Growth and Redistribution: The Hedging Perspective

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teaching materials

Kelly '56

assets $i \in I$, discrete time t

stationary portfolio $p(i) \in \Delta(I)$

- share $p(i)$ of wealth invested in each asset i each morning

each asset i has a gross return $r(i, \omega_t)$ in each period t

- iid states $\omega_t \sim q^0(\omega) \in \Delta(\Omega)$

investor maximizes long-run growth rate of her wealth:

$$\max_{p(i)} \mathrm{E}_{q^0(\omega)} \ln \left(\sum_i p(i) r(i, \omega) \right)$$

Example

$\Omega = I$

$\omega \sim q^0(\omega)$

exogenous return function: $r(i, \omega) = \text{const.} \times \mathbb{1}_{\omega = i}$

optimal allocation: $p^*(i) = q^0(i)$

in contrast, maximization of expected return leads to a.s. bankruptcy

Outcome Distribution

def: conditional outcome distribution

share of wealth in asset i at the end of a period with $\omega_t = \omega$:

$$o_p(i \mid \omega) := \frac{p(i) r(i, \omega)}{\sum_j p(j) r(j, \omega)}.$$

def: outcome distribution

share of wealth on i at the end of a random period:

$$o_p(i) := \mathsf{E}_{q^0(\omega)}\, o(i \mid \omega).$$

Kelly's Optimality Condition

First Order Condition

Optimal portfolio eliminates systematic redistribution of investments:

$$p^*(i) = o_{p^*}(i).$$

Macroeconomic Reinterpretation

individuals $i \in I$ with random gross returns $r(i, \omega_t)$

social planner maximizes long-run growth rate of aggregate wealth

we add two novel features:

- a constraint on the endowment distribution $p(i)$
- constrained control over the return function $r(i, \omega)$

the set $\mathcal{P}$ of feasible policies $\big(p(i), r(i, \omega)\big)$ needs not be a product set

- e.g. chosen endowment distribution may affect returns via incentives

Main Result

Proposition

Growth-maximizing endowment distribution p^* minimizes KL-divergence from the induced outcome distribution

$$p^*(i) \in \arg\min_{p(i)\in\mathcal{A}^*} \mathrm{KL}\left(o_{p^*,r^*}(i) \parallel p(i)\right),$$

where $\mathcal{A}^*$ is the set of feasible endowment distributions given optimized returns $r^*(i,\omega)$.

recall: $\mathrm{KL}\left(o(i) \parallel p(i)\right) = \mathrm{E}_{o(i)} \ln \frac{o(i)}{p(i)}$ is a pseudo-distance

note the myopicity: the outcome distribution is fixed to o_{p^*,r^*}

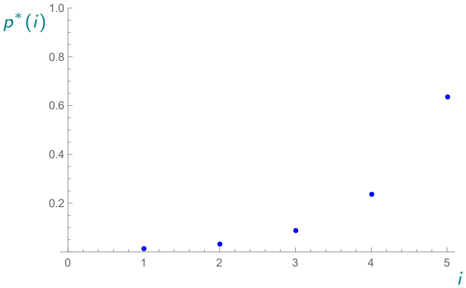
Example

no uncertainty

individual $i = 1, \ldots, 5$ has a return i

inequality constraint $\mathrm{H}\left(p(i)\right) \geq 1$

growth-optimal endowment distribution



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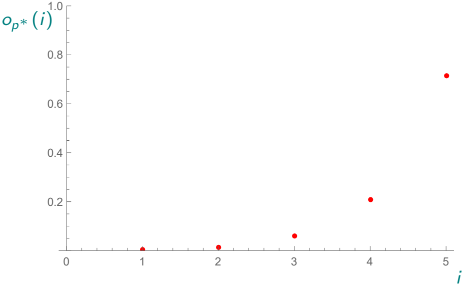


