leftrightarrow \\ L_Y &= \frac{\sigma b}{1-\sigma} N L_x, \quad \left(b \equiv \frac{e}{e-1} \right). \end{aligned} \tag{119}$$

From the labor market clearing condition (116) and (119) it follows that

$$\begin{aligned} L &= L_Y + N \left(L_{x_j} + L_{r_j} + f \right) \\ &= \left(\frac{\sigma b + 1 - \sigma}{1-\sigma} \right) N L_x + N L_r + N f \end{aligned} \tag{120}$$

Therefore,

$$L_x = \left(\frac{1-\sigma}{\sigma b + 1 - \sigma} \right) \left(\frac{L}{N} - L_r - f \right) \tag{121}$$

Let (f be such that) $N = \mathit{const} < +\infty$. From (117) and (119) it follows that

$$g_{L_Y} = g_{L_x} = g_{L_r} = 0. \tag{122}$$

Therefore, from the labor demand in ordinary goods production (108), ordinary goods production function, and the demand for the ordinary goods (102) it follows that

$$g_w = g_{P_Y} = g_C. \tag{123}$$

From the production function of consumption goods (101), (122), production function of capital goods and the basket of capital goods (104) it follows that

$$g_C = (1-\sigma) g_\lambda. \tag{124}$$

From (123), (124), optimal consumption path (100), productivity improvement process (109),and

(117) it follows that

$$\begin{aligned} L_r &= \frac{g_\lambda}{(\theta-1)(1-\sigma)g_\lambda+\rho+\alpha g_\lambda}L_x \Leftrightarrow \\ \frac{1}{\xi}g_\lambda &= \frac{g_\lambda}{(\theta-1)(1-\sigma)g_\lambda+\rho+\alpha g_\lambda}L_x \Leftrightarrow \\ L_x &= \frac{1}{\xi}\left\{\left[(\theta-1)(1-\sigma)+\alpha\right]g_\lambda+\rho\right\}. \end{aligned} \tag{125}$$

Denote

$$D \equiv \frac{1-\sigma}{\sigma b+1-\sigma} = \frac{(1-\sigma)(e-1)}{e+\sigma-1}.$$

From (121) and (125) it follows that

$$D\left(\frac{L}{N}-\frac{1}{\xi}g_\lambda-f\right) = \frac{1}{\xi}\left\{\left[(\theta-1)(1-\sigma)+\alpha\right]g_\lambda+\rho\right\} \tag{126}$$

This expression is equivalent to

$$g_\lambda(N) = \frac{\xi D\left(\frac{L}{N}-f\right)-\rho}{(\theta-1)(1-\sigma)+\alpha+D}. \tag{127}$$

The growth rate g_λ in (127) is the growth rate of productivity in capital good production given the number of firms N. This can be understood as the growth rate of productivity in partial equilibrium where the number of firms was set to N exogenously.[12] In general equilibrium the number of firms can be derived from zero profit condition

$$\pi = 0.$$

In turn the GDP growth rate is given by (124).

Bertrand versus Cournot: Toughness of competition

In symmetric equilibrium under Bertrand competition

$$e^B = \varepsilon-(\varepsilon-1)\frac{1}{N}, \tag{128}$$

and under Cournot competition

$$e^C = \frac{\varepsilon}{1+(\varepsilon-1)\frac{1}{N}}. \tag{129}$$

When $N<\infty$ both e^C and e^B are less than ε. In addition, it can be easily shown that for given number of firms $e^B > e^C$. Therefore,

$$\varepsilon > e^B > e^C.$$

Since given the number of firms the e^B is greater than e^C, it can be thought that under Bertrand competition the competition is tougher.

[12] One may assume that the government regulates the number of firms.

Let the number of firms be exogenously fixed. Consider the effect of toughening the competition in capital goods market on balanced growth in the economy, i.e., switching from Cournot (quantity) competition to Bertrand (price) competition. In order to analyze that effect consider the derivative of the g_λ with respect the elasticity of substitution perceived by capital good producing firms, i.e.,

$$\begin{aligned}
\frac{\partial}{\partial e} g_\lambda &= \frac{\partial D}{\partial e}\frac{\partial g_\lambda}{\partial D} \\
\frac{\partial D}{\partial e} &= \frac{\partial}{\partial e}\frac{(1-\sigma)(e-1)}{e+\sigma-1} = \frac{\sigma(1-\sigma)}{(e+\sigma-1)^2} > 0 \\
\frac{\partial g_\lambda}{\partial D} &= \frac{\xi\left(\frac{L}{N}-f\right)\left[(\theta-1)(1-\sigma)+\alpha\right]+\rho}{\left[(\theta-1)(1-\sigma)+\alpha+D\right]^2} > 0
\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial e} g_\lambda > 0.$$

This means that given the number of firms increasing the toughness of competition (note $e^B > e^C$) increases the growth rate of productivity in capital good production; thus, it increases also the growth rate of C. This would mean that increasing the toughness of competition increases the social welfare.

Intensifying the competition through higher number of firms

Similar to Smulders and van de Klundert (1995) consider the case when D does not change with N. In that case it is easy to notice that g_λ and the growth rate of C decline with the number of firms. This means that after some threshold increasing the number of firms can be undesirable from the social welfare perspective.

General equilibrium - the number of firms

Again, similar to Smulders and van de Klundert (1995) consider the case when D does not change with N. More precisely, let

$$D = \frac{(1-\sigma)(\varepsilon-1)}{\varepsilon+\sigma-1}.$$

In equilibrium the profits of any capital producing firm are zero, i.e.,

$$\pi_x = p_x x - w\left(L_x + L_r + f\right) = 0. \tag{130}$$

From the labor market clearing condition (116) and (110), (125) it follows that (130) is equivalent to

$$\begin{aligned}
\pi_x &= p_x x - w\left(\frac{L}{N} - \frac{\sigma b}{1-\sigma}L_x\right) \\
&= wbL_x - w\left(\frac{L}{N} - \frac{\sigma b}{1-\sigma}L_x\right) \\
&= w\left(\frac{b}{1-\sigma}L_x - \frac{L}{N}\right) \\
&= w\left(\frac{b}{1-\sigma}\frac{1}{\xi}\left\{\left[(\theta-1)(1-\sigma)+\alpha\right]g_\lambda + \rho\right\} - \frac{L}{N}\right) \\
&= 0.
\end{aligned}$$

Therefore, (130) is equivalent to

$$g_\lambda = \frac{\xi\frac{1-\sigma}{b}\frac{L}{N} - \rho}{(\theta-1)(1-\sigma)+\alpha}. \tag{131}$$

The number of firms then can be derived from (127) and (131), i.e.,

$$\begin{gathered}
\frac{\xi\frac{1-\sigma}{b}\frac{L}{N} - \rho}{(\theta-1)(1-\sigma)+\alpha} = \frac{\xi D\left(\frac{L}{N} - f\right) - \rho}{(\theta-1)(1-\sigma)+\alpha+D} \Leftrightarrow \\
\frac{(\theta-1)(1-\sigma)+\alpha+D}{(\theta-1)(1-\sigma)+\alpha}\left(\xi\frac{1-\sigma}{b}\frac{L}{N} - \rho\right) = \xi D\left(\frac{L}{N} - f\right) - \rho.
\end{gathered}$$

Denote

$$M = \frac{(\theta-1)(1-\sigma)+\alpha+D}{(\theta-1)(1-\sigma)+\alpha}.$$

Thus,

$$\begin{gathered}
M\left(\xi\frac{1-\sigma}{b}\frac{L}{N} - \rho\right) = \xi D\left(\frac{L}{N} - f\right) - \rho \Leftrightarrow \\
N^* = \frac{\xi\left(M\frac{1-\sigma}{b} - D\right)L}{(M-1)\rho - \xi D f}.
\end{gathered}$$

The growth rate of productivity g_λ^* follows from N^*.

Comparative statics

Exogenously fixed number of firms It has been shown already that the tougher competition, i.e., Cournot vs. Bertrand, implies higher growth rate of productivity improvement; thus, higher growth rate of C. At the same time, more intensive competition, i.e., higher number of firms, implies lower growth rate of productivity improvement; thus, lower growth rate of C.

The remaining comparative statics that are worth to emphasize are

- Increase in α decreases g_λ and g_C
- Increase in θ (or ρ) decreases both g_λ and g_C

- Increase in ξ increases the g_λ and g_C

Endogenously determined number of firms In this case the change in parameters can imply changes in the number of firms. This will be an additional channel that will affect the productivity growth rate and thus the growth rate of C.

The comparative statics for this case are left as an exercise.

The derivation of perceived elasticities of substitution For Bertrand competition consider

$$\frac{\partial x_j}{\partial p_{x_j}} = \frac{\partial}{\partial p_{x_j}} \frac{(1-\sigma)\, C}{P_X} \left(\frac{P_X}{p_{x_j}}\right)^{\varepsilon}.$$

$$\begin{aligned}
\frac{\partial x_j}{\partial p_{x_j}} &= (1-\sigma)\, C \frac{\partial}{\partial p_{x_j}} \left(p_{x_j}\right)^{-\varepsilon} P_X^{\varepsilon-1} \\
&= \frac{(1-\sigma)\, C}{P_X} \left[-\varepsilon P_X^{\varepsilon} \left(p_{x_j}\right)^{-\varepsilon-1} + (\varepsilon-1) \left(p_{x_j}\right)^{-\varepsilon} P_X^{\varepsilon} \frac{1}{\sum\limits_{j=1}^{N} p_{x_j}^{1-\varepsilon}} p_{x_j}^{-\varepsilon} \right] \\
&= \frac{x_j}{p_{x_j}} \left[-\varepsilon + \frac{(\varepsilon-1)\, p_{x_j}^{1-\varepsilon}}{\sum\limits_{j=1}^{N} p_{x_j}^{1-\varepsilon}} \right]
\end{aligned}$$

$\Rightarrow$

$$e_j^B = \varepsilon - \left[\frac{(\varepsilon-1)\, p_{x_j}^{1-\varepsilon}}{\sum\limits_{j=1}^{N} p_{x_j}^{1-\varepsilon}} \right].$$

For Cournot competition consider

$$\frac{\partial p_{x_j}}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{(1-\sigma)\, C}{X} \left(\frac{X}{x_j}\right)^{\frac{1}{\varepsilon}}.$$

$$
\begin{aligned}
\frac{\partial p_{x_j}}{\partial x_j} &= (1-\sigma)\, C X^{-1} \left[-\frac{1}{\varepsilon} X^{\frac{1}{\varepsilon}} (x_j)^{-\frac{1}{\varepsilon}-1} + (x_j)^{-\frac{1}{\varepsilon}} \left(\frac{1}{\varepsilon} - 1 \right) X^{\frac{1}{\varepsilon}-1} \frac{\partial}{\partial x_j} X \right] \\
&= \frac{p_{x_j}}{x_j} \left[-\frac{1}{\varepsilon} + x_j \left(\frac{1}{\varepsilon} - 1 \right) X^{-1} \frac{\partial}{\partial x_j} X \right] \\
&= \frac{p_{x_j}}{x_j} \left[-\frac{1}{\varepsilon} + \left(\frac{1-\varepsilon}{\varepsilon} \right) \frac{x_j^{\frac{\varepsilon-1}{\varepsilon}}}{\sum_{j=1}^{N} x_j^{\frac{\varepsilon-1}{\varepsilon}}} \right]
\end{aligned}
$$

$\Rightarrow$

$$
e_j^C = \frac{\varepsilon}{\left[1 + (\varepsilon - 1) \frac{x_j^{\frac{\varepsilon-1}{\varepsilon}}}{\sum_{j=1}^{N} x_j^{\frac{\varepsilon-1}{\varepsilon}}} \right]}.
$$

R&D based models of growth: The Grossman and Helpman (1991) model - vertical product innovation with creative destruction

Main Assumptions

- A multi-sector model of R&D-based endogenous growth that is driven by "creative destruction."

- There are two factors of production: a fixed amount of labor and fixed number, N, of capital good types. Within each variety j, capital goods differ in their quality.

- Qualities are of distance $q > 1$ of each other. The best quality within every sector j is q^{κ_j}, where $\kappa_j \in \mathbb{N} \cup \{0\}$. The initial quality is normalized to one ($q^k\big|_{\kappa=0} = 1$).

