Coordination Cycles*

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Abstract

Players repeatedly face a coordination problem in a dynamic global game. By choosing a risky action (invest) instead of waiting, players risk instantaneous losses as well as a loss of payoffs from future stages, in which they cannot participate if they go bankrupt. Thus, the total strategic risk associated with investment in a particular stage depends on the expected continuation payoff. High continuation payoff makes investment today more risky and therefore harder to coordinate on, which decreases today's payoff. Thus, expectation of successful coordination tomorrow undermines successful coordination today, which leads to fluctuations of equilibrium behavior even if the underlying economic fundamentals happen to be the same across the rounds. The dynamic game inherits the equilibrium uniqueness of the underlying static global game.

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1 Introduction

Strategic complementarities resulting in multiple equilibria are common in many economic situations.[1] Models with multiple equilibria and self-fulfilling beliefs have been suggested to explain sudden shifts of the economy from one state to another, but the weakness of early coordination models was their reduced predictive power. Without an additional selection principle, they could not discriminate among multiple equilibria, and thus such models severed the natural link between fundamentals and economic outcomes; see concluding remarks in Matsuyama (1991) for discussion of pros and cons of models with multiple equilibria. The global games literature, originating in Carlsson and van Damme (1993) filled the gap by showing that the multiplicity of equilibria in coordination games with complementarities is a peculiar consequence of an unrealistic assumption that the underlying economic fundamental is common knowledge. If observation of the fundamental is noisy, the multiplicity of equilibria is eliminated and the fundamental fully determines economic activity.

In static global games models, economic outcomes change only if the fundamental changes (possibly by a small amount). Thus although the global games framework solves the indeterminacy due to self-fulfilling beliefs, it leaves no place for endogenous fluctuations unconnected to the evolution of the underlying fundamentals, and hence it misses some of the attractive features of the older models based on self-fulfilling beliefs.

Some dynamic global games allow for a partial separation between the current economic fundamental and behavior — equilibrium behavior may differ across two rounds even if the fundamentals in these rounds are identical. The obvious reason is that, in dynamic models, the current fundamental is not a complete description of the economic environment; past or future rounds influence behavior as well. However, we are unaware of a dynamic global game that would allow for fully endogenous cycles. The change in behavior is always triggered by a change in the fundamental, which is exogenous. See Section 4 for a detailed review of the dynamic global game literature.

[1] Complementarities have been used to model search (Diamond, 1982); bank runs (Diamond and Dybvig, 1983); currency attacks (Obstfeld, 1996); or business cycles (Benhabib and Farmer, 1994). Cooper (1999) provides a survey of coordination problems in macroeconomics.

The dynamic link among the coordination problems in the model at hand is that each player, by her current action, influences not only her instantaneous payoff, but also her future participation in the game. For instance, a risky investment influences not only the instantaneous profit, but also investor's ability to participate in future projects: unsuccessful investment can lead to bankruptcy. Fear of bankruptcy may motivate an investor not to invest, especially in the days just before an expected boom. The amount of strategic risk associated with investment depends on the expected equilibrium outcome in the near future. It is higher before a boom than before a slump, and this negative link between tomorrow's and today's coordination leads to endogenous fluctuations of investment that will never converge to a steady state.

The model at hand inherits equilibrium uniqueness from the underlying static global game. Equilibrium behavior in each round is determined by an equilibrium threshold; players invest only if the current fundamental exceeds the threshold, and otherwise they wait. The thresholds, although uniquely determined, differ across periods. This can be interpreted as fluctuations of market sentiments; crises occur when these sentiments are too pessimistic — thresholds are too high — compared to the realized fundamentals. The model combines the equilibrium uniqueness of global games with the cyclicality of strategic delay models which analyze situations when players are motivated to delay investment to match the timing of others' investments. (e.g. Shleifer, 1986; Gale, 1995).

We introduce the basic model in an abstract setup in Section 2. To illustrate the economic intuition behind the main result, we apply the model to three economic problems in Section 3. We compare the model with other dynamic global games in Section 4. Section 5 covers certain technical generalizations of the basic model and Section 6 concludes.

2 The Basic Game

2.1 The Model

A continuum of players interacts in a sequence of coordination problems. Player $i \in [0, 1]$ chooses action $a^i \in \{0, 1\}$ in each round $t \in \{1, \ldots, T\}$; we briefly discuss infinite horizon at the end of 2.3. We will refer to action 1 as *investing* and to action 0 as *waiting*. The instantaneous payoff u_t^i of player i in round t is $u_t^i = 0$ if she waits and $u_t^i = \tau_t^i \pi(\theta_t, l_t)$ if she invests, where $\tau_t^i \geq 0$ is the degree of player i's *involvement* in the game at round t, θ_t is the economic fundamental at t, and l_t is the measure of players investing at t. Players maximize the sum of discounted instantaneous payoffs $\sum_{t=1}^{T} \delta^t u_t^i$.

The fundamentals θ_t are i.i.d. random variables with twice continuously differentiable c.d.f. $\Phi(\cdot)$ on the real line. We relax the assumption of i.i.d. fundamentals in Section 5.1. Players observe fundamentals with an error. Each player receives in each round t a private signal $x_t^i = \theta_t + \sigma \epsilon_t^i$, where the idiosyncratic errors ϵ_t^i are independent across players and rounds and drawn from a continuous p.d.f. $f(\cdot)$ with support on the real line. We will be interested in the equilibrium of the game in the limit as the size of the noise $\sigma \to 0$.

For the moment, let us keep the involvement levels τ_t^i fixed. We call the simultaneous move game with the set of players $I = [0, 1]$, action sets $A^i = \{0, 1\}$, payoff function $\pi(\theta, l)$, and the informational structure described above, the *static stage game*. We assume that the static stage game is a global game satisfying the following assumptions taken from Morris and Shin (2003):

A1 Action Monotonicity: $\pi(\theta, l)$ is weakly increasing in l.

A2 State Monotonicity: $\pi(\theta, l)$ is weakly increasing in θ.

A3 Strict Laplacian State Monotonicity: $\int_0^1 \pi(\theta, l) dl$ strictly increases in θ.[2]

A4 Uniform Limit Dominance: There exist $\underline{\theta} \in \mathbb{R} \cup \{-\infty, \infty\}$ and $\overline{\theta} \in \mathbb{R} \cup \{-\infty, \infty\}$ and $\epsilon > 0$ such that 1. $\pi(\theta, l) < -\epsilon$ for all $l \in [0, 1]$ and $\theta < \underline{\theta}$ and 2. $\pi(\theta, l) > \epsilon$ for all $l \in [0, 1]$ and $\theta > \overline{\theta}$.[3]

A5 Continuity: $\int_0^1 g(l) \pi(\theta, l) dl$ is continuous with respect to θ and densities $g(\cdot)$.

A6 Finite Expectations of Signals: $E[z] = \int_{-\infty}^{\infty} z f(z) dz$ is well-defined.

Each static stage game, if treated in isolation, is an identical coordination problem which has under complete information (for large set of fundamentals) two pure strategy equilibria in which either all or no players invest. The incomplete

[2]We use a slightly stronger version than Morris and Shin (2003), who only require that $\int_0^1 \pi(\theta, l) dl = 0$ has a unique solution.

[3]This is a slightly weaker version of the original assumption in Morris and Shin (2003). By allowing $\underline{\theta}$ and $\overline{\theta}$ to be $\pm\infty$, we admit also games in which one of the action is strictly dominant, and hence the solution is trivial.

information structure transforms the game into a global game which, in the limit $\sigma \to 0$, has a unique equilibrium characterized by a threshold signal above which players invest and below which they wait, see Proposition 2.2 in Morris and Shin (2003).

The dynamic link among different stage games is that the involvement τ_t^i is endogenous: τ_1^i is normalized to 1, and values in later rounds are defined recursively by $\tau_{t+1}^i = \tau_t^i b(a_t^i, \theta_t, l_t)$, with $0 \leq b(a_t^i, \theta_t, l_t)$. While we focus on economic situations where current activity "wears out" the player's involvement, $b(a_t^i, \theta_t, l_t) \leq 1$, it is not a necessary condition for Proposition 1. Define $\rho(\theta, l) = b(1, \theta, l) - b(0, \theta, l)$ and let it satisfy the following three assumptions:

A7 $\rho(\theta, l)$ is weakly increasing in l.

