Price Distortions in High-Frequency Markets

Jakub Steiner[1] Colin Stewart[2]

[1]University of Edinburgh and CERGE-EI

[2]University of Toronto

INEXC Conference
February 2014

Introduction

How does the ability to react quickly to changes in the market affect asset prices?

Price distortions due to bounded rationality may be magnified at high frequencies.

The use of coarse theories by some traders can have a large impact.

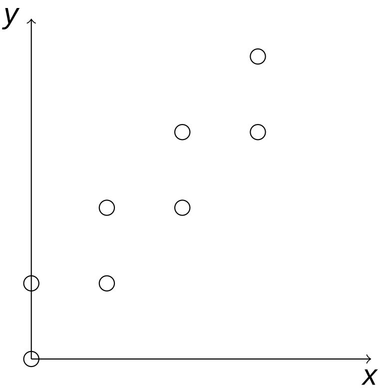
Main Idea

Short horizons make speculation the dominant force in price formation.

Speculation can magnify the effect on prices of mistakes in expectations.

Heterogeneity compounds the effect: distortions in prices can be large even if individual traders only make small errors.

Example



State space $\Omega = \{(x, y) \mid x \in \{0, \ldots, K\} \text{ and } y \in \{x, x+1\}\}$

Flow dividend $d(x, y) = (x + y)/(2K + 1)$.

Time $t = 0, \Delta, 2\Delta, \ldots$.

State ω_t evolves according to a continuous-time Markov process with transition rates

$$q\left(\omega, \omega'\right) = \frac{1}{2K + 2}.$$

State ω_t is publicly observed before trade takes place at time t.

Prices are determined as in Harrison and Kreps (1978).

Continuum of agents partitioned into groups $i = 1, \ldots, N$.
Agents within each group form identical expectations.

Agents are risk neutral.

Fixed supply, no short-selling.

Steady-state prices solve

$$P_\Delta(\omega) = \max_i \left\{ (1 - e^{-\Delta}) d(\omega) + e^{-\Delta} E^i \left[P_\Delta(\omega') \mid \omega \right] \right\}.$$

Agent i's reservation price depends on future prices only through the price in the next period because of risk neutrality.

Rational Benchmark

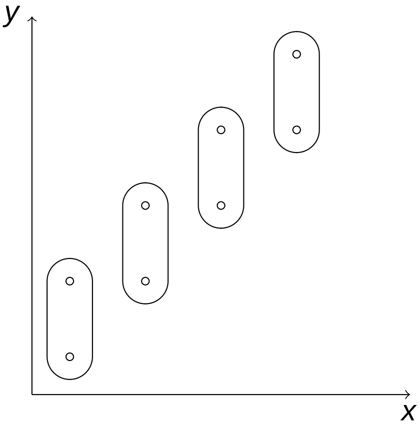
Suppose all agents have correct beliefs. Then

$$P_\Delta(\omega) = (1 - e^{-\Delta})d(\omega) + e^{-\Delta} E_{q_\Delta(\omega,\omega')} \left[P_\Delta(\omega') \mid \omega \right].$$

Solution for small Δ:

$$P(\omega) \approx \frac{d(\omega)}{2} + \frac{1}{4} = \text{expected discounted future dividends.}$$

Coarse Theory



x-theory

Agent's expectations are:

- coarse: they depend only on x;
- unbiased: correct on average within each category.

$\Rightarrow E^i = E_{q^x_\Delta}$ where

$$q^x_\Delta\big((x,y),\omega'\big) = \frac{q_\Delta\big((x,x),\omega'\big) + q_\Delta\big((x,x+1),\omega'\big)}{2}.$$

In this case,

$$|P^x_\Delta(\omega) - P_\Delta(\omega)| < \frac{1}{2K+1}.$$

Agent's expectations are:

- coarse: they depend only on x;
- unbiased: correct on average within each category.

