Growth and Redistribution: The Hedging Perspective
Online Appendix

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Section A characterizes the growth-optimal policy and the resulting pattern of wealth circulation for a special case. Section B extends this characterization to the general setting presented in the main text.

A Special Case: Rational Inattention

In Robson et al. (2023), we showed that the rational-inattention problem is equivalent to the planner's problem from the main text, in the special case of an unrestricted endowment distribution $p(i) \in \Delta(I)$ and a fixed return function $p(\omega \mid i)$—corresponding to Kelly's portfolio management problem.

To state the equivalence, recall that the rational inattention problem from Matějka and McKay (2015):

$$\max_{r(i,\omega)\in\Delta(I\times\Omega)} \quad \mathrm{E}_{r(i,\omega)}\, u(i,\omega) - \mathrm{I}_{r(i,\omega)} \qquad \text{(i)}$$

$$\text{s.t.} \quad r(\omega) = q^0(\omega).$$

In their framework, $i \in I$ denotes an action, and ω represents an uncertain payoff state. The decision-maker with utility function $u(i,\omega)$ selects a joint distribution $r(i,\omega)$ to maximize expected utility net of the information acquisition cost, quantified by the mutual information $\mathrm{I}_{r(i,\omega)}$. The marginal distribution $r(\omega)$ is constrained to match the prior distribution $q^0(\omega)$.

Proposition A (Robson et al. (2023))**.** *The joint distribution $r^*(i,\omega)$ solves the rational inattention problem* (i) *if and only if $p^*(i) = \sum_\omega r^*(i,\omega)$ is the optimal endowment distribution, and $q^*(i,\omega) = r^*(i,\omega)$ is the optimal pattern of wealth circulation in problem (8) of the main text, where the endowment distribution $p(i)$ is unconstrained and the return function is fixed to be $p(\omega \mid i) = \exp[u(i,\omega)]$.*

Proof. Since the endowment distribution $p(i)$ is unconstrained, Corollary 1 implies that $q^*(i) = p^*(i)$. Consequently, the objective in Problem (8) simplifies to:

$$\begin{aligned}\mathrm{KL}\left(q(i,\omega)\parallel p(i,\omega)\right) &= \mathrm{KL}\left(q(i)\parallel p(i)\right)+\sum_i q(i)\,\mathrm{KL}\left(q(\omega\mid i)\parallel p(\omega\mid i)\right)\\ &= \sum_i q(i)\,\mathrm{KL}\left(q(\omega\mid i)\parallel p(\omega\mid i)\right)\\ &= -\sum_i q(i)\Big(\mathrm{E}_{q(\omega|i)}\,u(i,\omega)+\mathrm{H}\left(q(\omega\mid i)\right)\Big)\\ &= -\left(\mathrm{E}_{q(i,\omega)}\,u(i,\omega)-\mathrm{I}_{q(i,\omega)}\right)-\mathrm{H}\left(q^0(\omega)\right).\end{aligned}$$

The first equality follows from the chain rule. The second equality uses that $\mathrm{KL}\left(q(i)\parallel p(i)\right)=0$ at the optimum. The third equality follows from the definition of KL-divergence and entropy, alongside the assumption $p(\omega\mid i)=\exp u(i,\omega)$. The last equality follows from the definition of mutual information, $\mathrm{I}_{q(i,\omega)}=\mathrm{H}\left(q(\omega)\right)-\sum_i q(i)\,\mathrm{H}\left(q(\omega\mid i)\right)$, with $q(\omega)=q^0(\omega)$. Therefore, the objectives of the minimization problem (8) and the rational inattention problem (i) differ only by a sign and the term $\mathrm{H}\left(q^0(\omega)\right)$, which is independent of the controls as needed. □

As a result, the characterization results from the rational inattention literature are applicable to the special case considered in this section. In particular, the optimal endowment distribution $p^*(i)$ and the optimal pattern $q^*(i,\omega)$ must satisfy the necessary fixed-point condition from Matějka and McKay (2015):

$$p^*(i) = \mathrm{E}_{q^0(\omega)}\,q^*(i\mid\omega) \tag{ii}$$

$$q^*(i\mid\omega) = \frac{p^*(i)e^{u(i,\omega)}}{\sum_{j\in I}p^*(j)e^{u(j,\omega)}}. \tag{iii}$$

Furthermore, the optimal endowment distribution $p^*(i)$ is characterized by the necessary and sufficient conditions for the rational inattention problem (i) from Caplin and Dean (2013):

$$\begin{aligned}\mathrm{E}_{q^0(\omega)}\frac{p(\omega\mid i)}{p^*(\omega)} &= 1 \text{ if } p^*(i)>0, \qquad \text{(iv)}\\ &\leq 1 \text{ if } p^*(i)=0,\end{aligned}$$

where $p^*(\omega) = \sum_i p^*(i)p(\omega \mid i)$ is the aggregate return in state ω.[1]