A8 $\rho(\theta, l)$ is weakly increasing in θ.

A9 $\int_0^1 g(l)\rho(\theta, l)dl$ is continuous with respect to θ and densities $g(\cdot)$.

High involvement is beneficial to players because the possibility of waiting ensures that the payoff 0 is always available. Thus investors benefit from high θ_t and l_t twice, through increases not only in the current profit, but also in the involvement in future rounds.

An example of a setup satisfying A7–A9 is a situation with players constrained only to one investment, $b(1, \theta, l) \equiv 0$, $b(0, \theta, l) \equiv 1$. As an alternative example, players are allowed to invest many times, but investing players may go bankrupt with a probability that decreases in the instantaneous payoff; that is, $b(1, \theta, l) \equiv r(\pi(\theta, l))$ with $0 \leq r(\cdot) \leq 1$ increasing, and $b(0, \theta, l) \equiv 1$. In the latter example τ_{t+1}^i is expected involvement, because bankruptcy is a random event.

The measure of players is kept constant across rounds; we assume that players who disappear from play are replaced by new entrants. This assumption is removed in Section 5.2. Finally, we assume a joint condition on the instantaneous payoff function $\pi(\cdot, \cdot)$ and the continuation function $\rho(\cdot, \cdot)$:

A10 $\tilde{\pi}(\theta, l) \equiv \pi(\theta, l) + \delta\rho(\theta, l)V$ satisfies the uniform limit dominance condition for all $V > 0$.

A sufficient condition for A10 to hold is that $\lim_{\theta \to +\infty} \pi(\theta, 0) = +\infty$ and $\lim_{\theta \to -\infty} \pi(\theta, 1) = -\infty$.[4] Another sufficient condition is that $\rho(\theta, l) \geq 0$ for θ

[4]This is satisfied by the commonly used payoff function $\pi(\theta, l) = \theta - 1 + l$.

above some $\overline{\theta}$ and $\rho(\theta, l) \leq 0$ below some $\underline{\theta}$.[5]

The information set of player i at t is

$$I_t^i = \{x_1^i, \ldots, x_t^i, \theta_1, \ldots, \theta_{t-1}, l_1, \ldots, l_{t-1}, a_1^i, \ldots, a_{t-1}^i\}$$

. A pure strategy $s = \{s_1, \ldots, s_T\}$ is a sequence of functions that assigns to each path of information sets $\{I_1^i, \ldots, I_T^i\}$ a path of actions $\{s_1(I_1^i), \ldots, s_T(I_T^i)\}$. However, as shown below, the equilibrium strategy has a simpler structure; the equilibrium action depends only on the current signal x_t^i.

2.2 The Solution

The interaction in the last round T is a static stage game, and thus it is a global game solvable by proposition 2.2 in Morris and Shin (2003). The unique strategy surviving iterated elimination of dominated strategies in the limit $\sigma \to 0$ is a threshold strategy:

$$s_T^*(x) = \begin{cases} 1 & \text{if } x > \theta_T^*, \\ 0 & \text{if } x < \theta_T^*, \end{cases}$$

with the threshold θ_T^* such that $s_T^*(\cdot)$ is the best response to the belief according to which the measure l_T of investing players is distributed uniformly on $[0, 1]$. Morris and Shin (2003) "dub such beliefs ... as being Laplacian, following Laplace's (1824) suggestion that one should apply a uniform prior to unknown events from the principle of insufficient reason." Such beliefs arise endogenously in global games for a player observing the threshold signal.

Given Laplacian beliefs, the threshold θ_T^* is the indifference point solving

$$\int_0^1 \pi(\theta_T^*, l)dl = 0,$$

which has a unique solution by A3.

Knowing the equilibrium strategy of the final stage game, we can compute the expected profit $\tau^i V_T$. In the limit $\sigma \to 0$, all players invest if and only if

[5]This is satisfied by the payoff functions in applications 3.1 and 3.2.

the fundamental $\theta_T > \theta_T^*$. In that case $l_T = 1$ and all receive $\tau_T^i[\pi(\theta_T, 1) + b(1, \theta_T, 1)V_{T+1}]$ where $V_{T+1} = 0$ as the game does not continue beyond the round T. If $\theta_T < \theta_T^*$ all players wait and receive $\tau_T^i V_{T+1} \equiv 0$. Thus we have

$$V_T = \int_{-\infty}^{\theta_T^*} \Big[\delta b(0, \theta, 0) \times V_{T+1}\Big] d\Phi(\theta) + \int_{\theta_T^*}^{+\infty} \Big[\pi(\theta, 1) + \delta b(1, \theta, 1) \times V_{T+1}\Big] d\Phi(\theta).$$

The stage game at $T - 1$ is again a static coordination problem but, in contrast to stage T, players also influence the continuation payoff at T by influencing τ_T^i. The reduced payoff at $T - 1$ is $b(0, \theta_{T-1}, l_{T-1})V_T$ for waiting, and $\pi(\theta_{T-1}, l_{T-1}) + b(1, \theta_{T-1}, l_{T-1})V_T$ for investing. We will refer to this interaction as the *modified* stage game. Its equilibrium is determined by the payoff differential between investing and waiting: $\tilde{\pi}_{T-1}(\theta, l) = \pi(\theta, l) + \delta\rho(\theta, l)V_T$.

The payoff differential $\tilde{\pi}_{T-1}(\cdot, \cdot)$ satisfies all of the assumptions of Proposition 2.2 in Morris and Shin (2003): 1. Action monotonicity is implied by A1 and A7. 2. State monotonicity is implied by A2 and A8. 3. Strict Laplacian state monotonicity is implied by A3 and A8. 4. Uniform limit dominance is assumed in A10. 5. Continuity is implied by A5 and A9. 6. Finite expectation of signals is assumed in A6.

The threshold at $T-1$ is again an indifference point of a player with Laplacian beliefs; it solves $\int_0^1 \tilde{\pi}_{T-1}(\theta_{T-1}^*, l)dl = 0$ and we can continue to solve earlier rounds.

Proposition 1. *For any $\epsilon > 0$, there exists $\overline{\sigma}$ such that for all $\sigma < \overline{\sigma}$, if strategy s survives iterated elimination of dominated strategies in the game Γ_σ, then $s_t(x_t) = 0$ for all $x_t \leq \theta_t^* - \epsilon$ and $s_t(x_t) = 1$ for all $x_t \geq \theta_t^* + \epsilon$ for all $t \in \{1, \ldots, T\}$, where $\theta_t^* \equiv \vartheta(V_{t+1})$ is the unique solution to*

$$\int_0^1 \Big[\pi(\theta_t^*, l) + \delta\rho(\theta_t^*, l)V_{t+1}\Big] dl = 0 \tag{1}$$

and V_t is detemined recursively by

$$V_t = G(V_{t+1}) \equiv \int_{-\infty}^{\vartheta(V_{t+1})} \delta b(0, \theta, 0)V_{t+1} d\Phi(\theta) + \int_{\vartheta(V_{t+1})}^{+\infty} \Big[\pi(\theta, 1) + \delta b(1, \theta, 1)V_{t+1}\Big] d\Phi(\theta), \tag{2}$$

together with the boundary condition $V_{T+1} = 0$.

Proof. Follows from the text above and from Proposition 2.2. in Morris and Shin (2003). □

2.3 Evolution of Thresholds

The thresholds are a function of the expected continuation values, $\theta_t^* = \vartheta(V_{t+1})$. Thus the equilibrium is fully determined by the continuation values V_t which evolve according to the mapping $V_t = G(V_{t+1})$. The long-run behavior of the system is determined by the fixed points of $G(\cdot)$ and their stability. For many setups, the mapping $G(\cdot)$ has a unique, unstable fixed point, in which case the thresholds necessarily fluctuate and the system never converges to a steady state.