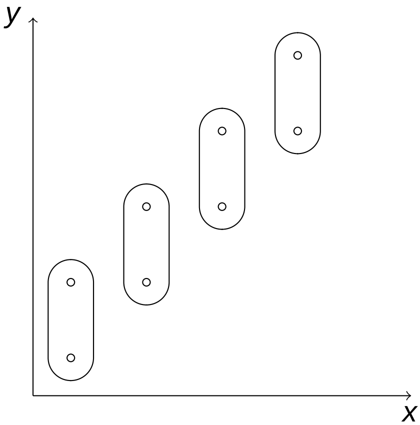
$\Rightarrow E^i = E_{q^x_\Delta}$ where

$$q^x_\Delta\big((x,y),\omega'\big) = \frac{q_\Delta\big((x,x),\omega'\big) + q_\Delta\big((x,x+1),\omega'\big)}{2}.$$

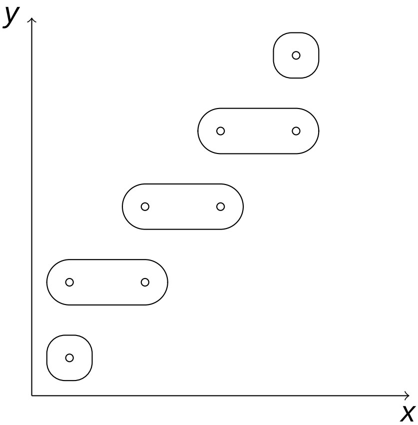
In this case,

$$|P^x_\Delta(\omega) - P_\Delta(\omega)| < \frac{1}{2K+1}.$$

Heterogeneous Theories



x-theory



y-theory

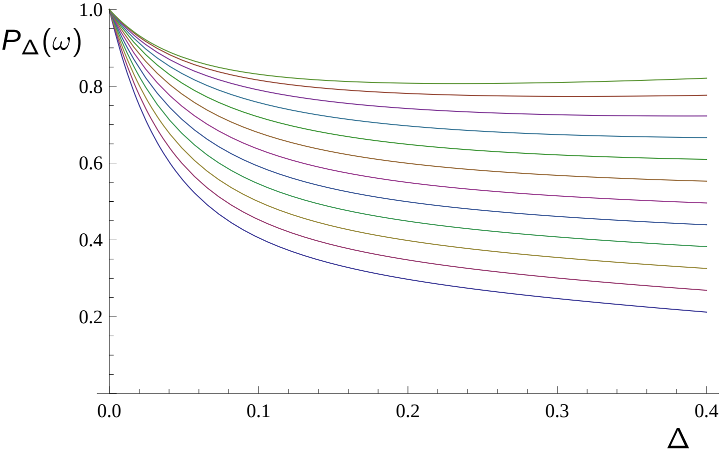
Agent's expectations are:

- coarse: measurable with respect to her partition;
- unbiased: correct when averaged within a category of her partition.

Define $q^y_\Delta(\omega, \omega')$ analogously to $q^x_\Delta(\omega, \omega')$.

Steady-state prices satisfy

$$P_\Delta(\omega) = (1 - e^{-\Delta})d(\omega) + e^{-\Delta} \max_{z \in \{x,y\}} E_{q^z_\Delta} \left[P_\Delta(\omega') \mid \omega \right].$$



$P_\Delta(\omega)$
1.0
0.8
0.6
0.4
0.2
0.0
0.1
0.2
0.3
0.4
Δ

Intuition

Why are prices very coarse even though theories are not?

Frequent trading shifts weights in the individual decisions toward speculation:

$$\text{dividend} \rightsquigarrow \text{resale price.}$$

When trading is frequent agents coordinate their reservation prices.

Agents can coordinate only if they condition only on information used by the whole population.

Market prices must be measurable in the finest common coarsening of the individual categorizations.

Model

Continuum of agents, divided into groups $1, \dots, N$ of sizes $\pi_1, \dots, \pi_N$.

Publicly observed state $\omega \in \Omega$ (finite) evolves according to homogeneous ergodic continuous-time Markov process q with stationary distribution ϕ.

Dividend function $d : \Omega \longrightarrow \mathbb{R}$.

Each group n employs a partition Π_n of Ω.

Trade occurs in periods $k = 0, 1, \dots$ corresponding to times $t = 0, \Delta, 2\Delta, \dots$.

Estimates of future prices are coarse but correct on average.

Agents in group n form estimates as if they believe the process is

$$m^n_\Delta(\omega,\omega') = \sum_{\omega''\in\Pi_n(\omega)} \phi\left(\omega'' \,|\Pi_n(\omega)\right) q_\Delta(\omega'',\omega').$$

Demands as in CARA-normal environment:

$$\alpha_n(\omega_k) = (1-e^{-\Delta})d(\omega_k) + e^{-\Delta}E^n\left[P(\omega_{k+1}) \mid \omega_k\right] - P(\omega_k).$$

Zero net supply $\Rightarrow$ steady-state prices satisfy

$$P_\Delta(\omega) = (1-e^{-\Delta})d(\omega) + e^{-\Delta}\sum_n \pi_n E_{m^n_\Delta}\left[P(\omega') \mid \omega\right].$$

Alternative interpretation: dynamic Morris and Shin (2002) beauty contest.