- The R&D sector produces "blueprints" for improved quality capital goods of each known variety. The input to the R&D production is investment in units of the final output. In sector j, where the highest quality is κ_j, the R&D expenditures are $Z_{j\kappa_j}$. The output of the R&D production is uncertain. The R&D expenditures result in a new capital good type $(\kappa_j + 1)$ with probability $p_{j\kappa_j}$. The technology of R&D production is linear in R&D investment,

$$p_{j\kappa_j} = \phi(\kappa_j) Z_{j\kappa_j}. \tag{132}$$

  As quality improves, new discoveries become more expensive in terms of the required investment of resources, i.e., $\phi'(\kappa_j) < 0$ gives diminishing returns to R&D input. As the probability depends only on the current quality level, it suggests that innovation occurs like a Poisson process.

  Note: Linearity implies absence of congestion. Innovation in each sector is "jumpy" (takes place in a discreet manner), however the existence of many sectors and the Law of Large Numbers ensures a smooth outcome at the aggregate level.

- The discovery of a better quality capital good of a particular type/variety provides an entrepreneur with monopoly rights over the use of the "blueprint." He produces the distinct type of capital good with a linear technology that transforms one unit of final output into one unit of capital good.

- There is free entry into the capital goods industry.

- In final goods sector, firms operate under perfect competition. They combine labor, L, with quality-adjusted input $\tilde{X}_j$ of every variety j of the existent (fixed) set of capital goods, $j \in \{1, ..., N\}$, i.e.,

$$Y = AL^{1-\alpha} \sum_{j=1}^{N} \tilde{X}_j^{\alpha}.$$

There is additive separability in all varieties of capital goods and all of them are used in the final goods production due to standard neoclassical function assumptions (Inada conditions).

- Different qualities of capital goods of a type j are perfect substitutes. Hence, if κ_j is the best quality known, the total input employed in the final goods sector of the jth type of capital good is:

$$\tilde{X}_j = \sum_{k=0}^{\kappa_j} q^k x_{jk}.$$

- It is assumed here that only the highest quality capital good survives of each type of capital good, hence

$$\tilde{X}_j = q^{\kappa_j} x_{j\kappa_j}.$$

  This assumption regarding the properties of the equilibrium path is to be verified below, by examining the conditions that support it.

  Note: The survival of the best quality only across sectors means that the model features technological obsolescence. This results in that the decision to conduct research in order to invent a better quality capital good is based on two forces. First, the discovery is to be overtaken by another researcher in the future (this decreases incentives for research). Second, the discovery of an improved quality capital good implies that there will be a transfer of the monopoly rent from the previous best-quality discovery owner (this increases the incentives for research).

- From the consumption-side, the representative HH chooses its consumption and assets to maximize its intertemporal utility: $U = \int_0^{+\infty} u(C_t) \exp(-\rho t)\, dt$ subject to the standard budget constraint.

Further assumptions

- All capital goods depreciate fully within one period.
- No population growth.

Market equilibrium

Final goods sector

The final good producers maximize their profits taking the price of their inputs, labor (w) and capital goods ($P_{j\kappa_j}, \forall j$) as given. Standard conditions imply that:

$$[L] \quad : \quad \frac{\partial Y}{\partial L} = (1-\alpha)\frac{Y}{L} = w \tag{133}$$

$$\left[x_{j\kappa_j}\right] \quad : \quad \frac{\partial Y}{\partial x_{j\kappa_j}} = A\alpha L^{1-\alpha} x_{j\kappa_j}^{\alpha-1} q^{\alpha\kappa_j} = P_{j\kappa_j}; \forall j \tag{134}$$

(Here prices of capital goods $P_{j\kappa_j}$ are in capital letters to avoid confusion with probability of discovery of new quality of type j capital good $p_{j\kappa_j}$.)

Capital goods sector

Each capital good producer, within every period maximizes its profits, $\pi_{j\kappa_j}$, by selecting its price, $P_{j\kappa_j}$, and quantity of production, $x_{j\kappa_j}$. For every unit of capital that it produces, it needs to use one unit of final goods that it "borrows" from HH at the current output price, i.e.,

$$\pi_{j\kappa_j} = P_{j\kappa_j} x_{j\kappa_j} - x_{j\kappa_j}$$

The monopolist takes as given the price of the output it uses in its production and the demand that its good is facing from the final goods producers, i.e., its problem is

$$\begin{aligned}
&\max_{x_{j\kappa_j}} \pi_{j\kappa_j} \\
&s.t. \\
&P_{j\kappa_j} = A\alpha L^{1-\alpha} x_{j\kappa_j}^{\alpha-1} q^{\alpha\kappa_j}
\end{aligned}$$

The FOC of this optimal program imply

$$\begin{aligned}
x_{j\kappa_j} &= LA^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} q^{\frac{\alpha}{1-\alpha}\kappa_j}, \\
P_{j\kappa_j} &= \frac{1}{\alpha}.
\end{aligned}$$

Note: It was assumed that only the best available quality is available within each type of capital good. Suppose instead that this was not the case and the second-best quality is also available. The marginal product of any two consecutive qualities differs by their quality difference, i.e., by a factor of q, which means that the price differential supported by the market equilibrium between the first and second highest quality is: $\frac{P_{j\kappa_j}}{P_{j\kappa_j-1}} = q$. Therefore, the monopoly producer of the second highest quality can at most charge $P_{j\kappa_j-1} = \frac{1}{\alpha q}$. If $\alpha q > 1$, then the second best quality producer cannot cover with such price its marginal cost of production and is driven out of the market. Alternatively, if $\alpha q < 1$, the result that only the leading technology survives within each variety can still be an equilibrium outcome, where the leader follows limit pricing. In such case it charges $P_j\kappa_j = q - \varepsilon$ ($\varepsilon \to +0$) and the next quality producer then could charge $1 - \frac{\varepsilon}{q} < 1$. Hereafter, assume that $\alpha q > 1$.

With the leading technology only, then the final output production in equilibrium is described by

$$
\begin{aligned}
Y &= AL^{1-\alpha}\sum_{j=1}^{N} q^{\alpha\kappa_j} x_{j\kappa_j}^{\alpha} \\
&= LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}\sum_{j=1}^{N} q^{\frac{\alpha}{1-\alpha}\kappa_j}.
\end{aligned}
$$

The "blueprint" production

Given the equilibrium outcome of the capital goods, labor and final output market, the next step is to examine the decision of the entrepreneurs to conduct R&D for the discovery of the κ_{j+1} quality of every type of capital good. Access to the market is free, therefore, every entrepreneur should be in the limit equating his cost in investing in R&D with his expected profit.

The cost of R&D is the investment in terms of current output, $Z_{j\kappa_j}$. The probability of a successful innovation of the quality $\kappa_j + 1$ is $p_{j\kappa_j}$ while the expected value from it is $V_{j\kappa_j+1}$. More rigorously, $V_{j\kappa_j+1}$ is the expected present value of the profit flows of the producer of $\kappa_j + 1$, which within every period is $\pi_{j\kappa_j+1} = \frac{1-\alpha}{\alpha} x_{j\kappa_j+1}$, until its position is overtaken by the discovery of the next quality of the same type of capital good. The latter depends on the probability of the discovery of the next higher quality, $p_{j\kappa_j+1}$. Therefore, the successful innovator of the κ_j+1 quality has $V_{j\kappa_j+1}$ that satisfies in equilibrium:

$$
\begin{aligned}
rV_{j\kappa_j+1} &= \pi_{j\kappa_j+1} - p_{j\kappa_j+1}V_{j\kappa_j+1} \\
&\Longrightarrow\ V_{j\kappa_j+1} = \frac{\pi_{j\kappa_j+1}}{r + p_{j\kappa_{j+1}}}
\end{aligned} \tag{135}
$$

The last condition is essentially an arbitrage condition. The entrepreneur should be indifferent between lending $V_{j\kappa_j+1}$ units of output and earning the market interest rate, i.e. $rV_{j\kappa_j+1}$ and holding the firm that provides him with a profit flow, $\pi_{j\kappa_j+1}$, and there is a probability that at the end of the period he loses the value of its firm because it is overtaken by a new higher quality product (i.e., its capital loss will be $-V_{j\kappa_j+1}$).

Therefore, given that there is free entry into capital goods sector in equilibrium

$$
p_{j\kappa_j}V_{j\kappa_j+1} = Z_{j\kappa_j}. \tag{136}
$$

From this condition and the R&D production function (132) together with (135) it follows that

$$
p_{j\kappa_j+1} = \phi(\kappa_j)\pi_{j\kappa_j+1} - r
$$

Therefore, the equilibrium probability of a new innovation is influenced by two forces

1. A higher quality of a type of capital variety implies higher demand for this type of capital and thereby higher profits for the successful innovator.

2. The marginal cost of discovering a higher quality capital variety increases.

When the positive effect dominates, newer sectors grow faster than older ones, i.e. there are increasing returns to scale. When the negative effect dominates there is convergence to à la Ramsey. When the two effects balance each other out (i.e. when there are CRS), there is balanced growth in all sectors.

Balanced growth path

All variables of the model need to grow at constant rates (BGP). A specification for $\phi(\kappa_j)$ that ensures balanced growth is the following

$$\phi(\kappa_j) = \frac{1}{\zeta} q^{-(\kappa_j+1)\frac{\alpha}{1-\alpha}}.$$

This implies that the free-entry condition boils down to

$$\frac{1}{\zeta}\frac{1-\alpha}{\alpha} L A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} = r + p_{j\kappa_j+1} = r + p.$$

Note that this specification was chosen in order to eliminate the asymmetry across the different sectors of the economy, given that it is sufficient for a constant probability of new innovations taking place across all sectors. The equilibrium R&D investment is

$$Z_{j\kappa_j} = \frac{p}{\frac{1}{\zeta} q^{-(\kappa_j+1)\frac{\alpha}{1-\alpha}}.} = q^{(\kappa_j+1)\frac{\alpha}{1-\alpha}} \left(\frac{1-\alpha}{\alpha} L A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} - \zeta r \right).$$

Note that there are scale effects in this model, as larger sectors spend more on R&D.

Define the aggregate quality index as

$$Q \equiv \sum_{j=1}^{N} q^{\kappa_j \frac{\alpha}{1-\alpha}}.$$

Then, the aggregate output and quantity of capital goods are given by

$$\begin{aligned} Y &= L A^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} Q, \\ X &= \sum_{j=1}^{N} X_{j\kappa_j} = L A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} Q. \end{aligned}$$

Aggregate R&D expenditures are

$$Z = \sum_{j=1}^{N} Z_{j\kappa_j} = \left(\frac{1-\alpha}{\alpha} L A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} - r\zeta \right) q^{\frac{\alpha}{1-\alpha}} Q.$$

Aggregate output, capital and R&D expenditures are proportional to the quality index Q, which is itself function of time. Therefore, in steady-state

$$g_Y = g_Z = g_X = g_Q \equiv g.$$

The resource constraint of this economy is

$$Y = C + X + Z,$$

implying that in the steady-state it is also true that

$$g_C = g.$$

On average in the economy in all different capital goods "industries", at every point in time, an innovation of a higher quality capital good takes place with probability p. Therefore, for a large N the Law of Large Numbers provides with the growth rate

$$g^* \approx E\left(\frac{\Delta Q}{Q}\right) = \frac{\sum_{j=1}^{N} p\left(q^{(\kappa_j+1)\frac{\alpha}{1-\alpha}} - q^{\kappa_j\frac{\alpha}{1-\alpha}}\right)}{\sum_{j=1}^{N} q^{\kappa_j\frac{\alpha}{1-\alpha}}} = p\left(q^{\frac{\alpha}{1-\alpha}} - 1\right)$$

This growth rate is implied from the production-side of the economy. It is negatively related to r since $p = \frac{1}{\zeta}\frac{1-\alpha}{\alpha}LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} - r$. On the other hand, the standard optimization condition for the representative household gives a positive relation between the growth rate of the economy and the real return on assets, $g = \frac{1}{\theta}(r-\rho)$. Therefore, the equilibrium interest rate and growth rate are

$$\begin{aligned} r^* &= \theta p\left(q^{\frac{\alpha}{1-\alpha}} - 1\right) + \rho \\ &= \frac{\theta\left(q^{\frac{\alpha}{1-\alpha}} - 1\right)\frac{1}{\zeta}\frac{1-\alpha}{\alpha}LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} + \rho}{1+\theta\left(q^{\frac{\alpha}{1-\alpha}} - 1\right)} \quad (137) \\ g^* &= \frac{1}{\theta}(r^*-\rho) \quad (138) \end{aligned}$$

TVC is satisfied for parameters that ensure $r > g$.