For instance, consider payoff $\pi(\theta, l) = \theta - 1 + l$ and let each player invest only once, $b(1, \theta, l) = 0$, $b(0, \theta, l) = 1$. The equations (1) and (2) simplify into

$$\theta_t^* = \vartheta(V_{t+1}) = \frac{1}{2} + \delta V_{t+1},$$

$$V_t = G(V_{t+1}) = \delta V_{t+1} \Phi\left(\frac{1}{2} + \delta V_{t+1}\right) + \int_{\frac{1}{2}+\delta V_{t+1}}^{+\infty} \theta d\Phi(\theta).$$

Let us plot a qualitative picture of the mapping $G(\cdot)$. An increase in V_{t+1} has two effects on V_t. The direct effect is positive: if the realization of θ_t happens to be low and players wait, then they receive a higher continuation payoff δV_{t+1}. However, the strategic effect is negative: the threshold θ_t^* increases because players have more to lose by investing if V_{t+1} is high. This decreases V_t because the players coordinate on investing at t less often. The relative size of the two effects depends on V_{t+1}. Let us, qualitatively, distinguish three regions of V_{t+1} given the distribution of fundamentals as on Figure 1.

1. A very low value of V_{t+1} means a very low threshold θ_t^*, and thus players almost always coordinate on investment; the continuation value thus affects V_T only minimally. Both effects are negligible. This is the plateau on the graph of $G(\cdot)$ in Figure 1.

2. An increase in V_{t+1} causes θ_t^* to fall into the main mass of the probability distribution $\Phi(\cdot)$, where the negative strategic effect is strong and it over-balances the positive direct effect. A small increase in V_{t+1} substantially

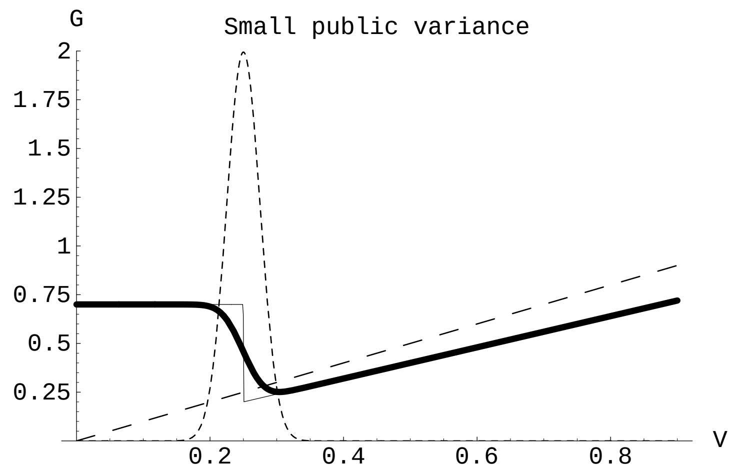

Figure 1: Thick line — mapping $G(\cdot)$ for a nonzero variance of priors $\tau > 0$. Thin line — mapping $G(\cdot)$ in the limit $\tau \to 0$. Dotted line — the probability density $\phi(\vartheta(V_{t+1}))$ at the threshold $\theta^*_T = \vartheta(V_{t+1})$. Dashed line — diagonal.

decreases the probability of successful coordination, and hence V_t sharply decreases.

3. Higher values of V_{t+1} lead to a threshold θ^*_t so high that players almost always coordinate on waiting. Hence, the strategic effect is negligible and the direct effect causes V_t to grow as δV_{t+1}. This is the region of the steady increase of $G(\cdot)$ on the right of Figure 1.

The stability of the fixed point depends on the region in which $G(\cdot)$ crosses the diagonal. The condition $\frac{1}{2} < E[\theta] < \frac{1}{2}/(1-\delta)$ ensures that, for $Var[\theta]$ sufficiently small, the fixed point is unique and unstable. In such a case, thresholds evolve in a regular cycle of fixed periodicity or in a chaotic path. Figure 2 depicts a numerical example of a fluctuating threshold path for particular parameters.[6] The booms and slumps are random events depending on the realized fundamentals, with booms more probable in rounds with low thresholds. Thus the rounds with low thresholds can be interpreted as times of investor's optimism — players (correctly) believe that their opponents will invest even if the realized fundamental happens to be low.

Equilibrium uniqueness holds for any T, and if the mapping $G(\cdot)$ has a unique

[6]Prior beliefs distribution $N(0.6, 0.01^2)$ and $\delta = 0.8$.

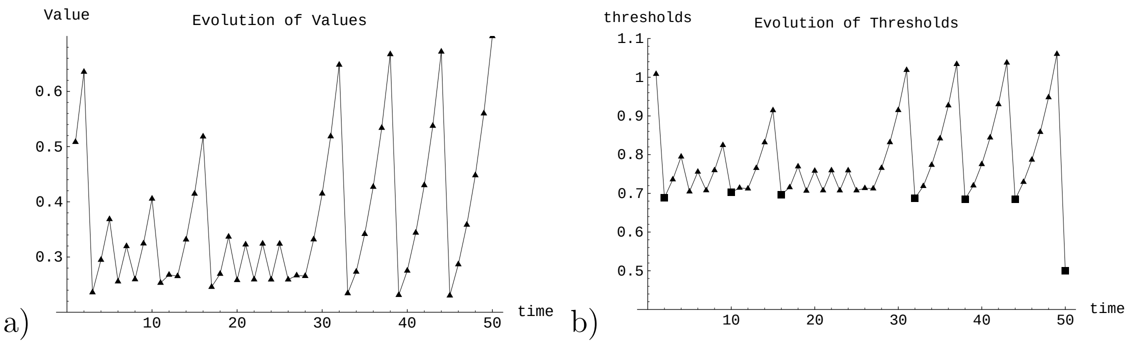

Figure 2: a) Evolution of the expected continuation values V_t generated by the mapping $G(\cdot)$. b) Evolution of the thresholds $\theta_t^* = \vartheta(V_{t+1})$. The symbol ■ denotes periods in which players coordinated on investment for one particular realization of random fundamentals $\{\theta_1, \ldots, \theta_T\}$.

and unstable fixed point, permanent fluctuations must occur even for very large, but finite T. Next, let us consider a game with infinite time horizon. In such a game, the equilibrium uniqueness result does not hold, as the boundary condition $V_{T+1} = 0$ is lost; nevertheless, the continuation values V_t still evolve according to the mapping $V_t = G(V_{t+1})$ in any equilibrium. Thus, in the case of a unique unstable fixed point, although we cannot specify a unique sequence of V_t, we know that the values and the thresholds fluctuate. Hence the main prediction of the model is the existence of fluctuations rather than any particular equilibrium path.

3 Applications

We present three simple models illustrating the basic framework within an economic context: a currency attack model building on Morris and Shin (1998), a model of co-moving crises, and a model of search cycles. While the solution of the first model is a straightforward application of Proposition 1, the other two models slightly generalize the basic framework. The crises model consists of two independent time series of coordination problems observed by a common pool of investors. The equilibrium actions happen to be correlated across the two series despite that the fundamentals are not. The model of search differs from the basic model in the details of the dynamic link among rounds.

3.1 Currency Attacks

In the first application we extend the Morris and Shin (1998)[7] model of currency crises by adding a continuation structure — unsuccessful speculators may go bankrupt and thus lose access to future profits. Morris and Shin consider a currency pegged to an exchange rate e^* which, if the government does not protect the peg, will float to a rate $\zeta(\theta_t) \leq e^*$, where the function $\zeta(\cdot)$ is continuous and increasing. A continuum of speculators with measure 1 decide whether or not to sell the currency short. The transaction cost of short-selling is c. If the currency is devaluated, short-selling pays a net profit $e^* - \zeta(\theta_t) - c$. The government defends the peg, but only if it is not too costly. The cost of defending increases with the measure of the short sales; the government will defend if the measure of attacking speculators is smaller than $a(\theta_t)$, which is continuous and increasing in the economic fundamental θ_t. The instantaneous payoff for not attacking is 0. The instantaneous payoff for attacking is summarized by

$$\pi(\theta_t, l_t) = \begin{cases} e^* - \zeta(\theta_t) - c & \text{if } a(\theta_t) < l_t, \\ -c & \text{if } a(\theta_t) \geq l_t. \end{cases} \tag{3}$$

The authors assume the existence of dominance regions.[8] The informational structure is that of a global game. The function $\pi(\theta_t, l_t)$ is weakly monotone[9] in θ_t and l_t, the Laplacian state is unique, and thus the static stage game is a global game.

We extend this model by assuming that an unsuccessful speculation results in bankruptcy with probability β. Alternatively, we could assume that managers responsible for the attack decision get fired if the attack fails (Chevalier and Ellison, 1999), in which case they miss bonuses based on future profits. The speculative capital of unsuccessful speculators is assumed to end up in the hands of other speculators after the bankruptcy, so the measure of the potential speculative capital is 1 in all rounds. Abandoning the peg makes further attacks impossible, so all players have zero future profits after the successful attack regardless of their

[7] See also Heinemann (2000).