Each agent i chooses an action P^i in each period.

Agent i's flow payoff is

$$-(1 - e^{-\Delta})(P^i - d(\omega_k))^2 - e^{-\Delta}(P^i - P_{k+1})^2,$$

where P_{k+1} is the average action in period $k + 1$.

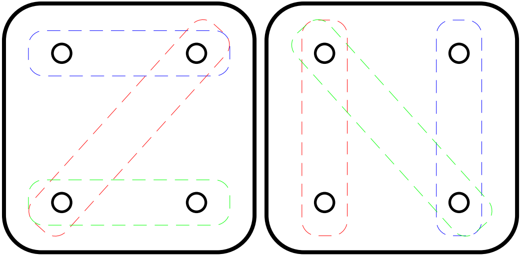
Yields the same steady-state prices.

Definition

The aggregate categorization is the meet (the finest common coarsening) of the individual categorizations Π_n.

Two states lie in the same aggregate category whenever a positive mass of agents fail to distinguish between them.

Aggregate categories may contain pairs of states that all agents distinguish between.



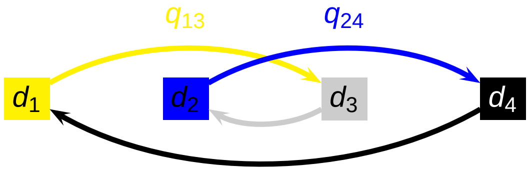
Proposition

1. *As $\Delta \to 0$, steady-state prices are constant within each aggregate category.*

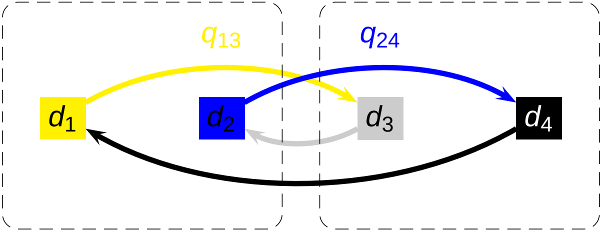
2. *The prices are equal to rational expectations prices for a simplified Markov process on aggregate categories.*

3. *Dividends and transition rates of the simplified process are averages of those of the original process, weighted according to ϕ.*

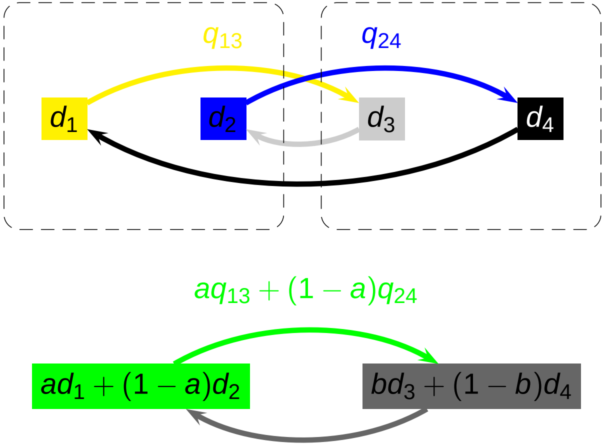
Illustration



Illustration



Illustration



Idea of Proof

Recall that steady-state prices satisfy

$$P_\Delta(\omega) = (1 - e^{-\Delta}) d(\omega) + e^{-\Delta} E \left[P_\Delta(\omega') \right],$$

where $E = \sum_n \pi_n E_{m_\Delta^n}$.

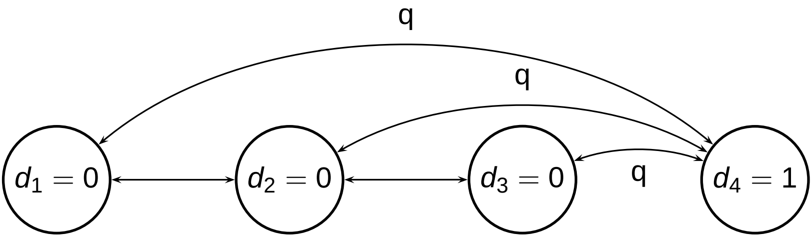
Iterating this equation gives

$$P_\Delta(\omega_0) = \sum_{k=0}^{\infty} e^{-k\Delta} E^k \left[(1 - e^{-\Delta}) d(\omega_k) \mid \omega_0 \right].$$

When Δ is small, expectations converge quickly within each aggregate category.