Further comments

The decentralized equilibrium in this model does not achieve first best allocations due to the following distortions

1. Monopoly pricing in the capital goods sector,
2. Obsolescence of the lower quality capital goods.

The first distortion implies that there is low investment and growth in the economy. The second implies that when a new innovation is made, then the successful innovator takes over the profits made by the previous leader in the particular type of capital variety. This "rat race" effect boosts the incentive to conduct R&D for the purpose of product innovation. At the same time though, for the same potential innovator there is positive probability that himself will be overtaken by the next discovery, which reduces the incentives for R&D. Because of discounting, as current profits matter more than future ones, the outcome is that in equilibrium there is more than optimal R&D investment, which would tend to increase growth. The net effect of the two distortions is ambiguous.

The decentralized growth rate will be equal to the social planner's one when relative prices are corrected and the successful innovators compensate their immediate predecessors for the loss of their monopoly rents. In the case that all innovations are conducted by the leaders in each sector, i.e. when there is no need for compensation, equilibrium R&D would be too low. If instead relative prices are corrected, but no compensation is offered, then R&D would be too high.

Scale effects

Scale effects can be eliminated assuming that $\phi(\kappa_j)$ is inversely proportional to L, e.g.,

$$\phi(\kappa_j) = \frac{1}{\zeta L} q^{-(\kappa_j+1)\frac{\alpha}{1-\alpha}}.$$

In such a case L will be eliminated in p. (Young, 1998; Dinopoulos and Thompson, 1998) eliminate scale effects in a similar way: they assume that higher L some way or another reduces the effect of research expenditures Z on the probability of success p.

Innovation by market leader

Thus far we have (reasonably) supposed that R&D is carried only by outsiders. Suppose now that the sector leader/firm can also carry R&D. Let $Z^o_{j\kappa_j}$ and $Z^l_{j\kappa_j}$ be the total R&D expenditures of outsiders and the leader, respectfully, and $Z_{j\kappa_j} = Z^o_{j\kappa_j} + Z^l_{j\kappa_j}$. Suppose for now that leaders and outsiders are equally productive/likely to produce innovation, therefore,

$$\begin{aligned} p^o_{j\kappa_j} &= Z^o_{j\kappa_j}\phi(\kappa_j), \\ p^l_{j\kappa_j} &= Z^l_{j\kappa_j}\phi(\kappa_j). \end{aligned}$$

Suppose that innovation by leaders and outsiders are independent. Therefore, the total probability of introduction of new product is

$$p_{j\kappa_j} = p^o_{j\kappa_j} + p^l_{j\kappa_j}.$$

This implies that the net benefit of outsiders from R&D is

$$p^o_{j\kappa_j} V_{j\kappa_j+1} - Z^o_{j\kappa_j} = Z^o_{j\kappa_j}\left\{\phi(\kappa_j) V_{j\kappa_j+1} - 1\right\}.$$

In turn, the net benefit of leaders is

$$p^l_{j\kappa_j} V_{j\kappa_j+1} - Z^l_{j\kappa_j} - p_{j\kappa_j} V_{j\kappa_j} = Z^l_{j\kappa_j} \left\{ \phi\left(\kappa_j\right) V_{j\kappa_j+1} - 1 \right\} - p_{j\kappa_j} V_{j\kappa_j}.$$

If $Z^o_{j\kappa_j} > 0$, i.e., outsiders carry research, then from the free entry condition it follows that $\phi\left(\kappa_j\right) E\left[V_{j\kappa_j+1}\right] - 1 = 0$. This implies that the net benefit of carrying research for the leader is negative. Therefore, the leader has no incentives to innovate. (This is because of product replacement effect: If the leader introduces new product it competes with its old product.)

This is Cournot-Nash equilibrium result where the leaders and outsiders take as given the actions (research effort) of all. However, since leaders earn profits, and therefore are comparably large, it might not seem reasonable to assume that they take the actions of outsiders as given. It seems more reasonable to make Stackelberg assumption: the leaders have a first mover advantage and internalize the effect of their R&D outlays on outsiders' R&D outlays. In such a case outsiders set $Z^o_{j\kappa_j}$ for a given $Z^l_{j\kappa_j}$ while the leader sets $Z^l_{j\kappa_j}$ taking into account the reaction functions for $Z^o_{j\kappa_j}$.

Further, for simplicity suppose that the leader has likelihood advantage over outsiders in performing R&D, i.e.,

$$\phi_o\left(\kappa_j\right) < \phi_l\left(\kappa_j\right).$$

This advantage can allow the leader to have positive net value from innovation. Moreover, since innovation from outsiders reduces net benefit of leader's R&D, sufficiently high advantage in R&D will motivate the leader to choose $Z^l_{j\kappa_j}$ so that it deters innovation by outsiders, i.e., $Z^o_{j\kappa_j} = 0$. Suppose that leader's advantage is sufficiently high so that $Z^o_{j\kappa_j} = 0$.

Denote $\phi_l\left(\right) = \phi\left(\right)$ and suppose again that

$$\begin{aligned} p_{j\kappa_j} &= \phi\left(\kappa_j\right) Z_{j\kappa_j}, \\ \phi\left(\kappa_j\right) &= \frac{1}{\zeta_l} q^{-\left(\kappa_j+1\right)\frac{\alpha}{1-\alpha}}, \end{aligned}$$

where $\frac{1}{\zeta_l}$ is R&D efficiency parameter of the leader.

The flow of monopoly profits is given by

$$\pi_{j\kappa_j} = \frac{1-\alpha}{\alpha} x_{j\kappa_j} = \left(\frac{1-\alpha}{\alpha} \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L \right) q^{\frac{\alpha}{1-\alpha}\kappa_j}.$$

In turn the value of the firm is

$$rV_{j\kappa_j} = \pi_{j\kappa_j} + p_{j\kappa_j} V_{j\kappa_{j+1}} - Z_{j\kappa_j} - p_{j\kappa_j} V_{j\kappa_j}, \tag{139}$$

where $p_{j\kappa_j} E\left[V_{j\kappa_{j+1}}\right] - Z_{j\kappa_j}$ term arises since at cost $Z_{j\kappa_j}$ the leader discovers new quality product and gains value $E\left[V_{j\kappa_{j+1}}\right]$ with probability $p_{j\kappa_j}$ but gaining $E\left[V_{j\kappa_{j+1}}\right]$ it looses $E\left[V_{j\kappa_j}\right]$.

It can be written equivalently as

$$V_{j\kappa_j} = \frac{\pi_{j\kappa_j} + p_{j\kappa_j} V_{j\kappa_{j+1}} - \frac{p_{j\kappa_j}}{\phi(\kappa_j)}}{r + p_{j\kappa_j}}.$$

In this setup the leader/monopolist in sector j decides on its own "entry" selecting $p_{j\kappa_j}$ to maximize $E\left[V_{j\kappa_j}\right]$. The fist order condition with respect to $p_{j\kappa_j}$ yields

$$V_{j\kappa_{j+1}} - V_{j\kappa_j} = \frac{1}{\phi\left(\kappa_j\right)} = \frac{Z_{j\kappa_j}}{p_{j\kappa_j}}. \tag{140}$$

Notice that this is the analogue of (136). Here, however, we have the difference between $E\left[V_{j\kappa_{j+1}}\right]$ and $E\left[V_{j\kappa_j}\right]$. This is because while innovating the leader looses its position of the previous product.

Now substitute (140) into (139) to find that

$$V_{j\kappa_j} = \frac{\pi_{j\kappa_j}}{r}.$$

Such a relation holds in this model since the leader stays in the market forever and has permanent profit flow. Therefore, the discount rate changes from $r + p_{j\kappa_j}$ to r. From this expression and (140) it follows that

$$r = \phi\left(\kappa_j\right)\left(\pi_{j\kappa_{j+1}} - \pi_{j\kappa_j}\right).$$

Using expressions for profit function and $\phi\left(\kappa_j\right)$ it follows that

$$r_l = \frac{1}{\zeta_l}\frac{1-\alpha}{\alpha}\alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L\left(1 - q^{-\frac{\alpha}{1-\alpha}}\right). \tag{141}$$

The growth rate of the economy then is given by

$$g_l = \frac{1}{\theta}\left(r_l - \rho\right). \tag{142}$$

The Social Planner's problem

The Social Planner maximizes the household's utility subject to resource constraint and production technologies

$$\begin{aligned}
&\max\left\{U = \int_0^{+\infty} \frac{C_t^{1-\theta} - 1}{1-\theta}\exp\left(-\rho t\right) dt\right\} \\
&s.t. \\
&AL^{1-\alpha}\sum_{j=1}^{N}\left(q^{\kappa_j} x_{j\kappa_j}\right)^{\alpha} = C + \sum_{j=1}^{N}\left(x_{j\kappa_j} + Z_{j\kappa_j}\right), \\
&p_{j\kappa_j} = \frac{1}{\zeta_l} q^{-\left(\kappa_j+1\right)\frac{\alpha}{1-\alpha}} Z_{j\kappa_j}.
\end{aligned}$$

Here $p_{j\kappa_j}$ is the leader's efficiency in innovation and we assume that $\frac{1}{\zeta_l}$ is no smaller than outsiders' efficiency in innovation since the Social Planner would assign R&D activity to the most efficient researcher.

To solve this problem it is convenient to figure out first static allocations. The optimal choices for x are given by

$$x_{j\kappa_j} = (\alpha A)^{\frac{1}{1-\alpha}} q^{\kappa_j \frac{\alpha}{1-\alpha}} L \; \forall j.$$

Substitution of these optimal rules into final goods production function yields

$$Y = \alpha^{\frac{\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} LQ,$$

where $Q = \sum_{j=1}^{N} q^{k_j \frac{\alpha}{1-\alpha}}$. This expression implies that the expected change of Y is given by expected change of Q,

$$\begin{aligned} E\left(\Delta Q\right) &= \sum_{j=1}^{N} p_{\kappa_j} \left[q^{(\kappa_j+1)\frac{\alpha}{1-\alpha}} - q^{\kappa_j \frac{\alpha}{1-\alpha}} \right] \\ &= \frac{Z}{\zeta_l} \left(1 - q^{-\frac{\alpha}{1-\alpha}}\right), \end{aligned}$$

Let the number of sectors be so large that $E\left(\Delta Q\right) = \dot{Q}$. Therefore, the Social Planner's dynamic problem can be written in the following manner

$$\begin{aligned} H^C &= \frac{C^{1-\theta} - 1}{1-\theta} + \lambda_Y \left(\frac{1-\alpha}{\alpha} \alpha^{\frac{1}{1-\alpha}} A^{\frac{1}{1-\alpha}} LQ - C - Z \right) \\ &\quad + \lambda_Q \frac{Z}{\zeta_l} \left(1 - q^{-\frac{\alpha}{1-\alpha}}\right) \end{aligned}$$

The optimal rules that follow from this problem are

$$\begin{aligned} [C] &: \; C^{-\theta} = \lambda_Y, \\ [Z] &: \; \lambda_Y = \lambda_Q \frac{1}{\zeta_l} \left(1 - q^{-\frac{\alpha}{1-\alpha}}\right), \\ [Q] &: \; \dot{\lambda}_Q = \lambda_Q \rho - \lambda_Y \frac{1-\alpha}{\alpha} \alpha^{\frac{1}{1-\alpha}} A^{\frac{1}{1-\alpha}} L. \end{aligned}$$

These imply that the growth rate in the economy is given by

$$g^{SP} = \frac{1}{\theta} \left[\frac{1}{\zeta_l} \frac{1-\alpha}{\alpha} \alpha^{\frac{1}{1-\alpha}} A^{\frac{1}{1-\alpha}} \left(1 - q^{-\frac{\alpha}{1-\alpha}}\right) L - \rho \right]. \tag{143}$$

In turn, the implicit rate of return is

$$r^{SP} = \frac{1}{\zeta_l} \frac{1-\alpha}{\alpha} \alpha^{\frac{1}{1-\alpha}} A^{\frac{1}{1-\alpha}} \left(1 - q^{-\frac{\alpha}{1-\alpha}}\right) L. \tag{144}$$

Comparing (144) and (141) reveals that $r^{SP} > r_l$ which implies that $g^{SP} > g_l$. This distortion stems from monopolistic pricing of the intermediate goods and can be alleviated with appropriate taxes/subsidies. In turn, if $\zeta_l = \zeta_o = \zeta$ then in decentralized equilibrium (137) and (138) prevail. The relationship between (137) and (144) - and therefore the relationship between (138) and (143) - is ambiguous

$$r^* = \frac{\theta\left(q^{\frac{\alpha}{1-\alpha}}-1\right)\frac{1}{\zeta}\frac{1-\alpha}{\alpha}\alpha^{\frac{2}{1-\alpha}}A^{\frac{1}{1-\alpha}}L+\rho}{1+\theta\left(q^{\frac{\alpha}{1-\alpha}}-1\right)}$$

$$r^{SP} = {}^{*}\frac{1}{\zeta}\frac{1-\alpha}{\alpha}\alpha^{\frac{1}{1-\alpha}}A^{\frac{1}{1-\alpha}}\left(1-q^{-\frac{\alpha}{1-\alpha}}\right)L.$$

This implies that innovation in decentralized equilibrium in this model can be either excessive or lower than in social optimum.