[8] The government devaluates for sufficiently bad fundamentals even without any speculators, and even a coordinated attack of all speculators will not lead to devaluation for sufficiently good fundamentals.

[9] $\pi(\theta, l)$ decreases with θ whereas, formally, Proposition 1 requires π increasing in θ. Such a situation can be accommodated by introducing $\tilde{\theta} = 1 - \theta$.

	$l_t > a(\theta_t)$	$l_t \leq a(\theta_t)$
Attack	0	$(1-\beta)$
Wait	0	1

Table 1: Continuation probability $b(a_t^i, l_t, \theta_t)$.

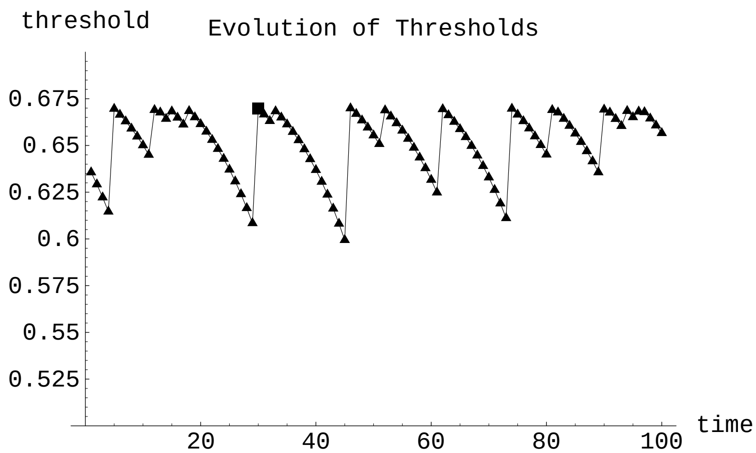

Figure 3: Evolution of the thresholds below which the speculators attack. The attack is more probable when the threshold is high. A (successful) attack has happened in the period denoted by the symbol ■, for a particular realization of the random fundamentals.

action. The continuation probabilities $b(a^i, \theta_t, l_t)$ are summarized in Table 1, and the process satisfies A7–A10.

Applying Proposition 1 we get:

Corollary 1. *Proposition 1 applies with thresholds* $\theta_t^* = \vartheta(V_{t+1})$ *solving equation*

$$[1 - a(\theta_t^*)][e^* - \zeta(\theta_t^*)] - a(\theta_t^*)\delta\beta V_{t+1} = c. \tag{4}$$

The evolution of expected future payoffs is determined by

$$V_t = G(V_{t+1}) = \int_{-\infty}^{\vartheta(V_{t+1})} (e^* - \zeta(\theta) - c)\, d\Phi(\theta) + \delta V_{t+1}[1 - \Phi(\vartheta(V_{t+1}))], \tag{5}$$

together with the boundary condition $V_{T+1} = 0$.

To illustrate the result, we study a numerical example with the exchange rate difference $e^* - \zeta(\theta)$ being constant and equal to 1. The function $a(\theta)$ describing the willingness of the government to protect the peg is set to $a(\theta) = \theta$. Equation (4) simplifies to $\vartheta(V_{t+1}) = \frac{1-c}{1+\delta\beta V_{t+1}}$ and (5) simplifies to $G(V_{t+1}) = \Phi(\vartheta(V_{t+1}))(1-c) + \delta V_{t+1}[1 - \Phi(\vartheta(V_{t+1}))]$. We plot the evolution of thresholds for particular parameters[10] in Figure 3. Periods with high thresholds are windows of probable attacks because the speculators (correctly) believe that others attack even if the fundamentals of the economy are quite high.

3.2 Emerging Markets Crises — Co-movement

The framework of Section 2 combines endogenous fluctuations of beliefs with equilibrium uniqueness and thus it is suitable for a study of beliefs-based contagion of crises. We consider two developing countries without any direct links but with a common set of investors. We assume that investment in either of the two countries may cause bankruptcy. The effect of bankruptcy on the unsuccessful investor is the same, regardless of the country in which the unsuccessful investment has been realized — future profits in *both* countries are lost. Thus, the willingness to risk investment at date t is influenced in both countries by the *common* value V_{t+1}. High future profits, regardless of the country in which they will be realized, undermine coordination in *both* countries today, which causes co-movement of the willingness to invest:

We consider two emerging market countries A and B with economic fundamentals $\theta_{A,t}$ and $\theta_{B,t}$ respectively, each with a continuum of investment opportunities of measure 1. There is a continuum of investors of measure 1, of which each observes two investment opportunities in each round — one in A and one in B. In each round t, each investor can invest in the project she observes in either country, in both, or in neither. The instantaneous payoff to an investor is the sum of returns from her current investments in A and B. The investments are, within each country, strategic complements. To keep the problem within the simple global games framework, we do not allow the players to choose the amount of their investment; they choose in each country only whether to invest one unit or not. We

[10]The prior distribution is $N(0.67, 0.001^2)$, $\beta = 0.5$, $c = 0.3$, $d = 0.9$.

assume simple return functions:

$$\pi_c(\theta, l) = \begin{cases} 1 - \gamma_c & \text{if } l > 1 - \theta, \\ -\gamma_c & \text{if } l \leq 1 - \theta, \end{cases}$$

where $\gamma_c > 0$ for $c \in \{A, B\}$. The payoff from not investing is 0. The return in A does not depend on investment $l_{B,t}$ or the fundamental $\theta_{B,t}$, and vice versa. The fundamentals $\theta_{A,t}$ and $\theta_{B,t}$, drawn from distributions $\Phi_A(\cdot)$ and $\Phi_B(\cdot)$, respectively, are independent across countries and times, so the instantaneous payoffs and the distributions cannot themselves explain any correlation in economic outcomes.

We now introduce a continuation structure which leads to correlation in investment in the otherwise independent countries. Investment in country c causes bankruptcy of the investor with probability b_c. More precisely, if a player does not invest in either of the countries, then the probability of bankruptcy is 0; if she invests only in country c, then the probability is b_c; if in both countries, then the probability is $b_A + b_B$. We do not model a detailed mechanism of bankruptcy. The bankruptcy is a black box for distress that a company (or manager) may meet in an emerging market and which may constrain the company's (manager's) future activities. Bankruptcy, although it was caused by a problem in *one* country, precludes the player from operating in *both* A and B in all future rounds. We assume that events in A and B that lead to bankruptcy are independent.

The modified stage game of country $c \in \{A, B\}$ in period t is described by the payoff difference between investing and not investing:

$$\tilde{\pi}_{c,t}(\theta_{c,t}, l_{c,t}) = \pi_c(\theta_{c,t}, l_{c,t}) - b_c \delta V_{t+1}, \tag{6}$$

which constitutes two independent global games, one for each country. These can be solved in each round, so we can again solve the game backwards.

Proposition 2. *The game has, for generic parameter values, a unique equilibrium (in the limit of precise signals $\sigma \to 0$). Investors invest in country $c \in \{A, B\}$ at date t if and only if the fundamentals $\theta_{c,t}$ are above the threshold $\theta^*_{c,t}$ where*

$$\theta^*_{c,t} \equiv \vartheta_c(V_{t+1}) = \gamma_c + b_c \delta V_{t+1} \tag{7}$$

if $\gamma_c + b_c\delta V_{t+1} < 1$*, and* $\vartheta_c(V_{t+1}) = +\infty$ *otherwise.*

The evolution of V_t *is determined by*

$$
\begin{aligned}
V_t = G(V_{t+1}) \quad \equiv \quad & \Phi_A(\vartheta_A(V_{t+1}))\delta V_{t+1} + \Phi_B(\vartheta_B(V_{t+1}))\delta V_{t+1} + \\
& (1 - \Phi_A(\vartheta_A(V_{t+1})))[1 - \gamma_A + (1 - b_A)\delta V_{t+1}] + \\
& (1 - \Phi_B(\vartheta_B(V_{t+1})))[1 - \gamma_B + (1 - b_B)\delta V_{t+1}],
\end{aligned} \tag{8}
$$

together with the boundary condition $V_{T+1} = 0$.