B General Case

An iterative procedure generalizing the Blahut-Arimoto algorithm characterizes the growth-optimal policy and the corresponding pattern of wealth circulation. To initiate this process, let $q_0(i,\omega)$ be an arbitrary interior joint distribution, satisfying $\sum_i q_0(i,\omega) = q^0(\omega)$. The procedure is then defined recursively as follows:

$$p_k(i,\omega) \in \operatorname*{arg\,min}_{p(i,\omega)\in\tilde{\mathcal{P}}} \quad \mathrm{KL}\left(q_k(i,\omega) \parallel p(i,\omega)\right) \tag{v}$$

$$q_k(i,\omega) \in \operatorname*{arg\,min}_{q(i,\omega)\in\Delta(I\times\Omega)} \quad \mathrm{KL}\left(q(i,\omega) \parallel p_{k-1}(i,\omega)\right) \quad \text{s.t.} \quad q(\omega) = q^0(\omega). \tag{vi}$$

Proposition B. *Assume that $\tilde{\mathcal{P}}$ is convex and includes a policy p such that $p(i,\omega) > 0$ for all pairs (i,ω). Then, $p_k(i,\omega)$ converges to the growth-maximizing policy $p^*(i,\omega)$, and $q_k(i,\omega)$ converges to the corresponding pattern of circulation $q^*_{p^*}(i,\omega)$.*

Proof. The result follows from Theorem 3 and the subsequent Remark in Csiszar and Tusnady (1984). Their result requires the condition that (in their notation) $p_0 \in P$ is positive for precisely those $x \in X$ where there exist $p \in P$ and $q \in Q$ such that $p(x)q(x) > 0$. This condition holds in our setting because $q_0(i,\omega)$ (the counterpart of their p_0) is assumed to be interior, and $\tilde{\mathcal{P}}$ is assumed to contain an interior $p(i,\omega)$, providing the counterparts of their p and q with $p(x)q(x)$ also interior. Their result thus guarantees that our $p_k(i,\omega)$ converges to the growth-maximizing policy for those (i,ω) where $q^*(i,\omega) > 0$. In our specific setting, $p_k(i,\omega)$ also converges for those (i,ω) where $q^*(i,\omega) = 0$ because $q_k(i \mid \omega) = p_{k-1}(i \mid \omega)$, and hence $p_k(i \mid \omega)$ converges to 0, leading to $p_k(i,\omega) \to 0$ as well. $\square$

Proposition B implies that the optimal policy and the corresponding pattern of wealth circulation must form a fixed point of the system (v, vi). This

[1] The optimality conditions of Caplin and Dean for the rational inattention problem are identical to those for Kelly's portfolio management problem. See Theorem 16.2.1 in Cover and Thomas (2006). This coincidence has motivated our interest in stochastic growth.

generalizes the necessary conditions (ii, iii) from Matějka and McKay (2015). Finally, the necessary and sufficient conditions from Caplin and Dean (2013) extend as follows.

Proposition C (Sufficient and Necessary Conditions)**.** *Assume that the set $\tilde{\mathcal{P}}$ of feasible policies is convex. A policy $p^*(i,\omega)$ maximizes the economy's growth rate if and only if*

$$\mathrm{E}_{q^0(\omega)} \frac{p(\omega)}{p^*(\omega)} \leq 1 \text{ for all } p(i,\omega) \in \tilde{\mathcal{P}}, \tag{vii}$$

where $p^*(\omega) = \sum_i p^*(i,\omega)$ and $p(\omega) = \sum_i p(i,\omega)$ denote the aggregate returns in state ω under the policies p^* and p, respectively.

Proof. Fix an individual j and a state $\tilde{\omega}$. The derivative of the growth rate with respect to the policy is given by:

$$\begin{aligned} \partial_{p(j,\tilde{\omega})} \sum_{\omega} q^0(\omega) \ln\Big(\sum_i p(i,\omega)\Big)\Big|_{p=p^*} &= \frac{q^0(\tilde{\omega})}{\sum_i p^*(i,\tilde{\omega})} \\ &= \frac{q^0(\tilde{\omega})}{p^*(\tilde{\omega})}. \end{aligned}$$

Since the objective function, $\sum_\omega q^0(\omega)\ln(\sum_i p(i,\omega))$, is concave in p, and the feasible set $\tilde{\mathcal{P}}$ is convex, the set of maximizers remains unchanged by linearizing the objective around the optimum. Therefore, $p^*(i,\theta)$ is a growth-maximizing policy if and only if it maximizes $\sum_{i,\omega} \frac{q^0(\omega)}{p^*(\omega)} p(i,\omega)$ over $\tilde{\mathcal{P}}$. Summing this objective across i yields (vii). □

References

Caplin, A. and M. Dean (2013). Behavioral implications of rational inattention with Shannon entropy. Technical report, National Bureau of Economic Research.

Cover, T. M. and J. A. Thomas (2006). *Elements of Information Theory* (Second ed.). New York: John Wiley and Sons.

Csiszar, I. and G. Tusnady (1984). Information geometry and alternating minimization procedures. *Statistics and Decisions, Dedewicz 1*, 205–237.

Matějka, F. and A. McKay (2015). Rational inattention to discrete choices: A new foundation for the multinomial logit model. *American Economic Review 105*(1), 272–298.

Robson, A., L. Samuelson, and J. Steiner (2023). Decision theory and stochastic growth. *American Economic Review: Insights 5*(3), 357–376.