As already highlighted monopolistic pricing tends to reduce r. There are, however, other forces that affect interest rate in this setting. These forces stem from not-full property rights on successes in R&D. First, since part of the returns for innovating firms come from expropriation of other firms' business, r^* is too high from social point of view. Second, since each firm has only temporary benefits from innovation r^* is too low from social point of view.[13]

Contrast with Romer (1990) The horizontal expansion in the capital good varieties model may be better suited for large-scale inventions (e.g. the ones implying the establishment of a new industry). The vertical expansion of every variety of capital goods is better suited to account for the smaller and gradual improvements in the quality of the capital goods.

On the one hand, both of these endogenous growth models share key features and thereby predictions. In both models, the production-side equilibrium implies that the interest rate has a negative growth effect as it reduces the present discounted value of the expected profits from innovations. Also, both models "suffer" from scale-effects. The engine of long-run growth is the R&D production that is conducted given market-based incentives. Noteworthy, the organizational capital/institutional level of the economy, as captured by the term A in the aggregate production function, has not only a level effect (as in Solow), but a permanent growth effect.

On the other hand, there are important differences in their specifications. Romer's model lacks the feature of creative destruction as it assumes that the different varieties are not direct substitutes or complements. As a result, in the decentralized equilibrium the R&D effort cannot be too high.

[13] Notice that this second effect is discounted since the entry of competitors happens in later periods.

Technology Diffusion and Growth - Barro and Sala-i-Martin (1997)

In neoclassical growth theories we had an inference that countries can converge over time. In endogenous growth theories that we have studied, however, countries would not converge over long time horizon. This inference might not hold though in multi-country setting where there is technological diffusion.

If imitation and application of discoveries are cheaper than innovation than imitating/follower countries can catch up to the innovating/leader countries. Such a mechanism can generate convergence albeit diminishing returns to R&D do not apply.

Consider a world where technology spills over from a leading economy (call it country 1) to a follower economy (call it country 2). Let the economy of country 1 be characterized with a version of Romer (1990) model where the technology maps one to one to the number of capital goods types that have been invented in this country. Capital goods are initially used in final goods production in country 1. The owners of blueprints of capital goods are the monopolistic providers of their goods for use in country 1. In order to acquire a blueprint entrepreneurs need to invest in R&D. Assume that country 2 imitates the products that have been discovered in country 1, but does not invent any product. The imitation of these goods is costly (e.g., because of adaptation issues). This cost is similar to the R&D expenditure in country 1 except that it is lower. The entrepreneur who incurs this cost is assumed to become the monopoly provider of the capital good for use in country 2. Assume that imitators pays no fees to holders of the capital goods in country 1; hence, firms in country 1 receive no compensation for the use of their inventions in country 2.

Final goods produced in these countries are identical and can be traded over the borders. Assume that trade is balanced between these countries at every point in time (e.g., because there is no global capital market.). Thus, in effect, the countries can be treated as closed economies except for the spillovers of technology.

Main assumptions

- Country 1:

  1. R&D sector - produces "blueprints" of new varieties/types of capital goods $\dot{N}_1$. R&D production uses S amount of final goods and has a production technology

  $$\dot{N}_1 = \eta_1 S_1,$$

  where $\eta_1 > 0$ is the efficiency of the discovery process.

  2. Capital goods sector - The sector uses the blueprints and produces intermediate capital goods for the final goods production. It is characterized by monopolistic competition. There is free-entry in the market of new blueprints. Entrepreneurs compete for patent that provides them with infinite-horizon property rights on a new blueprint. The production of a unit of capital goods/varieties requires a unit of final goods.

3. Final goods sector - Final goods producers employ labor L_1 and capital goods $x_1(i), i \in [0, N_1]$ in the production

$$Y_1 = A_1 L_1^{1-\alpha} \int_0^{N_1} x_1^{\alpha}(i)\, di.$$

- Country 2:

1. R&D sector - produces "blueprints" of new varieties/types of capital goods $\dot{N}_2$. R&D production uses S amount of final goods and has a production technology

$$\dot{N}_2 = \eta_2 S_2,$$

where $\eta_2 > 0$ is the efficiency of the discovery process.

2. Imitation sector - Imitation copies blueprints from $[0, N_1]$. Therefore, it can happen if $N_2 < N_1$.[14] R&D investments are in terms of final goods. The efficiency of R&D process is $\nu_2$which negatively depends on N_2/N.[15]

   - Such a property would hold, for instance, when capital goods of country 1 varied in terms of how costly they were to adapt to country 2. The goods that were easier to imitate would then be copied first.

   Assume, further, that $\nu_2 \left[N_2(0)/N_1(0)\right] > \eta_2$ so that it is profitable to imitate initially and $\nu_2(1) = \eta_2$ [note that we implicitly assume here that $N_2(0) < N_1(0)$; let as well $N_2(0) > 0$].

3. Final goods sector - Final goods producers employ labor L_2 and capital goods $x_2(i), i \in [0, N_2]$ in the production

$$Y_2 = A_2 L_2^{1-\alpha} \int_0^{N_2} x_2^{\alpha}(i)\, di.$$

Further assumptions

- In both countries

1. All capital goods depreciate fully within one period in final goods production (i.e., capital goods x can be thought to represent capital services)
2. No growth in population, i.e., $L_1, L_2 = const$
3. From the consumption-side, the representative HH chooses its consumption and next period assets to maximize its lifetime utility $U = \int_0^{+\infty} u(C_t) \exp(-\rho t)\, dt$, subject to the standard budget constraint

[14] Notive that we assume that the N_2 goods are a subset of the N_1 goods.
[15] There is no trade in intermediate goods.

Market equilibrium

Country 1

From the problems of final goods and capital goods producers it follows that

$$
\begin{aligned}
&w_1 = (1-\alpha)\frac{Y_1}{L_1}, \\
&p_{x_1(i)} \equiv p_{x_1} = \frac{1}{\alpha}, \\
&x_1(i) \equiv x_1 = A_1^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L_1, \\
&\pi_{x_1} = \frac{1-\alpha}{\alpha} A_1^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L_1, \\
&y_1 = \frac{Y_1}{L_1} = A_1^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} N_1.
\end{aligned}
$$

In turn, from the problem of R&D firms is

$$\max_S V_{x_1}\dot{N} - S,$$

which implies that in equilibrium

$$V_{x_1} = \frac{1}{\eta_1}.$$

Therefore, from Hamilton-Jacobi-Bellman equation it follows that

$$
\begin{aligned}
r_1 &= \eta_1 \pi_{x_1} \\
&= \eta_1 \frac{1-\alpha}{\alpha} A_1^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L_1.
\end{aligned}
$$

In turn from the consumer optimization problem, budget constraint, and final goods production function it follows that all aggregate quantities (Y_1, C_1, N_1) grow at constant rate

$$g_1 = \frac{1}{\theta}\left(\eta_1 \frac{1-\alpha}{\alpha} A_1^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L_1 - \rho\right).$$

This implies that country 1 is on balanced growth path from time 0 onwards, i.e., there is no transition.

Country 2

From the problems of final goods and capital goods producers it follows that

$$\begin{aligned} w_2 &= (1-\alpha)\frac{Y_2}{L_2}, \\ p_{x2(i)} &\equiv p_{x_2} = \frac{1}{\alpha}, \\ x_2(i) &\equiv x_2 = A_2^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}L_2, \\ \pi_{x_2} &= \frac{1-\alpha}{\alpha}A_2^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}L_2, \\ y_2 &= \frac{Y_2}{L_2} = A_2^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}N_2. \end{aligned} \tag{145}$$

From R&D sector it follows that in equilibrium

$$V_{x_1} = \nu_2\left(N_2/N_1\right).$$

Therefore, from Hamilton-Jacobi-Bellman equation it follows that

$$r_2 = \nu_2\pi_{x_2} - \frac{\dot{\nu}_2}{\nu_2}.$$

From consumer optimization problem then it follows that

$$g_{C_2} = \frac{1}{\theta}\left(\nu_2\pi_{x_2} - \frac{\dot{\nu}_2}{\nu_2} - \rho\right).$$

Balanced growth path

On balanced growth path from $r = \nu_2\pi_{x_2} - \frac{\dot{\nu}_2}{\nu_2}$ and $\pi_{x_2} = \frac{1-\alpha}{\alpha}A_2^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}L_2$ it follows that ν_2 is constant and N_2 and N_1 grow at the same rate. Therefore, $r_2 = r_1$ and

$$\nu_2\pi_{x_2} = \eta_1\pi_{x_1}.$$

Using the formulas for profit functions then implies that in case when $\nu_2 > \eta_2$ on balanced growth path

$$\nu_2 = \eta_1\left(\frac{A_1}{A_2}\right)^{\frac{1}{1-\alpha}}\frac{L_1}{L_2}.$$

Since in this case it needs to be so that $\nu_2 > \eta_2$ the following inequality should hold

$$1 < \frac{\eta_1}{\eta_2}\left(\frac{A_1}{A_2}\right)^{\frac{1}{1-\alpha}}\frac{L_1}{L_2}.$$

Notice that in this case country 2 is always imitating and country 1 is always innovating (for country 1 there is no one to imitate from.) Since in this case on balanced growth path $N_2 < N_1$ country 2 always lags behind country 1 in terms of development level if

$$\frac{y_2}{y_1} = \left(\frac{A_2}{A_1}\right)^{\frac{1}{1-\alpha}} \frac{N_2}{N_1} < 1. \tag{146}$$

A sufficient condition for this is $A_2 \leq A_1$.

Transition dynamics in country 2

In equilibrium, in country 2 and Y_2 and N_2 are proportional. Denote

$$\hat{N} = \frac{N_2}{N_1},$$
$$\chi_2 = \frac{C_2}{N_2}.$$

Here the first one is state-like variable while the second is control-like variable.