Proof. The incentive $\tilde{\pi}_{c,t}(\theta_{c,t}, l_{c,t})$ to invest in country c described by (6) satisfies Assumptions A1, A2, A3, A5. The noise distribution $f(\cdot)$ satisfies A6. The modified payoff $\tilde{\pi}_{c,t}(\cdot, \cdot)$ exhibits dominance regions for any $V_{t+1} > 0$ except when $1 - \gamma_c - \delta b_c V_{t+1} = 0$. Thus, unless $V_{t+1} = \frac{1-\gamma_c}{\delta b_c}$, which happens only for non-generic parameters, the dominance regions exist. Therefore, coordination problems of both countries at each stage are global games, and the thresholds[11] $\vartheta_c(V_{t+1})$ are the solutions to the equation $\int_0^1 \tilde{\pi}_{c,t}(l, \theta)dl = 0$, which gives (7). Equation (8) describes that, in the limit of precise signals, all players invest if and only if $\theta_{c,t} > \theta^*_{c,t} = \vartheta_c(V_{t+1})$, in which case they receive the instantaneous payoff $1 - \gamma_c$ and go bankrupt with probability b_c; players wait if $\theta_{c,t} < \vartheta_c(V_{t+1})$ and receive δV_{t+1}. □

We have generated the random fundamentals and marked the crises during which investors do not invest by ■, see Figure 4.[12] The fluctuations of the thresholds and the occurrence of the crises are correlated across the two countries despite the lack of direct links between them. A high threshold above the country's average fundamental means that crisis is probable, as investors will (correctly) believe that others invest only if the realized fundamental is high. The high thresholds can thus be interpreted as pessimistic market sentiments.

The effect is related to changes in the degree of strategic risk caused by a wealth increase, and the implied decrease of absolute risk aversion studied in Goldstein and Pauzner (2003). In contrast to this approach, our fluctuations

[11] Or investing is trivially dominated if $V_{t+1} > \frac{1-\gamma_c}{\delta b_c}$.

[12] $\gamma_A = 0.3$, $\gamma_B = 0.5$, prior beliefs distribution in country A is $N(0.78, 0.002^2)$; in country B $N(0.82, 0.001^2)$, $\delta = 0.9$, $b_A = 0.15$, $b_B = 0.1$.

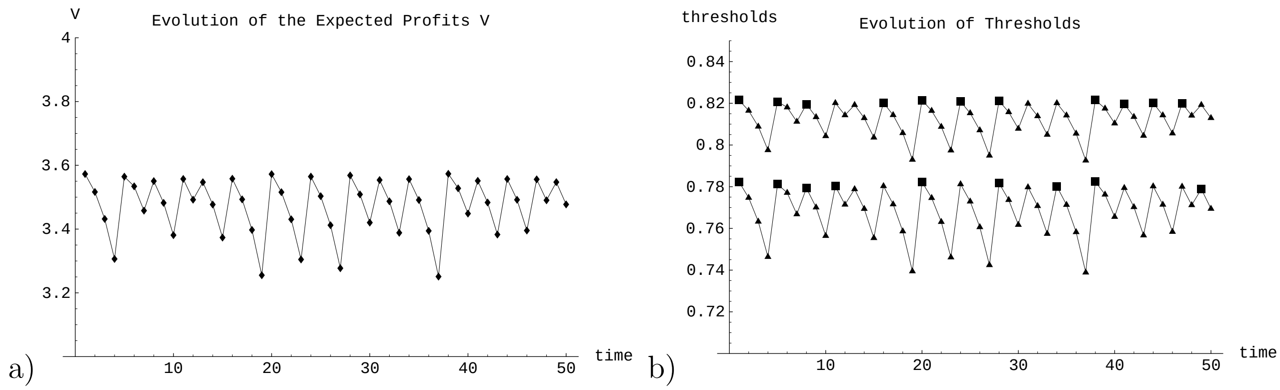

Figure 4: a) The evolution of the future expected profits V_{t+1} is common for both countries. b) As a consequence, the evolution of the thresholds θ_A^* and θ_B^* is correlated. The probability of crisis is high when the thresholds are high. The symbol ■ denotes a crisis for one particular realization of random fundamentals.

of strategic risk are caused by changes in the lottery rather than by changes in risk attitudes; our players are risk-neutral. Another difference is that our model, compared to Goldstein and Pauzner, has a reverse causality: profits tomorrow influence strategic risk today, whereas wealth accumulated yesterday influences risk aversion today in the Goldstein and Pauzner model. A crisis in A at t is not caused by a crisis in B at t or earlier in our model. Rather, the correlation of crises is caused by the commonality of the expected future profit. Thus, the outcome of our model is a contagion in the broad sense of an excess co-movement, but not in a narrow sense requiring a causal link from an earlier crisis to a later one.

3.3 Fluctuations of Search Activity

In this section we study a model of one-sided search in which, unlike in the two previous applications, the ability to search tomorrow is not decreased by any outcome of the search today. Rather, the incentive to search is determined by the current expected difference in continuation values between employed and unemployed players. It is easier to find a partner if potential partners are actively searching for a partnership than in a society where nobody else searches, see Diamond (1982). High future search activity decreases the current incentive to search, because it makes long term unemployment unlikely and thus decreases the disadvantage of being currently unemployed. The model again exhibits a negative link between future and current activity levels, which leads to endogenous fluctuations in search activity and in the unemployment level.

There is a continuum of identical players, each player needing a partner to produce. Players receive an instantaneous payoff of 1 in each round in which they have a partner, in which case we call them employed. After the payoff is received, the partnership survives into the next period with probability $0 < p < 1$, or dissolves with probability $1-p$. Players without a partner receive 0 instantaneous payoff, we call them unemployed, and they can search for a partner by incurring (stochastic) cost θ_t drawn from a c.d.f. $\Phi(\cdot)$.[13]

An unemployed player who searches at t finds a partner with probability $m_t = m(l_t)$, where l_t is the relative share of searching players among the unemployed ones, not their absolute number,[14] which effectively renormalizes the measure of players to 1 in each round. The function $m(l_t)$ is assumed to be increasing, and for simplicity, we let $m(l_t) = l_t$.

Proposition 3. *The game has a unique equilibrium (in the limit of precise signals $\sigma \to 0$). Let $V_{e,t}$, $V_{u,t}$ be the expected continuation values for employed and unemployed players respectively, and let $\Delta_t = V_{e,t} - V_{u,t}$ be the expected payoff advantage of an employed player.*

The threshold search cost θ_t^*, below which unemployed players search, is

$$\theta_t^* = \vartheta(\Delta_{t+1}) = \frac{\delta \Delta_{t+1}}{2}. \tag{9}$$

The evolution of the expected payoff advantage Δ_t of having a partner is determined by

$$\Delta_t = G(\Delta_{t+1}) = 1 + \int_{-\infty}^{\vartheta(\Delta_{t+1})} \theta d\Phi(\theta) + [p - \Phi(\vartheta(\Delta_{t+1}))]\delta\Delta_{t+1}, \tag{10}$$

[13] We assume that costs are sometimes prohibitively high which implies the existence of the right dominance region, and that the costs are sometimes negative, which implies the existence of the left dominance region. This can be justified by government paying a subsidy for the search, which exceeds the true costs, or by an intrinsic motivation exceeding pecuniary costs.

[14] This can be justified in the following way: Unemployed players first simultaneously decide whether to prepare for future production by incurring cost θ_t. They are afterwards randomly matched to pairs and partnership is formed if both members of a pair are prepared.

together with the boundary condition

$$\Delta_{T+1} = 0.$$

Proof.

Lemma 1. $\Delta_t > 0$ *for all* $t = 1, \ldots T$.