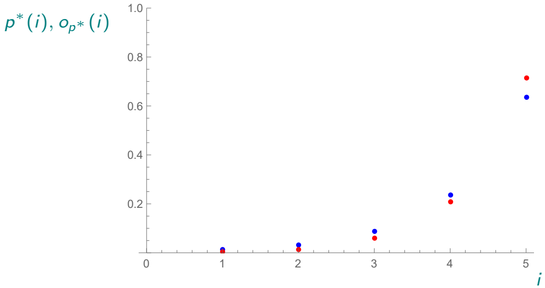
Example

no uncertainty

individual $i = 1, \ldots, 5$ has a return i

inequality constraint $\mathrm{H}\left(p(i)\right) \geq 1$

the 'closest' feasible endowment distribution to o_p^* coincides with p^*



Change of the Notation

relabel the return function $r(i, \omega)$ as $p(\omega \mid i)$

- generalized (non-normalized) conditional distribution

define (generalized) joint distribution

$$p(i, \omega) = p(i)p(\omega \mid i)$$

it is a (sufficient statistics for) the policy

- so we refer to $p(i, \omega)$ as policy

Auxiliary Problem

known as Variational Autoencoder in machine learning

let $\tilde{\mathcal{P}}$ be the set of feasible policies $p(i, \omega)$

Proposition

Policy $p^*(i, \omega)$ maximizes the growth rate if and only if solves

$$\min_{q(i,\omega),p(i,\omega)} \quad \mathrm{KL}\left(q(i,\omega) \parallel p(i,\omega)\right)$$

$$\text{s.t.} \quad q(i,\omega) \in \Delta(I \times \Omega)$$

$$q(\omega) = q^0(\omega)$$

$$p(i,\omega) \in \tilde{\mathcal{P}},$$

together with some $q^*(i, \omega)$.

studied in machine learning in the context of cognition/perception

Dynasties of Dollars

interpretation of $q(i,\omega)$

a dynasty originates in \$1 at $t = 1$

it moves from one individual to another

it multiplies by the return of its current owner

define pattern (of circulation) $q(i,\omega)$ as:

- the empirical frequency of a dynasty being in hands of i and state ω

all patterns $q(i,\omega)$ s.t: $q(\omega) = q^0(\omega)$ are followed by some dynasties

Wealth of a Pattern

informal proof

fix a policy p

define wealth of pattern q as the total wealth of all dynasties following it

wealth of pattern q grows at the rate:

$$\mathrm{E}_{q(i,\omega)} \ln p(\omega \mid i) - \mathrm{E}_{q^0(\omega)} \mathrm{KL}(q(i \mid \omega) \parallel p(i))$$

- growth rate of each dynasty circulating according the patter q
- rate of decline in the measure of such dynasties (Sanov's Theorem)

the fastest growing pattern q_p^* dominates

Wealth of a Pattern

informal proof

fix a policy p

define wealth of pattern q as the total wealth of all dynasties following it

wealth of pattern q grows at the rate:

$$-\,\mathsf{KL}\left(q(i,\omega)\parallel p(i,\omega)\right)+\text{const.}$$

- growth rate of each dynasty circulating according the patter q
- rate of decline in the measure of such dynasties (Sanov's Theorem)

the fastest growing pattern q_p^* dominates

Optimal Path

define joint outcome distribution as $o_p(i, \omega) = q^0(\omega) o_p(i \mid \omega)$

- prob. that a random end-of-period dollar is in hands of i in state ω

lemma

Joint outcome distribution equals the optimal pattern:

$$o_p(i, \omega) = q_p^*(i, \omega).$$

intuition:

- almost all wealth circulates according to $q_p^*(i, \omega)$.

Proof of The Main Result

growth-maximizing policy solves

$$\min_{p(i,\omega)} \quad \mathrm{KL}\left(q^*(i,\omega) \parallel p(i,\omega)\right)$$

$$\text{s.t.} \quad p(i,\omega) \in \tilde{\mathcal{P}}.$$

Proof of The Main Result

growth-maximizing policy solves

$$\min_{p(i)} \left\{ \mathrm{KL}\left(q^*(i) \parallel p(i)\right) + \sum_i q^*(i)\, \mathrm{KL}(q^*(\omega \mid i) \parallel p(\omega \mid i)) \right\}$$

$$\text{s.t.} \quad p(i) \in \mathcal{A}^*.$$

Proof of The Main Result

growth-maximizing policy solves

$$\min_{p(i)} \left\{ \mathrm{KL}\left(o_{p*}(i) \parallel p(i)\right) + \sum_i q^*(i)\,\mathrm{KL}(q^*(\omega \mid i) \parallel p(\omega \mid i)) \right\}$$

s.t. $\quad p(i) \in \mathcal{A}^*.$