Let

$$\nu_2\left(\hat{N}\right) = \eta_2 \hat{N}^{-\sigma}, \sigma > 0.$$

Therefore, on balanced growth path

$$\hat{N} = \left[\frac{\eta_1}{\eta_2}\left(\frac{A_2}{A_1}\right)^{\frac{1}{1-\alpha}} \frac{L_2}{L_1}\right]^{\frac{1}{\sigma}}.$$

The growth rate of χ_2 is

$$g_{\chi_2} = g_{C_2} - g_{N_2},$$

where

$$g_{C_2} = \frac{1}{\theta}\left(\eta_2 \frac{\pi_{x_2}}{\hat{N}^{\sigma}} + \sigma g_{\hat{N}} - \rho\right) \tag{147}$$

and g_{N_2} can be figured out from the resource constraint

$$Y_2 = C_2 + N_2 x_2 + \nu_2 \dot{N}_2.$$

Using the expressions for Y_2, x_2, and π_{x_2}

$$g_{N_2} = \eta_2 \frac{1}{\hat{N}^{\sigma}}\left(\frac{1+\alpha}{\alpha}\pi_{x_2} - \chi_2\right).$$

This implies that

$$\begin{aligned} g_{\hat{N}} &= \eta_2 \frac{1}{\hat{N}^{\sigma}} \left(\frac{1+\alpha}{\alpha} \pi_{x_2} - \chi_2 \right) - g_1, \\ g_{\chi_2} &= \left[\frac{1}{\theta} + \left(\frac{1}{\theta}\sigma - 1 \right) \frac{1+\alpha}{\alpha} \right] \eta_2 \frac{1}{\hat{N}^{\sigma}} \pi_{x_2} \\ &\quad + \left(1 - \frac{1}{\theta}\sigma \right) \eta_2 \frac{1}{\hat{N}^{\sigma}} \chi_2 - \frac{1}{\theta} \left(\rho + \sigma g_1 \right). \end{aligned}$$

This is a two-dimensional autonomous system of differential equations in $\hat{N}$ and χ_2. The Jacobian of the system is

$$J = \begin{pmatrix} \frac{\partial \frac{\partial \chi_2}{\partial t}}{\partial \chi_2} & \frac{\partial \frac{\partial \chi_2}{\partial t}}{\partial \hat{N}} \\ \frac{\partial \frac{\partial \hat{N}}{\partial t}}{\partial \chi_2} & \frac{\partial \frac{\partial \hat{N}}{\partial t}}{\partial \hat{N}} \end{pmatrix}$$

The elements of the Jacobian matrix evaluated around the steady-state are

$$\begin{aligned} \frac{\partial \frac{\partial \chi_2}{\partial t}}{\partial \chi_2} &= \left(1 - \frac{1}{\theta}\sigma \right) \eta_2 \frac{1}{\hat{N}^{\sigma}} \chi_2, \\ \frac{\partial \frac{\partial \chi_2}{\partial t}}{\partial \hat{N}} &= -\sigma \frac{1}{\theta} \left(\rho + \sigma g_1 \right) \chi_2 \hat{N}^{-1}, \\ \frac{\partial \frac{\partial \hat{N}}{\partial t}}{\partial \chi_2} &= -\eta_2 \frac{1}{\hat{N}^{\sigma}} \hat{N}, \\ \frac{\partial \frac{\partial \hat{N}}{\partial t}}{\partial \hat{N}} &= -\sigma g_1. \end{aligned}$$

Therefore, the determinant of the Jacobian evaluated around the steady-state is

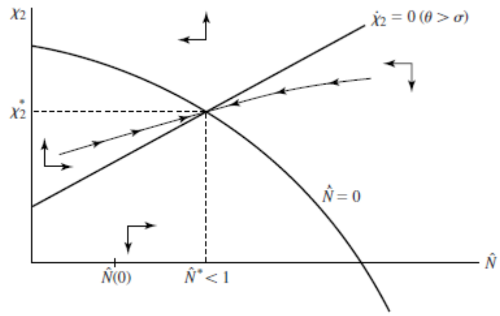
$$\det J = -\frac{1}{\eta_2 \hat{N}^{\sigma}} \chi_2 \frac{1}{\theta} \sigma \left(\theta g_1 + \rho \right) < 0.$$

Since it is negative the eigenvalues have alternating signs and the system is saddle-path stable.

The loci are

$$\begin{aligned} \left[g_{\hat{N}} = 0 \right] &: \ \chi_2^1 = \frac{1+\alpha}{\alpha} \pi_{x_2} - \frac{1}{\eta_2} g_1 \hat{N}^{\sigma}, \\ \left[g_{\chi_2} = 0 \right] &: \ \chi_2^2 = \frac{\rho + \sigma g_1}{\theta - \sigma} \frac{1}{\eta_2} \hat{N}^{\sigma} - \frac{1 - \frac{1+\alpha}{\alpha} \left(\theta - \sigma \right)}{\theta - \sigma} \pi_{x_2}. \end{aligned}$$

The first locus is downward sloping. Without loss of generality letting $\theta > \sigma$ the second locus is upward sloping. The phase diagram of this dynamic system is offered in the following figure.



Consider the first locus. Given that $\frac{\partial \frac{\partial \hat{N}}{\partial t}}{\partial \chi_2} < 0$ for a given $\hat{N}$ if χ_2 is greater/lower than χ_2^1 then $\hat{N}$ declines/increases. Given that $\frac{\partial \frac{\partial \chi_2}{\partial t}}{\partial \chi_2} > 0$ for a given $\hat{N}$ if χ_2 is greater/lower than χ_2^2 then χ_2 increases/declines.

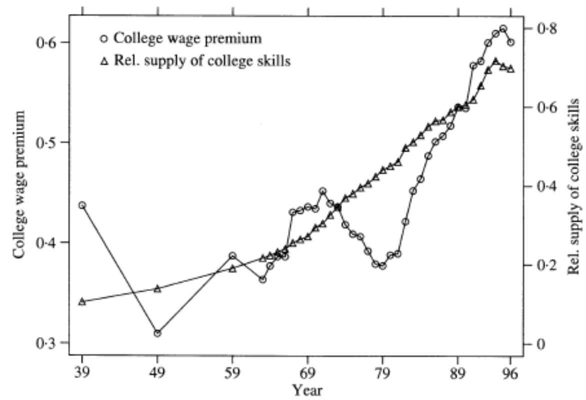
Since $\hat{N} = \frac{N_2}{N_1}$ always rises toward its steady-state value, from the expression for $g_{\hat{N}}$ it follows that $g_{\hat{N}}$ falls to 0. Therefore, during transition country 2 grows at a higher rate than country 1: imitation is in percentage terms greater than innovation. The growth rate slows down during transition since the cost of imitation rises.[16]

From (145) and (146) it follows then that the growth rate of per-capita output in country 2 declines over time to g_1. Therefore, there is a convergence pattern in which the follower country's output per-capita grows faster than the leader's, but the differential in the growth rates diminishes the more the follower catches up. The follower country never catches up exactly, however.

[16] Increase in the cost of imitation represents a type of declining returns to imitation. This is similar to the diminishing returns to capital accumulation in the neoclassical growth models.

Directed Technical Change: Acemoglu (2002) - horizontal product innovation

The following plot, taken from Acemoglu (2002), shows that in the US in spite of increasing supply of college graduates (high-skills) relative to non-graduates (low-skill) wage premium of college graduates (relative wage of college graduates) has increased.



Relative Supply of College Skills and College Premium

FIGURE 1

The behaviour of the (log) college premium and relative supply of college skills (weeks worked by college equivalents divided by weeks worked by noncollege equivalents) between 1939 and 1996. Data from March CPSs and 1940, 1950 and 1960 censuses

This is puzzling since conventional theory, with decreasing demand function, implies that increasing supply of college graduates should have reduced wage premium. Acemoglu in Acemoglu (2002), amongst other things, provides theoretical explanation behind such a relationship. Explanation is based on induced and directed/biased technical change which can generate increasing long-run demand curve.

Definition 2 *Consider a production function $F(L, Z; A)$, where L and Z are inputs and A measures technology (let $\partial F/\partial A > 0$). Technological change is L-biased if*

$$\frac{\partial \frac{\partial F/\partial L}{\partial F/\partial Z}}{\partial A} > 0.$$

This means that such a technological change increases marginal product of L more than the marginal product of Z. Analogously is defined Z-biased technological change. Notice that biased technical change is distinct from factor augmenting technological change. For example, technology is L-augmenting if $F(L, Z; A)$

To see the difference between these two notions take a special CES case of production function

$$Y = F(L, Z; A_L, A_Z) = \left[\gamma (A_L L)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)(A_Z Z)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where $\gamma \in (0, 1)$ and $\varepsilon \geq 0$. Clearly, when (1) $\varepsilon = 0$ this function attains Leontief form, (2) $\varepsilon = 1$ this function attains Cobb-Douglas form, and (3) $\varepsilon = +\infty$ this function is linear in L and Z. In the latter case L and Z are perfect substitutes. In turn, when $\varepsilon > 1$ call L and Z gross-substitutes since in such a case increasing the price of one of the factors everything else held fixed increases the demand for the other factor. In turn, when $\varepsilon \in (0,1)$ call L and Z gross-complements since in such a case increasing the price of one of the factors everything else held fixed reduces the demand for the other factor.

In this production function A_L is L-augmenting and A_Z is Z-augmenting. However, whether technology (or technological change) is biased depends on the elasticity of substitution between factor inputs.

$$\frac{\partial Y/\partial Z}{\partial Y/\partial L} = \frac{1-\gamma}{\gamma}\left(\frac{A_Z}{A_L}\right)^{\frac{\varepsilon-1}{\varepsilon}}\left(\frac{Z}{L}\right)^{-\frac{1}{\varepsilon}},$$

For example,

$$\frac{\partial\frac{\partial Y/\partial Z}{\partial Y/\partial L}}{\partial A_Z} = \frac{\varepsilon-1}{\varepsilon}\frac{1-\gamma}{\gamma}\left(\frac{A_Z}{A_L}\right)^{\frac{\varepsilon-1}{\varepsilon}}\left(\frac{Z}{L}\right)^{-\frac{1}{\varepsilon}}\frac{1}{A_Z}.$$

Therefore, if $\varepsilon < 1$ then technological change in terms of A_Z is L-biased in the sense that it increases the marginal product of L more than marginal product of Z. In contrast, if $\varepsilon > 1$ it is Z-biased in the sense that it increases the marginal product of Z more than marginal product of L. The reason why in case $\varepsilon < 1$ technological change in terms of A_Z is L-biased is quite intuitive. First, consider Cobb-Douglas case where factor inputs have to have equal shares and have elasticity of substitution of 1. In such a case, increases in A_Z raise the demand for both inputs in equal amounts. Meanwhile, in case when $\varepsilon < 1$ labor L and Z are gross-complements and increases in A_Z raise the demand for L more than for Z. Whereas, in case when $\varepsilon > 1$ labor L and Z are gross-substitutes and increases in A_Z raise the demand for Z more than for L.

Notice that for given technology levels relative marginal product declines with relative factor input, e.g.,

$$\frac{\partial\frac{\partial Y/\partial Z}{\partial Y/\partial L}}{\partial Z/L} < 0.$$

This is the usual *substitution effect*, which leads to downward sloping (relative) demand curve. However, if changes in $\frac{Z}{L}$ induce positively correlated changes in $\frac{A_Z}{A_L}$ and c, i.e.,

$$\frac{\partial\frac{\partial Y/\partial Z}{\partial Y/\partial L}}{\partial Z/L} = \frac{\partial Y/\partial Z}{\partial Y/\partial L}\left[\frac{\varepsilon-1}{\varepsilon}\left(\frac{A_Z}{A_L}\right)^{-1}\frac{\partial A_Z/A_L}{\partial Z/L} - \frac{1}{\varepsilon}\left(\frac{Z}{L}\right)^{-1}\right]$$

where $\frac{\partial A_Z/A_L}{\partial Z/L} > 0$ (or if $\varepsilon < 1$ and $\frac{\partial A_Z/A_L}{\partial Z/L} < 0$), then the (relative) demand curve would be at least flatter. Moreover, if the first term in the square brackets is larger than the second term then the (relative) demand curve would be upward sloping.

In the model presented below $\frac{\partial A_Z/A_L}{\partial Z/L} > 0$ holds when $\varepsilon > 1$ and $\frac{\partial A_Z/A_L}{\partial Z/L} < 0$ holds when $.\varepsilon < 1$.

Main assumptions

- A multi-sector model of R&D-driven growth with horizontal innovations.

- Two factors of production: labor L and Z, which can be high-skill labor.
- Four types of intermediate goods: Y_L, Y_Z, x_L and x_Z.
- Intermediate goods Y_L and Y_Z have production functions

$$Y_L = \frac{1}{1-\beta}\left(\int_0^{A_L} x_L(j)^{1-\beta}\,dj\right)L^{\beta}, \tag{148}$$
$$Y_Z = \frac{1}{1-\beta}\left(\int_0^{A_Z} x_Z(j)^{1-\beta}\,dj\right)Z^{\beta}, \tag{149}$$

  where $\beta \in (0,1)$, and A_L and A_Z are the number of different types of x_L and x_Z intermediate goods.

- Each type of intermediate good x_s ($s = L, Z$) is supplied by a monopolist, which sets price χ_s and has marginal cost ψ in terms of final goods.
- Final goods are produced using intermediate goods Y_L and Y_Z with a CES production function

$$Y = \left[\gamma Y_L^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)\,Y_Z^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}. \tag{150}$$

  The price of final goods is normalized to 1.

- The R&D sector produces "blueprints" for new types of intermediate goods x_L and x_Z. The input to the R&D production is investment in units of the final output. There is free entry into R&D process

$$\dot{A}_L = \eta_L R_L, \tag{151}$$
$$\dot{A}_Z = \eta_Z R_Z. \tag{152}$$

- The discovery of a types of intermediate good provides an entrepreneur with monopoly right.
- From the consumption-side, the representative HH own and supplies L and Z inelastically and chooses its consumption and assets to maximize its life-time utility: $U = \int_0^{+\infty} \frac{C_t^{1-\theta}-1}{1-\theta}\exp(-\rho t)\,dt$ subject to the standard budget constraint.