Proof. (Lemma 1) In the appendix. $\square$

Fix t and suppose $V_{e,t+1}$, $V_{u,t+1}$ are uniquely determined. The employed players face no decisions and receive the expected payoff

$$V_{e,t} = 1 + p\delta V_{e,t+1} + (1-p)\delta V_{u,t+1}. \quad (11)$$

Unemployed players face a coordination problem characterized by the payoff

$$u(a_t, l_t, \theta_t) = \begin{cases} l_t \delta V_{e,t+1} + (1-l_t)\delta V_{u,t+1} - \theta_t & \text{if } a_t^i = 1, \\ \delta V_{u,t+1} & \text{if } a_t^i = 0. \end{cases}$$

An unemployed player's incentive to search is

$$\tilde{\pi}_t(\theta_t, l_t) \equiv u(1, l_t, \theta_t) - u(0, l_t, \theta_t) = \delta \Delta_{t+1} l_t - \theta_t.$$

Thus the modified stage game satisfies A1-6 and can be solved as a global game.[15] The threshold $\vartheta(V_{t+1})$ in equation (9) is the unique solution of $\int_0^1 \tilde{\pi}_t(\theta, l) dl = 0$. Given the threshold, we may express the expected profits of an unemployed player as

$$V_{u,t} = \delta V_{e,t+1} \Phi(\vartheta(\Delta_{t+1})) - \int_{-\infty}^{\vartheta(\Delta_{t+1})} \theta d\Phi(\theta) + \delta V_{u,t+1}[1 - \Phi(\vartheta(\Delta_{t+1}))]. \quad (12)$$

The function $G(\Delta_{t+1})$ in (10) can be found by subtracting (12) from (11). $\square$

We compute the evolution of Δ_t for particular[16] parameters, see Figure 5. The

[15] $\tilde{\pi}_t(\theta_t, l_t)$ decreases in θ instead of increasing as required in A2, but this can be accommodated by introducing $\tilde{\theta} = 1 - \theta$.

[16] The prior beliefs distribution is $N(1.3, 0.05^2)$, $p = 0.95$, $\delta = 0.9$; the ratio of partnerless players at $t = 0$ is 0.1.

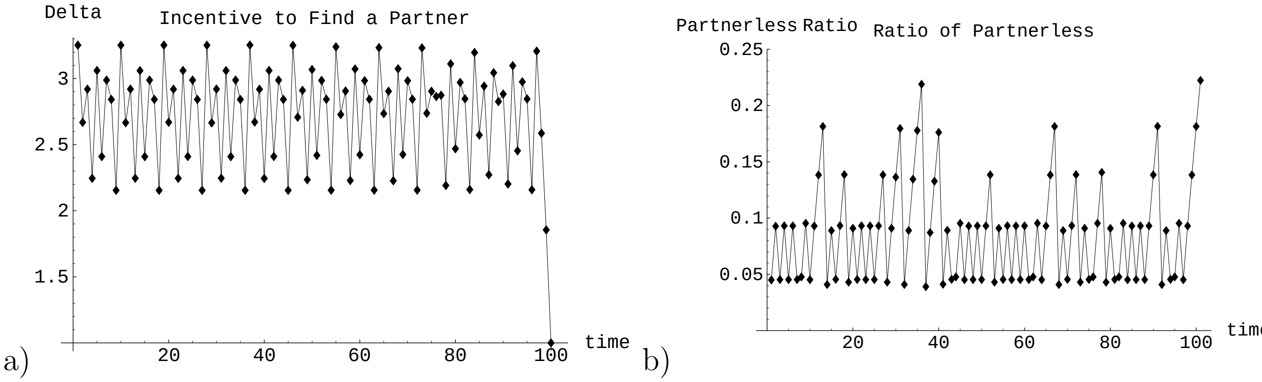

Figure 5: a) Evolution of the advantage of being partnered $\Delta_t = V_{e,t} - V_{u,t}$. b) Evolution of the unemployment level for a particular realization of the random fundamentals. A decrease is more probable when Δ_{t+1} is high.

value of Δ_t oscillates. When tomorrow's advantage Δ_{t+1} of being employed is high, players coordinate on searching even if the search costs are relatively high. Thus, the probability that the unemployment level falls at t is increasing in Δ_{t+1}.

Like Diamond (1982), we have limited ourselves to a one-sided search model, in which we do not distinguish the roles of employers and employees; rather, any pair of players can form a productive pair. The advantage is that we stay within the framework of simple global games in which all players are the same *ex ante*. Both our model and Diamond's model admit fluctuations in the measure of partnerless players, which Diamond interprets as unemployment fluctuations. However, whereas the fluctuations are a possible outcome of Diamond's model, they are a necessary outcome in our model.

4 Dynamic Global Games Literature

The dynamic global games literature can be organized according to the assumed intertemporal links.

Frankel and Pauzner (2000), Burdzy, Frankel and Pauzner (2001) and Frankel and Burdzy (2005), study a series of coordination problems in which fundamentals evolve according to a stochastic process and players experience frictions in changing their actions. These models have a unique equilibrium in which the actions may depend not only on the current fundamental, but also on the current level of investment. The models thus, under some specifications of parameters, exhibit hysteresis. Oyama (2004) assumes complementarities between actions of successive generations in an OLG model. This model also exhibits hysteresis; the equilibrium strategy is characterized by two thresholds $\theta^* < \theta^{**}$. During a boom, players wait only if the fundamental falls below θ^*, but afterwards they return to investing only after the fundamental rises above θ^{**}, and vice versa. Because of the hysteresis, the coordination outcome can differ across two instances in these models even if the fundamentals happen to be identical. However, fluctuations of equilibrium behavior occur only after an exogenous, though possibly small shock to the economic fundamental; the economy cannot shift during a period in which the fundamental happens to be stationary.

Morris and Shin (1999)[17] consider another dynamic link: past fundamentals serve as a public signal for the current period. Such a public signal influences equilibrium behavior if the noise is non-vanishing. The thresholds θ_t^* thus fluctuate, but they are exogenously determined by the realizations of θ_{t-1}.

Angeletos, Hellwig and Pavan (forthcoming) allow players to postpone decisions on an attack for the sake of acquiring additional information about the fundamental. The economic fundamental is kept constant, but private beliefs evolve as the players observe the outcomes of previous attacks and receive new private signals. The failure of a more aggressive attack yesterday conveys a more pessimistic signal about the current fundamental, and hence hinders today's attack, compared to a less aggressive attack yesterday — this negative link allows for equilibrium fluctuations in the intensity of attacks. The fluctuations can be entirely fundamental-independent, and driven solely by changes in beliefs, thanks to the exogenous arrival of new information.

The next subsection is devoted to the part of the dynamic global games literature that is most closely related to the paper at hand.

[17] See also Chamley (1999) who lets players receive information about previous fundamentals by observing the history of investment levels.

4.1 Recursive Global Games

Giannitsarou and Toxvaerd's (2003) general study of recursive global games is a natural benchmark for the model at hand. As in our model, players interact in a finite series of interrelated stage games. The state of a player is described by an idiosyncratic variable τ_t^i, and the state of the economic environment by common variables θ_t and w_t. The informational structures of this and our model are identical and typical of the global games literature; players observe θ_t with small idiosyncratic errors, but more generally than our basic model, the authors allow θ_t to follow a Markov process. The variables τ_t^i and w_t are defined recursively:

$$\tau_{t+1}^i = b(\tau_t^i, a_t^i, \theta_t),\ w_{t+1} = c(w_t, l_t, \theta_t).$$

The authors assume that the transition functions b and c are increasing in all arguments, the instantaneous payoff function $u(a^i, \theta, l, \tau^i, w)$ is supermodular in all pairs of arguments, and the distribution of θ_t stochastically increases in θ_{t-1}.

The authors solve the game by backwards induction. The modified payoff at each stage is

$$u(a_t^i, \theta_t, l_t, \tau_t^i, w_t) + \delta V_{t+1}(\theta_t, l_t, \tau_t^i, w_t),$$

where $V_{t+1}(\theta_t, l_t, \tau_t^i, w_t)$ is the expected continuation payoff conditional on its arguments. The main result is that the modified stage game in each round is a global game with a unique equilibrium solvable by the techniques of Frankel, Morris, and Pauzner (2003).

The result rests on the assumption that all of the intertemporal links in the model are of a positive nature. For instance, today's high investment increases future investment, because high future w will increase future motivation to invest. In turn, high future investment motivates players to increase current investment in order to increase the future involvements τ^i. Thus, the positive intertemporal links further strengthen the contemporaneous complementarity.

Interestingly, there are dynamic economic processes that violate the positive links assumed in Giannitsarou and Toxvaerd (2003), and yet each modified stage game is a global game with a unique equilibrium. Toxvaerd (2004) studies a dynamic interaction in which the static stage game violates the complementarity in each round, and yet the modified stage games, taking into account the continuation payoff, are global games. Another example that deviates from the positive dynamic links in Giannitsarou and Toxvaerd (2003) and yet is solvable by the iteration of global games techniques is the model at hand. The future involvement τ^i does not increase with a player's action; on the contrary, it "wears out" if the player currently invests. This negative link — tomorrow's high investment motivates players to invest less today — distinguishes the paper at hand from both Giannitsarou and Toxvaerd (2003) and Toxvaerd (2004), in which the link between tomorrow's investment and today's investment is positive. Thus while these models result in an endogenous growth or decline, our model offers a framework suitable for an analysis of endogenous cycles.