Example

Categorization can lead to excess variation in prices (at least in some states).

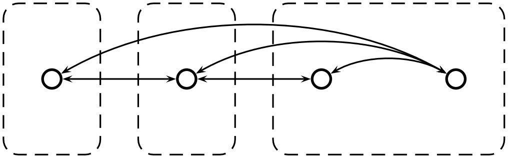


Transition probabilities are symmetric and the probability of transitioning to state ω_4 is the same across the other states.

The transition probability between ω_1 and ω_3 is zero.

For rational expectations, the distinction among ω_1, ω_2, and ω_3 is irrelevant.

Suppose that some agents categorize ω_3 and ω_4 together.



High-frequency prices in states ω_3 and ω_4 become identical.

Prices in ω_1 and ω_2 differ.

Categorization has spillovers across states.

General Price Mechanism

Let

$$P^n_\Delta(\omega) = (1 - e^{-\Delta})d(\omega) + e^{-\Delta}E_{m^n_\Delta}\left[P_\Delta(\omega') \mid \omega\right].$$

Market price $P_\Delta(\omega)$.

Assume there exists $\mu \in (0, 1]$ such that

$$\max_n P^n_\Delta(\omega) \geq P_\Delta(\omega) \geq (1 - \mu)\min_n P^n_\Delta(\omega) + \mu \max_n P^n_\Delta(\omega)$$

for every ω and Δ.

This holds for Harrison and Kreps (1978), and for the reduced form model.

Main Result

Proposition

Limit prices are constant on each aggregate category.

Lemma

Agents coordinate their expectations in the limit:

$$\lim_{\Delta \to 0} \left(P^n_\Delta(\omega) - P^m_\Delta(\omega)\right) = 0 \textit{ for all } \omega, m, \textit{ and } n.$$

The proposition follows from the lemma together with the observation that expectations are constant on individual categories.

Main Result

Proposition

Limit prices are constant on each aggregate category.

Lemma

Agents coordinate their expectations in the limit:

$$\lim_{\Delta \to 0} (P_\Delta^n(\omega) - P_\Delta^m(\omega)) = 0 \text{ for all } \omega, m, \text{ and } n.$$

The proposition follows from the lemma together with the observation that expectations are constant on individual categories.

Sketch of Proof of Lemma

Suppose not. Let $P^{\underline{n}}(\underline{\omega})$ be a minimal limit reservation price across all states in which expectations are not coordinated.

By assumption,

$$P(\underline{\omega}) \geq (1-\mu)P^{\underline{n}}(\underline{\omega}) + \mu \max_n P^n(\underline{\omega}) > P^{\underline{n}}(\underline{\omega}).$$

By minimality of $(\underline{n}, \underline{\omega})$,

$$P^n(\omega) \geq P^{\underline{n}}(\omega) = P^{\underline{n}}(\underline{\omega})$$

for every n and $\omega \in \Pi_{\underline{n}}(\underline{\omega})$.

But then $P(\omega) \geq P^{\underline{n}}(\omega)$ for every $\omega \in \Pi_{\underline{n}}(\underline{\omega})$, implying that $E^n[P(\omega')|\underline{\omega}] > P^{\underline{n}}(\underline{\omega})$, a contradiction.

Weighted Probabilities

Instead of categorization, suppose that some agents overweight small probabilities according to

$$m^i_\Delta(\omega,\omega') = \frac{\lambda(q_\Delta(\omega,\omega'))}{\sum_{\omega''}\lambda(q_\Delta(\omega,\omega''))},$$

with $\lim_{p\to 0^+}\frac{\lambda(p)}{p} = \infty$.

This assumption holds for weighting functions commonly used in prospect theory (e.g., Prelec (1998), Gonzalez and Wu (1999)).

Remaining agents use correct probabilities.

Proposition

If $q(\omega,\omega') > 0$ for every ω and ω' then limit prices are constant.

Risk Averse Agents

Random lifespan with fixed expected duration (i.e. trade for number of periods proportional to $1/\Delta$).

CARA utility from consumption at time of exit.

State space as in leading example.

Agents are equally divided between the two categorizations.

Proposition

Limit prices are constant.

Conclusion

At high frequency, expectations of the resale price become dominant in price formation.

Prices become sensitive to errors in expectations by some agents.

When agents use coarse theories, market aggregation is typically much coarser than any individual theory.

Prices can

- fail to respond to relevant information,
- overreact to information.