Further assumptions

- All intermediate goods depreciate fully within one period.
- No population growth.

Market equilibrium

Final goods sector

Final goods producers maximize their profits taking the prices of their inputs (p_L and p_Z) as given.

$$\max_{Y_l, Y_h} Y - p_L Y_L - p_Z Y_Z,$$
$$s.t.$$
$$(150)$$

Standard first order conditions imply that

$$[Y_Z] \quad : \quad p_Z = \frac{Y}{Y_Z} \frac{(1-\gamma) Y_Z^{\frac{\varepsilon-1}{\varepsilon}}}{\gamma Y_L^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) Y_Z^{\frac{\varepsilon-1}{\varepsilon}}}, \tag{153}$$

$$[Y_L] \quad : \quad p_L = \frac{Y}{Y_L} \frac{\gamma Y_L^{\frac{\varepsilon-1}{\varepsilon}}}{\gamma Y_L^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) Y_Z^{\frac{\varepsilon-1}{\varepsilon}}}. \tag{154}$$

Denote the ratio of p_Z and p_L by p,

$$p = \frac{p_Z}{p_L} = \frac{1-\gamma}{\gamma} \left(\frac{Y_Z}{Y_L}\right)^{-\frac{1}{\varepsilon}}. \tag{155}$$

Clearly, greater is $\frac{Y_Z}{Y_L}$ lower is the relative price p, where *ceteris paribus* $\frac{Y_Z}{Y_L}$ increases with $\frac{Z}{L}$.

Since the price of final goods is normalized to 1 from (153), (154), (155), and (150) it follows that

$$1 = \left[\gamma^{\varepsilon} p_L^{-(\varepsilon-1)} + (1-\gamma)^{\varepsilon} p_Z^{-(\varepsilon-1)}\right]^{\frac{1}{\varepsilon-1}}. \tag{156}$$ [17]

Y_Z and Y_L intermediate goods sectors

Intermediate goods producers maximize profits taking prices of their inputs (w_Z, w_L, $\chi_Z(j)$, and $\chi_L(j)$) as given. Producers of Y_L intermediate goods solve the following problem

$$\max_{L, x_L(j)} p_L Y_L - w_L L - \int_0^{A_L} \chi_L(j)\, x_L(j)\, dj,$$
$$s.t.$$
$$(148)$$

[17] One way to get this expression is as follows: (1) divide (150) to Y_L and use (154) and (155) to express Y and Y_L ratio in terms of prices, (2) use (155) to express Y_Z and Y_L ratio in terms of prices.

Standard first order conditions imply that

$$\begin{aligned} [L] &: \quad w_L = p_L \beta \frac{Y_L}{L}, && (157) \\ [x_L(j)] &: \quad \chi_L(j) = p_L x_L(j)^{-\beta} L^{\beta}. && (158) \end{aligned}$$

The problem of the producers of Y_Z intermediate goods is similar and resulting optimal rules are

$$\begin{aligned} [Z] &: \quad w_Z = p_Z \beta \frac{Y_Z}{Z}, && (159) \\ [x_Z(j)] &: \quad \chi_Z(j) = p_Z x_Z(j)^{-\beta} Z^{\beta}. && (160) \end{aligned}$$

Notice that the demands for intermediate goods x increase with factor inputs L and Z, increase with prices p_L and p_Z, and decline with $\chi_L(j)$ and $\chi_Z(j)$. The demands increase with factor inputs since higher amount of factor inputs implies "more workers to use x." Higher prices of the produce, meanwhile, imply higher marginal product of all inputs, therefore, increase the demand for x. The negative relationship between x and χ represents standard downward sloping demand curve.

x_Z and x_L intermediate goods sectors

The production of a unit of an intermediate good $x_s(j)$ $(S = L, Z)$ requires ψ units of final goods. These intermediate goods producers maximize profits internalizing demand functions. To simplify the notation assume that $\psi = 1 - \beta$.

The producers of $x_L(j)$ solves the following problem

$$\begin{aligned} &\max_{x_L(j)} \pi_L(j) = \chi_L(j) x_L(j) - \psi x_L(j) \\ &s.t. \\ &(158). \end{aligned}$$

First order condition gives then

$$\chi_L(j) = 1, \qquad (161)$$

which implies that $\chi_L(j) \equiv \chi_L$. The problem of the producer of $x_Z(j)$ is similar and gives

$$\chi_Z(j) \equiv \chi_Z = 1. \qquad (162)$$

These expressions imply that $x_L(j) \equiv x_L$ and $x_Z(j) \equiv x_Z$.

Therefore, from (161), (162), (158), and (160) it follows that the profits of firms that produce intermediate goods x are

$$\begin{aligned} \pi_L &= \beta p_L^{\frac{1}{\beta}} L, && (163) \\ \pi_Z &= \beta p_Z^{\frac{1}{\beta}} Z. && (164) \end{aligned}$$

In turn, the values of these firms are given by standard conditions:

$$\begin{aligned} rV_L &= \pi_L + \dot{V}_L, && (165) \\ rV_Z &= \pi_Z + \dot{V}_Z. && (166) \end{aligned}$$

At least on balanced growth path the value of the firms should not appreciate and r is constant. Therefore, values of these firms are simply proportional to profits. This implies that, for example, V_L and incentive to innovate in L-intensive sector grows with p_L and L. This is because higher p_L implies higher price of product and higher L implies more workers that can use the product. The first one is *price effect* and the second one is *market size effect*.

Digression

To gain more intuition and delve into these effects consider the case when $\dot{V}_L = \dot{V}_Z = 0$ and use (155), (148), (149), (158), (160), (161), and (162) to obtain

$$p = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon\beta}{\sigma}} \left(\frac{A_Z Z}{A_L L}\right)^{-\frac{\beta}{\sigma}}, \tag{167}$$

where $\sigma = \varepsilon - (\varepsilon - 1)(1-\beta)$ measures the substitutability between factor inputs Z and L. Notice that $\sigma > 1 \Leftrightarrow \varepsilon > 1$ (and $\sigma < 1 \Leftrightarrow \varepsilon < 1$). This means that these factor inputs are gross-substitutes (complements) if and only if intermediate goods Y_L and Y_Z are gross-substitutes (complements).

Further, use (165), (166) to get that

$$\frac{V_Z}{V_L} = p^{\frac{1}{\beta}} \frac{Z}{L} = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}} \left(\frac{A_Z}{A_L}\right)^{-\frac{1}{\sigma}} \left(\frac{Z}{L}\right)^{\frac{\sigma-1}{\sigma}}. \tag{168}$$

In this expression $p^{\frac{1}{\beta}}$ illustrates price effect and $\frac{Z}{L}$ illustrates market size effect. Clearly, when $\sigma > 1$ (or equivalently $\varepsilon > 1$) market size effect on relative values dominates price effect. In turn, when $\sigma < 1$ (or equivalently $\varepsilon < 1$) price effect on relative values dominates market size effect.

Regarding relative demand for factors, from (155), (148), (149), (157), (159), (161), and (162) it follows that

$$\begin{aligned} w_L &= p_L \beta \frac{1}{1-\beta} A_L \left(p_L^{\frac{1}{\beta}} L\right)^{1-\beta} L^{\beta-1}, \\ w_Z &= p_Z \beta \frac{1}{1-\beta} A_Z \left(p_Z^{\frac{1}{\beta}} Z\right)^{1-\beta} Z^{\beta-1}, \\ \frac{w_Z}{w_L} &= p^{\frac{1}{\beta}} \frac{A_Z}{A_L} = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}} \left(\frac{A_Z}{A_L}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{Z}{L}\right)^{-\frac{1}{\sigma}}. && (169) \end{aligned}$$

For a given $\frac{A_Z}{A_L}$ relative demand for Z declines with $\frac{Z}{L}$. This is the usual substitution effect: increasing Z relative to L induces substitution of Z for L and reduces relative marginal product of Z. Notice that $\frac{w_Z}{w_L}$ increases with $\frac{A_Z}{A_L}$ if $\sigma > 1$ (i.e., if $\sigma > 1$ increases in $\frac{A_Z}{A_L}$ are Z-biased) and declines with it if $\sigma < 1$ (i.e., if $\sigma < 1$ increases in $\frac{A_Z}{A_L}$ are L-biased)

When $\frac{V_Z}{V_L}$ is constant in equilibrium (which will hold because of arbitrage as discussed below), increasing $\frac{Z}{L}$ induces increase in $\frac{A_Z}{A_L}$ in case when $\sigma > 1$ and induces decline in $\frac{A_Z}{A_L}$ in case when $\sigma < 1$ - this illustrates induced technological change. Because of this effect, increase in $\frac{Z}{L}$ has two effects on the relative demand. It reduces relative demand for a given technological level - which is clearly substitution effect. However, it also increases $\left(\frac{A_Z}{A_L}\right)^{\frac{-1}{\sigma}}$ for any value of σ and, therefore, relative demand curve is flatter when technology responds to changes in $\frac{Z}{L}$ - this illustrates induced-biased technological change.

R&D processes

Free entry into R&D implies that marginal gain and cost are equal. Therefore,

$$V_L = \frac{1}{\eta_L}, \tag{170}$$
$$V_Z = \frac{1}{\eta_Z}. \tag{171}$$

These expressions imply that $\dot{V}_L = \dot{V}_Z = 0$ and

$$\eta_L \pi_L = \eta_Z \pi_Z \tag{172}$$

which indicates that in equilibrium investments in x_L or x_Z should be equally profitable.

The household side

The representative household solves the following problem

$$\max_C U = \int_0^{+\infty} \frac{C_t^{1-\theta} - 1}{1-\theta} \exp(-\rho t)\, dt$$
$$s.t.$$
$$\dot{K} = rK + w_L L + w_Z Z - C \tag{173}$$

The resulting optimal rule is the standard Euler equation

$$\frac{\dot{C}}{C} = \frac{1}{\theta}(r - \rho). \tag{174}$$

Balanced growth path

Denoting $\eta = \frac{\eta_Z}{\eta_L}$ and using the ratio of values in (168) it follows that in equilibrium

$$\frac{A_Z}{A_L} = \eta^\sigma \left(\frac{1-\gamma}{\gamma}\right)^\varepsilon \left(\frac{Z}{L}\right)^{\sigma-1}. \tag{175}$$

From this expression and (175) in turn it follows that

$$\frac{w_Z}{w_L} = \eta^{\sigma-1}\left(\frac{1-\gamma}{\gamma}\right)^{\varepsilon}\left(\frac{Z}{L}\right)^{\sigma-2}. \tag{176}$$

Since it is always the case that $\sigma - 2 > -\frac{1}{\sigma}$ [or equivalently $(\sigma-1)^2 > 0$] relative demand with induced technological change (176) is flatter than relative demand when technology levels are given (169). Moreover, when $\sigma > 2$ this relative demand curve is upward sloping.

From, (173) and (174) it follows that the growth rate of final output is

$$g = \frac{1}{\theta}(r-\rho).$$

Using (165) and (170) gives an expression for r,

$$r = \eta_L \beta p_L^{\frac{1}{\beta}} L.$$

In turn, from (156), (167), and (175) an expression for p_L,

$$\begin{aligned} 1 &= \gamma^{\varepsilon} p_L^{-(\varepsilon-1)} + (1-\gamma)^{\varepsilon} p_Z^{-(\varepsilon-1)}, \\ p_Z &= p_L \eta^{-\beta}\left(\frac{Z}{L}\right)^{-\beta}. \end{aligned}$$

$$p_L = \left[\gamma^{\varepsilon} + (1-\gamma)^{\varepsilon}\left(\eta^{-\beta}\left(\frac{Z}{L}\right)^{-\beta}\right)^{-(\varepsilon-1)}\right]^{\frac{1}{\varepsilon-1}}. \tag{177}$$

Therefore, the growth rate of final output is

$$g = \frac{1}{\theta}\left\{\beta\left[\gamma^{\varepsilon}\eta_L^{(\varepsilon-1)\beta}L^{(\varepsilon-1)\beta} + (1-\gamma)^{\varepsilon}\eta_Z^{(\varepsilon-1)\beta}Z^{(\varepsilon-1)\beta}\right]^{\frac{1}{\varepsilon-1}\frac{1}{\beta}} - \rho\right\}.$$

State dependent R&D processes

Notice that the relative productivity of R&D processes η shows up in relative values of performing different types of R&D, in (176) and in the rate of growth of the economy. Therefore, it ends up being important for our discussion. In turn, the ratio of values of performing different types of R&D does not depend on A_Z and A_L since in this (lab-equipment) specification of R&D both processes of R&D depend on Y symmetrically and the ratio of marginal products is

$$\frac{\partial A_Z}{\partial R_Z} / \frac{\partial A_L}{\partial R_L} = \eta,$$

Y_Z and Y_L enter symmetrically into final goods production and are linear in A_Z and A_L.