5 Generalizations

In the next two sections, we enrich the basic model from Section 2 with some of the structure of Giannitsarou and Toxvaerd (2003). We allow for a Markov process in the evolution of θ_t in Section 5.1. Changes in the number of players are allowed in Section 5.2, which can be understood within Giannitsarou's and Toxvaerd's framework as a change in an aggregate state variable w_t. As stressed in the previous section, our model violates Giannitsarou's and Toxvaerd's (2003) assumption of the positive intertemporal links: the number of future participating players, and the individual involvement decrease with current investment. Consequently, these generalizations are not straightforward, and we need to impose certain restrictions to obtain equilibrium uniqueness. We impose a restriction on the degree of the intertemporal dependence of fundamentals in Section 5.1, and we limit the number of players to two in Section 5.2. For the sake of simplicity, we assume a simple evolution of the involvement τ_t^i: players are allowed to invest only once, so that $b(1, \theta, l) = 0$ and $b(0, \theta, l) = 1$ in both Sections 5.1 and 5.2. While we examine the two generalizations in the two independent extensions of the basic model, these can be easily combined into one.

5.1 Relaxing the I.I.D. Nature of Shocks

The fundamental θ_t follows a Markov process; θ_t is distributed according to $\Phi(\theta_t; \theta_{t-1})$, with a starting distribution $\Phi_0(\theta_1)$ in Round 1. As in the basic model from Section 2, θ_{t-1} is observable at t. We assume

A11 $\Phi(\cdot; \theta'_{t-1})$ weakly first order stochastically dominates $\Phi(\cdot; \theta_{t-1})$ whenever $\theta'_{t-1} \geq \theta_{t-1}$.

The equilibrium continuation value $V_{t+1}(\theta_t)$ is no longer a constant as it was in the case of i.i.d. fundamentals. Rather, it depends on θ_t through the influence on the distribution of θ_{t+1} and the modified stage payoff $\tilde{\pi}_t(\theta, l) = \pi(\theta, l) - \delta V_{t+1}(\theta)$ may fail to exhibit strategic complementarities. In order to assure the state monotonicity of the modified payoff, $V_{t+1}(\theta)$ must not increase in θ too quickly compared to $\pi(\theta, l)$. For this reason, we impose additional assumptions on the instantaneous payoff function:

A12 The derivative $\frac{\partial}{\partial\theta}\pi(\theta, l)$ is bounded below by $m > 0$ and above by $n < +\infty$.

A13 $\pi(\theta, 1) - \pi(\theta, 0)$ is bounded above by $q < +\infty$.

The modified payoff $\tilde{\pi}_t(\theta, l)$ has a left-hand dominance region because $\tilde{\pi}_t(\theta, l) < \pi(\theta, l)$. One possible way to assure existence of a right-hand dominance region is to assume:

A14 There exists $\overline{\Phi}(\cdot)$ which first order stochastically dominates $\Phi(\cdot; \theta_{t-1})$ for any θ_{t-1}.

As a consequence, $V_t(\theta_{t-1})$ is bounded and A12 assures that $\lim_{\theta\to+\infty} \pi(\theta, 0) = +\infty$, and hence the right-hand dominance region exists.

Proposition 4. *Suppose A1–A6 and A11–A14. There exists $k > 0$, independent of the length of the game T, such that if $\frac{\partial}{\partial\theta_{t-1}} E[\theta_t|\theta_{t-1}] < k$ and $-\frac{\partial}{\partial\theta_{t-1}}\Phi(\theta_t; \theta_{t-1}) < k$, then the game has a unique equilibrium (in the limit $\sigma \to 0$). The thresholds θ_t^* and continuation values $V_t(\cdot)$ are uniquely determined by*

$$\int_0^1 \pi(\theta_t^*, l)dl - \delta V_{t+1}(\theta_t^*) = 0, \tag{13}$$

$$V_t(\theta_{t-1}) = \int_{-\infty}^{\theta_t^*} \delta V_{t+1}(\theta) d\Phi(\theta; \theta_{t-1}) + \int_{\theta_t^*}^{+\infty} \pi(\theta, 1) d\Phi(\theta; \theta_{t-1}), \tag{14}$$

together with the boundary condition $V_{T+1}(\theta) \equiv 0$, and the continuation value

$V_t(\theta_{t-1})$ *increases in* θ_{t-1} *for each* $t = 2, \dots T$.

Proof. Define $\overline{V}$ to be the solution to

$$\overline{V} = \int_{-\infty}^{+\infty} \max[\delta \overline{V}, \pi(\theta, 1)] d\overline{\Phi}(\theta). \tag{15}$$

This value is well defined because $\frac{\partial}{\partial \overline{V}} \int_{-\infty}^{+\infty} \max[\delta \overline{V}, \pi(\theta, 1)] d\overline{\Phi}(\theta)$ is below δ and the right-hand side of (15) is positive for $\overline{V} = 0$; hence (15) has a unique solution.

We solve the game by induction. Fix t, and suppose $V_{t+1}(\theta) \leq \overline{V}$ and $\delta \frac{\partial}{\partial \theta} V_{t+1}(\theta) < m$, both for all $\theta \in \mathbb{R}$. Then the modified stage game at t with the payoff $\tilde{\pi}_t(\theta, l) = \pi(\theta, l) - \delta V_{t+1}(\theta)$ satisfies all of the six global game assumptions. Therefore, it has a unique equilibrium described by (13) and (14).

For the induction argument, we need to show that $V_t(\theta) \leq \overline{V}$ and $\delta \frac{\partial}{\partial \theta} V_t(\theta) < m$. $V_t(\theta) \leq \overline{V}$ holds because the integrand in (14) is weakly smaller than the integrand in (15). We prove that $\delta \frac{\partial}{\partial \theta} V_t(\theta) < m$ in the next two lemmas.

Lemma 2. *Suppose* $t(\theta') - t(\theta) \geq u(\theta') - u(\theta)$ *whenever* $\theta' \geq \theta$. *Define* $T(\theta) \equiv \int_{-\infty}^{\infty} t(\tilde{\theta}) d\Phi(\tilde{\theta}; \theta)$ *and* $U(\theta) \equiv \int_{-\infty}^{\infty} u(\tilde{\theta}) d\Phi(\tilde{\theta}; \theta)$. *Then, under A11,*

$$T(\theta') - T(\theta) \geq U(\theta') - U(\theta), \tag{16}$$

whenever $\theta' \geq \theta$.

Proof. In the appendix. □

We have $V_t(\theta) = \int_{-\infty}^{+\infty} v_t(\tilde{\theta}) d\Phi(\tilde{\theta}; \theta)$, where

$$v_t(\theta) = \begin{cases} \pi(\theta, 1) & \text{if } \theta > \theta_t^*, \\ \delta V_{t+1}(\theta) & \text{if } \theta < \theta_t^*. \end{cases}$$

Next, we define $v'_t(\cdot)$ which increases more quickly than $v_t(\cdot)$:

$$v'_t(\theta) = \begin{cases} \pi(\theta_t^*, 1) + n(\theta - \theta_t^*) & \text{if } \theta > \theta_t^*, \\ \pi(\theta_t^*, 0) + n(\theta - \theta_t^*) & \text{if } \theta < \theta_t^*. \end{cases}$$

Lemma 3. $v'_t(\theta') - v'_t(\theta) \geq v_t(\theta') - v_t(\theta)$ *whenever* $\theta' \geq \theta$.

Proof. In the appendix. □

According to Lemma 2, $\delta \frac{\partial}{\partial\theta} V_t(\theta)$ is smaller than $\delta \frac{\partial}{\partial\theta} V_t'(\theta)$, where $V_t'(\theta) = \int_{-\infty}^{+\infty} v_t'(\tilde{\theta}) d\Phi(\tilde{\theta};\theta)$.

$$\delta \frac{\partial}{\partial \theta_{t-1}} V_t'(\theta_{t-1}) = \delta n \frac{\partial}{\partial \theta_{t-1}} E[\theta|\theta_{t-1}] - \delta q \frac{\partial}{\partial \theta_{t-1}} \Phi(\theta_t^*;\theta_{t-1}), \tag{17}$$

and (17) is, for $k > 0$ sufficiently small, smaller than m.