In case, however, there are fixed resources employed in R&D and there are externalities in R&D as in Romer (1990) the inference might depend also on other factors. For example, let S_Z and S_L

be research labor employed in R&D for Z and L respectively and R&D processed given by

$$\dot{A}_L = \eta_L A_L^{\frac{1+\delta}{2}} A_Z^{\frac{1-\delta}{2}} S_L, \qquad (178)$$

$$\dot{A}_Z = \eta_Z A_L^{\frac{1-\delta}{2}} A_Z^{\frac{1+\delta}{2}} S_Z \qquad (179)$$

where $\delta \in [0,1)$ measures the degree of within-sector spillovers - or state dependence of the R&D process. The supply of research labor is constant

$$S = S_L + S_Z.$$

Clearly, in this case the relative marginal product of inputs in R&D processes is

$$\frac{\partial A_Z}{\partial S_Z} / \frac{\partial A_L}{\partial S_L} = \eta \left(\frac{A_Z}{A_L} \right)^{\delta}.$$

This expression does not depend on $\frac{A_Z}{A_L}$ if $\delta = 0$ and is linear in $\frac{A_Z}{A_L}$ if $\delta = 1$.[18] Call the former no state dependence and the latter extreme state dependence.

This implies that

$$V_L \eta_L A_L = V_Z \eta_Z A_Z.$$

On balanced growth path $\dot{V}_L = \dot{V}_Z = 0$ and this expression simplifies to

$$\pi_L \eta_L A_L^{\delta} = \pi_Z \eta_Z A_Z^{\delta}, \qquad (180)$$

which is analogous to (172). Combining this expression with (163), (164), and (155) gives the analogue of (175)

$$\frac{A_Z}{A_L} = \eta^{\frac{\sigma}{1-\sigma\delta}} \left(\frac{1-\gamma}{\gamma} \right)^{\frac{\varepsilon}{1-\sigma\delta}} \left(\frac{Z}{L} \right)^{\frac{\sigma-1}{1-\sigma\delta}}. \qquad (181)$$

Parameter δ appears in this expression since it governs how changes in A_L and A_Z affect productivities in R&D and therefore incentives to innovate. Clearly, when $\delta = 0$ (181) and (175) are equivalent. Meanwhile, when $\delta > 0$ the reaction of $\frac{A_Z}{A_L}$ to changes in $\frac{Z}{L}$ is amplified since changes in $\frac{Z}{L}$ lead to changes in $\frac{A_Z}{A_L}$ which change incentives to innovate and alter $\frac{A_Z}{A_L}$ again (e.g., $\sigma > 0$ and $\frac{Z}{L} \uparrow \Rightarrow \frac{A_Z}{A_L} \uparrow$ but since $\frac{A_Z}{A_L} \uparrow$ productivity of engaging in Z-complementary innovation increases relative to A_L, therefore, more effort is directed there -i.e., in free-entry equilibrium π_Z falls relative to π_L - and $\frac{A_Z}{A_L} \uparrow$ more).[19]

Substituting (181) into (169) gives

$$\frac{w_Z}{w_L} = \eta^{\frac{\sigma-1}{1-\sigma\delta}} \left(\frac{1-\gamma}{\gamma} \right)^{\frac{\varepsilon(1-\delta)}{1-\sigma\delta}} \left(\frac{Z}{L} \right)^{\frac{\sigma-2+\delta}{1-\sigma\delta}}.$$

[18] Notice that the specification of R&D process is selected to generate balanced growth path.

[19] Notice that in the limiting case when $\delta = 1$ the relationship between $\frac{A_Z}{A_L}$ and $\frac{Z}{L}$ does not depend on σ and $\frac{A_Z}{A_L}$ declines with $\frac{Z}{L}$.

It can be shown that this model displays stable dynamics when $\sigma < \frac{1}{\delta}$. In case when $\delta = 1$ clearly $\sigma > \frac{1}{\delta}$. Moreover, $\frac{w_Z}{w_L}$ increases with $\frac{Z}{L}$ when $\sigma > 2 - \delta$. Such a condition, together with $\sigma < \frac{1}{\delta}$, can be satisfied for δ-s from $[0, 1)$.

Further applications and discussion

The model presented above can be applied to explain persistent income gap between developed and less-developed countries (e.g., OECD countries vs. Armenia, Georgia, etc). In few words, let Z represent high-skill labor and L represent low-skill labor. Less developed countries usually use/adopt technologies created in developed countries and have lower Z/L ratio. In such a case, since these technologies are created for Z/L ratio in developed countries they are less appropriate for production in less-developed countries, although it might be still optimal to use/adopt those technologies.

For simplicity, consider the setup without state-dependence (i.e., $\delta = 0$) and two countries called "North" - developed country with Z/L - and "South" - less-developed country with Z'/L', and let

$$\frac{Z}{L} > \frac{Z'}{L'}.$$

Assume that South does not engage in R&D and can costlessly copy technologies invented in North. However, firms producing intermediate goods x in South have marginal cost $(1-\beta)\,\kappa^{\frac{-\beta}{1-\beta}}$. It seems natural to think that $\kappa < 1$ so that production of machines in South is costlier than in North.

This implies that the price of intermediate goods x in South is $\kappa^{\frac{-\beta}{1-\beta}}$. Therefore, from (158), (160), (148) and (149) it follows that

$$\frac{Y'}{Y} = \frac{\kappa\left\{\gamma\left[A_L L'\left(p'_L\right)^{\frac{1-\beta}{\beta}}\right]^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)\left[A_Z Z'\left(p'_Z\right)^{\frac{1-\beta}{\beta}}\right]^{\frac{\varepsilon-1}{\varepsilon}}\right\}^{\frac{\varepsilon}{\varepsilon-1}}}{\left[\gamma\left(A_L L p_L^{\frac{1-\beta}{\beta}}\right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)\left(A_Z Z p_Z^{\frac{1-\beta}{\beta}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}}.$$

In North, prices can be expressed in terms of parameters using (167) and (177) treating $\frac{A_Z}{A'_L}$ as exogenous. In turn, in South prices can be expressed using

$$\begin{aligned} p'_L &= \left[\gamma^{\varepsilon} + (1-\gamma)^{\varepsilon}\left(\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon\beta}{\sigma}}\left(\frac{A_Z}{A_L}\frac{Z'}{L'}\right)^{-\frac{\beta}{\sigma}}\right)^{-(\varepsilon-1)}\right]^{\frac{1}{\varepsilon-1}} \\ p' &= \frac{p'_Z}{p'_L} = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon\beta}{\sigma}}\left(\frac{A_Z}{A_L}\frac{Z'}{L'}\right)^{-\frac{\beta}{\sigma}}. \end{aligned}$$

Therefore,

$$\frac{Y'}{Y} = \kappa \left[\frac{1+\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}} \left(\frac{A_Z}{A_L}\frac{Z'}{L'}\right)^{\frac{\sigma-1}{\sigma}}}{1+\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}} \left(\frac{A_Z}{A_L}\frac{Z}{L}\right)^{\frac{\sigma-1}{\sigma}}} \right]^{\frac{1-\beta}{\sigma-1}} \times \left[\frac{1+\left(\frac{1-\gamma}{\gamma}\right)^{\frac{(1-\beta)(\varepsilon-1)+\sigma\beta}{\sigma\beta}} \left(\frac{A_Z Z'}{A_L L'}\right)^{\frac{(\varepsilon-1)}{\varepsilon}\frac{\sigma-(1-\beta)}{\sigma}}}{1+\left(\frac{1-\gamma}{\gamma}\right)^{\frac{(1-\beta)(\varepsilon-1)+\sigma\beta}{\sigma\beta}} \left(\frac{A_Z Z}{A_L L}\right)^{\frac{(\varepsilon-1)}{\varepsilon}\frac{\sigma-(1-\beta)}{\sigma}}} \right]^{\frac{\varepsilon}{\varepsilon-1}} \frac{L'}{L}.$$

and

$$\partial\frac{Y'}{Y}/\partial\frac{A_Z}{A_L} = \frac{1-\beta}{\sigma}\frac{Y'}{Y}\frac{1}{1+\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}}\left(\frac{A_Z}{A_L}\frac{Z'}{L'}\right)^{\frac{\sigma-1}{\sigma}}} \frac{\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}}\left(\frac{A_Z}{A_L}\right)^{\frac{-1}{\sigma}}\left(\frac{Z'}{L'}\right)^{\frac{\sigma-1}{\sigma}}\left[1-\left(\frac{Z}{L}/\frac{Z'}{L'}\right)^{\frac{\sigma-1}{\sigma}}\right]}{1+\left(\frac{1-\gamma}{\gamma}\right)^{\frac{\varepsilon}{\sigma}}\left(\frac{A_Z}{A_L}\frac{Z}{L}\right)^{\frac{\sigma-1}{\sigma}}}$$
$$+\frac{\sigma-(1-\beta)}{\sigma}\frac{Y'}{Y}\frac{1}{1+\left(\frac{1-\gamma}{\gamma}\right)^{\frac{(1-\beta)(\varepsilon-1)+\sigma\beta}{\sigma\beta}}\left(\frac{A_Z Z'}{A_L L'}\right)^{\frac{(\varepsilon-1)}{\varepsilon}\frac{\sigma-(1-\beta)}{\sigma}}}$$
$$\times\frac{\left(\frac{1-\gamma}{\gamma}\right)^{\frac{(1-\beta)(\varepsilon-1)+\sigma\beta}{\sigma\beta}}\left(\frac{A_Z}{A_L}\right)^{\frac{(\varepsilon-1)}{\varepsilon}\frac{\sigma-(1-\beta)}{\sigma}-1}\left(\frac{Z'}{L'}\right)^{\frac{\varepsilon-1}{\varepsilon}\frac{\sigma-(1-\beta)}{\sigma}}\left[1-\left(\frac{Z}{L}/\frac{Z'}{L'}\right)^{\frac{\varepsilon-1}{\varepsilon}\frac{\sigma-(1-\beta)}{\sigma}}\right]}{1+\left(\frac{1-\gamma}{\gamma}\right)^{\frac{(1-\beta)(\varepsilon-1)+\sigma\beta}{\sigma\beta}}\left(\frac{A_Z Z}{A_L L}\right)^{\frac{(\varepsilon-1)}{\varepsilon}\frac{\sigma-(1-\beta)}{\sigma}}}.$$

Clearly, therefore, if $\sigma > 1$ then $\partial\frac{Y'}{Y}/\partial\frac{A_Z}{A_L} < 0$ and if $\sigma < 1$ (more precisely $\sigma < 1-\beta$) then $\partial\frac{Y'}{Y}/\partial\frac{A_Z}{A_L} > 0$. This means that when products are gross substitutes increase in $\frac{A_Z}{A_L}$ increases income gap. This happens because in this case increase in $\frac{A_Z}{A_L}$ favors more Z than L and Z is less abundant than L in South compared to North. In turn, when products are gross complements then increase in $\frac{A_Z}{A_L}$ reduces income gap because South is more abundant of L.

Directed Technical Change: Acemoglu (1998) - vertical product innovation

Main assumptions

- A multi-sector model of R&D-driven growth with vertical innovations.
- Two factors of production: high-skill labor H and low-skill labor L.
- Four types of intermediate goods: Y_h, Y_l, x_h and x_l.
- Intermediate goods Y_h and Y_l are complementary.
- Intermediate goods x_h and x_l have varying quality.
- The highest qualities are are summarized by an index q_s, $s = h, l$. The difference between best and second best quality is equal to $\lambda > 1$ for $s = h, l$. The initial quality is normalized to one.
- The R&D sector produces "blueprints" for improved quality of intermediate goods. The input to the R&D production is investment in units of the final output.
- If total amount of R&D expenditure in sector s is z then the probability of discovery of new quality in that sector is

  $$z\phi(z). \tag{182}$$

  As quality improves, new discoveries become more expensive in terms of the required investment of resources, i.e., $\phi'(z) < 0$ gives diminishing returns to R&D input. As the probability depends only on the current quality level, it suggests that innovation occurs like a Poisson process. Assume further that this probability is non-decreasing in total effort z and

  $$\lim_{z\to 0}\phi(z) = +\infty; \lim_{z\to +\infty}\phi(z) = 0.$$

  Note: Linearity implies absence of congestion. Innovation in each sector is "jumpy" (takes place in a discreet manner), however the existence of many sectors and the Law of Large Numbers ensures a smooth outcome at the aggregate level.

- The marginal cost of inventing q_s quality intermediate good is Bq_s, $B > 0$. There is free entry into R&D process.
- The discovery of a better quality intermediate good of a particular type provides an entrepreneur with monopoly rights over the use of the "blueprint." He produces the distinct good with a linear technology that transforms one unit of final output into one unit of good.
- There is free entry into the intermediate goods industry.

- The final goods sector operates under perfect competition. It combines Y_h and Y_l in CES production function

$$Y = \left(Y_l^{\rho} + \gamma Y_h^{\rho}\right)^{\frac{1}{\rho}}, \tag{183}$$

  where $\gamma > 0$, $\rho \leq 1$ and the elasticity of substitution between Y_h and Y_l is $\frac{1}{1-\rho}$. This means that when $\rho < 0$ intermediate goods Y_h and Y_l are more complementary than under Cobb-Douglas - gross-complements - and when $\rho > 0$ intermediate goods Y_h and Y_l are more substitutable than under Cobb-Douglas - gross-substitutes. The price of final goods is normalized to 1.

- There is mass one of each intermediate goods producers.

- Y_h employs high-skill labor and Y_l employs low-skill labor.

- The $i-$th producer of Y_s, $s = h, l$, has production technology

$$y_s(i) = A_s(i)\, n_s(i)^{\beta}, \tag{184}$$

  where $\beta \in (0, 1)$, $n_s(i)$ is labor input and $A_s(i)$ is labor productivity.

- Labor productivity in each sector is

$$A_s(i) = \frac{1}{1-\beta} \int_0^1 q_s(j)\, x_s(i,j)^{1-\beta}\, dj, \tag{185}$$

  where it is assumed here that only the highest quality survives of each type of intermediate good. (This will be equilibrium outcome.)

- From the consumption-side, the representative HH supplies labor inelastically and chooses its consumption and assets to maximize its intertemporal utility: $U = \int_0^{+\infty} c_t \exp(-\eta t)\, dt$ subject to the standard budget constraint.

Further assumptions

- All intermediate goods depreciate fully within one period.

- No population growth.

Market equilibrium

Final goods sector

Final goods producers maximize their profits taking the prices of their inputs (p_h and p_l) as given.

$$\max_{Y_l, Y_h} Y - p_h Y_h - p_l Y_l,$$

Standard first order conditions imply that

$$[Y_h] \quad : \quad p_h = \frac{Y}{Y_h}\frac{\gamma Y_h^{\rho}}{Y_l^{\rho}+\gamma Y_h^{\rho}}, \tag{186}$$

$$[Y_l] \quad : \quad p_l = \frac{Y}{Y_l}\frac{Y_l^{\rho}}{Y_l^{\rho}+\gamma Y_h^{\rho}}. \tag{187}$$

Denote the ratio of p_h and p_l by p,

$$p = \frac{p_h}{p_l} = \gamma \left(\frac{Y_l}{Y_h}\right)^{1-\rho}. \tag{188}$$

Y_h and Y_l intermediate goods sectors

Intermediate goods producers maximize profits taking prices of their inputs (w_h, w_l, and $\chi_s(j)$) as given.- using small letters for individual producers and capital letters for aggregate variables -

$$\max_{n_s(i),x_s(i,j)} p_s y_s(i) - w_s n_s(i) - \int_0^1 \chi_s(j)\, x_s(i,j)\, dj,$$

Standard first order conditions imply that

$$[n_s(i)] \quad : \quad w_s = p_s \beta \frac{y_s(i)}{n_s(i)}, \tag{189}$$

$$[x_s(i,j)] \quad : \quad \chi_s(j) = p_s q_s(j)\, x_s(i,j)^{-\beta}\, n_s(i)^{\beta}. \tag{190}$$

Labor marker clearing requires that

$$\int_0^1 n_s(i)\, di = \begin{cases} H & s = h, \\ L & s = l. \end{cases},$$

Since there is a unit mass of producers, expressing $x_s(i,j)$ in terms of $n_s(i)$ from (190) and using that in (189) gives that

$$n_s(i) \equiv N_s,$$
$$x_s(i,j) \equiv X_s(j),$$

where $N_s = H$ for $s = h$ and $N_s = L$ for $s = l$..Therefore, $A_s(i) \equiv A_s$ and wage premium (ratio of w_h to w_l) is

$$\frac{w_h}{w_l} = \gamma \left(\frac{A_h}{A_l}\right)^{\rho} \left(\frac{H}{L}\right)^{-(1-\rho\beta)}. \tag{191}$$

Assume that in short-run A_h and A_l are fixed, clearly then

$$\frac{\partial \frac{w_h}{w_l}}{\partial \frac{H}{L}} < 0$$

which is the usual substitution effect showing that for a given technology relative demand curve for skills is downward sloping. Moreover, when $\rho > 0$

$$\frac{\partial \frac{w_h}{w_l}}{\partial \frac{A_h}{A_l}} > 0.$$

This means that when Y_h and Y_l are gross-substitutes wage premium of high-skill labor increases with the productivity of high-skill labor. This happens because in such a circumstance demand for low-skill labor declines relative to the demand for high-skill labor.

The aggregate demands for intermediate goods $x_s(j)$, in turn, are

$$X_s(j) = \left[\frac{p_s q_s(j)}{\chi_s(j)}\right]^{\frac{1}{\beta}} N_s. \tag{192}$$

x_h and x_l intermediate goods sectors

The production of a unit of an intermediate good $x_s(i,j)$ requires $q_s(j)$ units of final goods. Intermediate goods producers maximize profits given (192).

$$\max_{x_s(i,j)} \pi_s(i,j) = \chi_s(j) x_s(i,j) - q_s(j) x_s(i,j)$$
$$s.t.$$
$$(192).$$

First order condition is then

$$\chi_s(j) = \frac{q_s(j)}{1-\beta}.$$

This is a standard condition implying that price is constant mark-up over marginal cost.

Assume that within each quality, if there are more than one firms producing it, competition is Bertrand type. Further, assume that each quality which is below the highest one can be produced by a continuum of firms (i.e., qualities below highest one don't have patent protection and there is free entry). The marginal cost of firms that produce the second highest quality would be then $\frac{q_s(j)}{\lambda}$. Meanwhile, according to (192) it has marginal product which is $\lambda^{-\frac{1}{\beta}}$ times lower than the marginal product of the highest quality. Therefore, the maximum price that second best quality producer can set is

$$\frac{q_s(j)}{1-\beta}\lambda^{-\frac{1}{\beta}}.$$

Then, the second highest quality, as well as all qualities below, aren't produced in case when the marginal cost is higher than this price. This is equivalent to

$$\lambda > (1-\beta)^{-\frac{\beta}{1-\beta}}.$$[20]

This implies that

$$\begin{aligned} X_s(j) &= \left[(1-\beta) p_s N_s^{\beta}\right]^{\frac{1}{\beta}}, \\ \pi_s(j) &= \frac{\beta}{1-\beta} q_s(j) X_s(j), \\ A_s(i) &= (1-\beta)^{\frac{1-2\beta}{\beta}} \left[p_s N_s^{\beta}\right]^{\frac{1-\beta}{\beta}} Q_s. \end{aligned}$$

where

$$Q_s = \int_0^1 q_s(j)\, dj$$

are the average qualities of intermediate inputs used in high- and low-skill industries.

In turn, the discounted value of holding patent on the highest quality in each s sector is

$$rV_s(j) = \pi_s(j) - z_s(j)\,\phi(z_s(j))\,V_s(j) + \dot{V}_s(j). \tag{193}$$

This is standard condition. On the left-hand side it has discounted value of holding patent. On the right-hand side it has instantaneous profits, probability of instantaneous discovery of new quality of intermediate good in the same sector, and capital gain.

R&D processes

Free entry into R&D implies that marginal gain and cost are equal. Therefore,

$$\phi(z_s(j))\,V_s(j) = Bq_s(j). \tag{194}$$

Balanced growth path

On balanced growth path quantities and prices grow at constant rates. From (183) it follows that

$$g_Y = SHg_{Y_l} + (1-SH)\,g_{Y_h},$$

where $SH = \frac{Y_l^{\rho}}{Y_l^{\rho}+\gamma Y_h^{\rho}}$. This implies that as long as $SH \in (0,1)$ on balanced growth path

$$g_Y = g_{Y_l} = g_{Y_h}.$$

[20] Clearly, there is a problem here. Marginal product differences should be computed from (192). According to it second highest has marginal product which is λ^{-1} times lower than the marginal product of the highest quality. Therefore, the maximum price that second best quality producer can set is $\frac{q_s(j)}{1-\beta}\lambda^{-1}$.

Denote these growth rates by g and notice that this equality implies that

$$g_{A_s} = g_{Q_s} = g,$$

for $s = h, l$. This holds because on balanced growth path the only source of growth is R&D and price of intermediate goods proportionally increases with quality.

On balanced growth path capital gains are zero and from (193) and (194) it follows that

$$B\left[\frac{r + z_s(j)\,\phi(z_s(j))}{\phi(z_s(j))}\right] = \frac{\beta}{1-\beta}\left[(1-\beta)\,p_s N_s^{\beta}\right]^{\frac{1}{\beta}}. \tag{195}$$

From (184), (185), (188) and (190) it follows that

$$p = \gamma^{\beta\nu}\left(\frac{Q_h}{Q_l}\right)^{-\beta(1-\rho)\nu}\left(\frac{H}{L}\right)^{-\beta(1-\rho)\nu} \tag{196}$$

where $\nu = \frac{1}{1-(1-\beta)\rho}$.

Since there is a continuum of high-skill and low-skill intensive inputs, $z_s\phi(z_s)$ is the rate of technical improvement in both sectors. Hence, $(\lambda - 1)\,z_s\phi(z_s)$ is the rate of growth of Q_s. Since $g_{Q_s} = g$ it has to be that $z_h = z_l$. Therefore, from (195) and (196) it follows that

$$p = \left(\frac{H}{L}\right)^{-\beta}, \tag{197}$$

$$\frac{Q_h}{Q_l} = \gamma^{\frac{1}{1-\rho}}\left(\frac{H}{L}\right)^{\frac{\beta\rho}{1-\rho}}. \tag{198}$$

Notice that

$$\frac{\partial\frac{Q_h}{Q_l}}{\partial\frac{H}{L}} > 0.$$

Because of this "market-size" effect long-run relative demand for skills is flatter than short-run demand. It might be also upward sloping. Using (191) it follows that

$$\frac{w_h}{w_l} = \gamma^{\frac{1}{1-\rho}}\left(\frac{H}{L}\right)^{\frac{\beta\rho^2}{1-\rho}-(1-\rho\beta)}.$$

In this relative demand curve the second term, $-(1-\rho\beta)$, is the usual substitution effect. Clearly, if the first term is positive long-run relative demand curve is upward sloping.

Regarding, the existence and uniqueness of balanced growth path, preferences are linear, which means that $r=\eta$. Moreover, combining (195) with (183), (186), and (188) gives

$$p_h=\gamma\left[\left(\frac{Y_l}{Y_h}\right)^{\rho}+\gamma\right]^{\frac{1}{\rho}-1},$$

$$p_h=\gamma\left[\gamma^{-\frac{\rho}{1-\rho}}\left(\frac{H}{L}\right)^{-\beta\frac{\rho}{1-\rho}}+\gamma\right]^{\frac{1-\rho}{\rho}},$$

$$B\left[\frac{\eta+z\phi\left(z\right)}{\phi\left(z\right)}\right]=\beta\left(1-\beta\right)^{\frac{1-\beta}{\beta}}\left(L^{\beta\frac{\rho}{1-\rho}}+\gamma^{\frac{1}{1-\rho}}H^{\frac{\rho}{1-\rho}\beta}\right)^{\frac{1-\rho}{\rho}\frac{1}{\beta}}.$$

The left-hand side of this expression is strictly increasing with z whereas right-hand side is constant. This establishes existence and uniqueness of balanced growth path as long as allocations are such that no allocation becomes negative.

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