The induction assumption that $V_{t+1}(\theta) < \overline{V}$ and $\delta \frac{\partial}{\partial\theta} V_{t+1}(\theta) < m$ holds trivially for $t = T$ because $V_{T+1}(\theta) \equiv 0$. □

5.2 Evolving Number of Players

Two players repeatedly interact in the stage games with i.i.d. fundamentals. The static stage game payoff function $\pi(\theta, a^{-i})$ satisfies all of the global games assumptions. In particular, $\pi(\theta, a^{-i})$ satisfies A1, A2, A4, A5, A6. Assumption A3 must be modified for a finite set of players:
A3' $\frac{1}{2}[\pi(\theta,0) + \pi(\theta,1)]$ strictly increases in θ.

The investing players *are not replaced* by new entrants. Thus, if a player waits and her opponent invests, the player finds herself in a nonstrategic situation in the subsequent round with the instantaneous payoff for investing being $\pi(\theta, 0)$.

Proposition 5. *Suppose A1, A2, A3', A4, A5, A6 and A10. The game with evolving number of players has a unique equilibrium (in the limit $\sigma \to 0$). Each player at t conditions her investment $s^*(x_t^i, r)$ on her signal x_t^i and the number of remaining opponents $r \in \{0, 1\}$:*

$$s_T^*(x_t^i, r) = \begin{cases} 1 & \text{if } x_t^i > \theta_T^*(r), \\ 0 & \text{if } x_t^i < \theta_T^*(r). \end{cases}$$

The thresholds $\theta_t^*(r)$ are the unique solutions of

$$\sum_{a^{-i}=0}^{r} \left[\pi(\theta_t^*(r), a^{-i}) - \delta V_{t+1}(r - a^{-i})\right] = 0, \tag{18}$$

where

$$V_t(r) = \delta V_{t+1}(r)\Phi\left(\theta_t^*(r)\right) + \int_{\theta_t^*(r)}^{\infty} \pi(\theta, r) d\Phi(\theta), \tag{19}$$

with the boundary condition $V_{T+1}(r) = 0$*, for* $r = 0, 1$*.*

Proof. Fix t and suppose that the equilibrium continuation values at t satisfy $V_{t+1}(1) \geq V_{t+1}(0)$. Then the modified stage game with payoff function $\tilde{\pi}_t(\theta, a^{-i}) = \pi(\theta, a^{-i}) - \delta V_{t+1}(1 - a^i)$ satisfies A1, A2, A3', A4, A5 and A6. The modified stage game at t for two player interaction is thus a global game, and its equilibrium threshold solves (18) with $r = 1$. In the case that the opponent has invested before t, the player solves the nonstrategic problem and the optimal threshold is given by (18) with $r = 0$. The expected continuation payoff at t is given by (19).

It remains to show that $V_t(1) \geq V_t(0)$:

$$V_t(0) = \int_{-\infty}^{+\infty} \max[\pi(\theta, 0), \delta V_{t+1}(0)] d\Phi(\theta), \tag{20}$$

while $V_t(1)$ is weakly larger than

$$\int_{-\infty}^{+\infty} \max\left[\frac{\pi(\theta, 0) + \pi(\theta, 1)}{2}, \delta \frac{V_{t+1}(1) + V_{t+1}(0)}{2}\right] d\Phi(\theta), \tag{21}$$

because the players optimize under Laplacian beliefs. Both arguments of the max operator are weakly larger in (21) than in (20).

The initial statement that $V_{t+1}(1) \geq V_{t+1}(0)$ is trivially true for $t = T$, as $V_{T+1} = 0$, which closes the induction. □

The generalization of Proposition 5 to a larger number of players is not straightforward. The complication is that the continuation values $V_t(r)$ may fail to increase in the number of remaining opponents. In a static game, three players coordinate on investment more easily than two players, because the expected number of investing players under the Laplacian belief is larger in the first case. At the same time, in the dynamic game, the expected continuation values are larger for three players than for two, which makes coordination harder in the latter case. As a result, $V_t(r)$ may be non-monotone even if $V_{t+1}(r)$ is not, which may lead to multiple equilibria, as the modified stage game may fail to satisfy the action monotonicity condition. Thus, for a large number of players, the equilibrium thresholds may be uniquely determined for some stages at the end of the game, but multiple equilibria may exist in the earlier stage games.

6 Summary

Bankruptcy is worse prior to a boom than prior to a slump. Searching for a job today is more important if tomorrow's search prospects look grim, than if tomorrow looks bright. We have formalized this ideas in a dynamic global game model which consists of a series of simple static global games. The non-trivial dynamic link among the rounds is that players influence not only their instantaneous payoff, but also their ability to participate in future projects. Successful coordination tomorrow increases the strategic risk associated with bankruptcy today, and thus makes today's investment more risky. Coordination tomorrow thus undermines today's coordination, creating a negative feedback effect between tomorrow and today, which leads to cycles. The dynamic model inherits attractive features of static global games: it is dominance solvable and thus, in the unique equilibrium, fluctuations unconnected to economic fundamentals not only may happen, but are a certain outcome of the model.

The unique equilibrium with a chaotic path should not be taken as a literal prediction of behavior, because the slightest error in computation of thresholds would multiply greatly after a few iterations. It is extremely difficult to coordinate on such a chaotic equilibrium, and yet no other equilibrium exists. We believe that such a chaotic equilibrium of perfectly rational players is a benchmark for a dynamic system of boundedly rational agents. That the system of perfectly rational agents necessarily fluctuates suggests that a boundedly rational dynamic system would fluctuate as well.

A Appendix

Proof. **Lemma 1** We prove the statement by induction. Suppose $\Delta_{t+1} \equiv V_{e,t+1} - V_{u,t+1} \geq 0$. Then $\delta V_{u,t+1} \leq V_{u,t} \leq \delta V_{e,t+1}$. Using $V_{e,t} = 1 + \delta p V_{e,t+1} + \delta(1-p) V_{u,t+1}$, we get

$$\Delta_t \leq 1 + \delta p \Delta_{t+1},$$

$$\Delta_t \geq 1 - \delta(1-p)\Delta_{t+1}.$$

Let us denote the maximum and minimum of $\{\Delta_u\}_{u=t}^T$ by M and m. The equations imply

$$M \leq 1 + \delta p M,$$

$$m \geq 1 - \delta(1-p)M,$$

which gives

$$m \geq 1 - \frac{\delta(1-p)}{1-\delta p},$$

and the right-hand side is positive for all $0 < p < 1$ and $0 < \delta < 1$. $\square$

Proof. **Lemma 2** Inequality (16) is equivalent to

$$\int t(\tilde{\theta})d\Phi(\tilde{\theta};\theta') - \int t(\tilde{\theta})d\Phi(\tilde{\theta};\theta) \geq \int u(\tilde{\theta})d\Phi(\tilde{\theta};\theta') - \int u(\tilde{\theta})d\Phi(\tilde{\theta};\theta),$$

which in turn is equivalent to

$$\int [t(\tilde{\theta}) - u(\tilde{\theta})]d\Phi(\tilde{\theta};\theta') \geq \int [t(\tilde{\theta}) - u(\tilde{\theta})]d\Phi(\tilde{\theta};\theta). \tag{22}$$

The function $t(\cdot) - u(\cdot)$ is weakly increasing and thus, as $\Phi(\tilde{\theta};\theta')$ stochastically dominates $\Phi(\tilde{\theta};\theta)$, (22) holds. $\square$

Proof. **Lemma 3**
1. The statement holds above θ_t^* because $\frac{\partial}{\partial\theta}\pi(\theta, l) < n$.
2. The statement holds below θ_t^* because $\frac{\partial}{\partial\theta}\delta V_{t+1} < m \leq n$.
3. Both v_t and v_t' are discontinuous at θ_t^*. The discontinuity is larger in the case of v' than v because $\pi(\theta_t^*, 1) - \pi(\theta_t^*, 0) > \pi(\theta_t^*, 1) - \delta V_{t+1}(\theta^*) = \pi(\theta_t^*, 1) - \int_0^1 \pi(\theta_t^*, l)dl$. The last equality is implied by (13). $\square$

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