Communication Can Destroy Common Learning*

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Abstract

We show by example that communication can cause common knowledge acquisition to fail. In the absence of communication, agents acquire approximate common knowledge of some parameter, but with communication they do not.

1 Introduction

The significance of common knowledge in determining equilibrium outcomes of games has become well established since the seminal work of Lewis (1969). In settings where players acquire information over time, an important question is whether (approximate) common knowledge of certain events will eventually be attained. An interesting recent paper by Cripps, Ely, Mailath, and Samuelson (2008, henceforth CEMS) identifies conditions under which a parameter becomes common knowledge if agents privately learn the value of the parameter over time. They refer to common knowledge acquisition as "common learning."

In addition to private learning, economic agents frequently acquire information through communication. Intuitively, one might think that introducing communication could only help to achieve common learning since it improves the information agents have about each other's knowledge and beliefs. This intuition is false. We show by example that communication can cause common learning to fail. Our example exhibits common learning of an underlying parameter if agents do not communicate, but when communication is introduced according to a particular protocol, common learning does not occur. Moreover, the failure of common learning is profound; approximate common knowledge of the parameter fails uniformly across all periods in every state.

In the example, two agents, 1 and 2, independently observe the value of some underlying parameter, possibly at stochastic times. If the agents do not communicate, then the value of the parameter becomes approximate common knowledge since each agent eventually assigns high probability to the other agent having observed the parameter.

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In addition to direct observation of the parameter, agents communicate according to the following variant of the Rubinstein (1989) email game. When agent 1 observes the value of the parameter, she sends a message to agent 2, which is privately received following some stochastic delay.Upon receipt, agent 2 sends a confirmation message to agent 1, which is again subject to stochastic delay. Agent 1 in turn sends a confirmation to agent 2, and so on. There is no other communication. All communication is truthful and consists only of each agent (partially) reporting her own information to the other agent.

Under this communication protocol, common learning of the parameter fails (for some delay distributions). With communication, if agent 2 has not received the first message from agent 1, it is no longer true that she assigns high probability to agent 1 having observed the parameter, even after many periods. Although the unconditional probability that agent 1 has observed the parameter becomes high, the probability conditional on the first message not having been received is bounded away from 1. Therefore, until she receives the first message, agent 2 faces second-order uncertainty, that is, uncertainty about agent 1's beliefs about the parameter. Since agent 1 is uncertain of the time at which this message is received, she faces third-order uncertainty until she receives agent 1's confirmation. Continuing in this fashion, some higher order uncertainty persists regardless of how many messages have been delivered.

The Rubinstein (1989) email game showed that communication can have a double-edged effect on common knowledge acquisition. In the email game, agent 1 observes a parameter, sends a message informing agent 2 of the parameter, agent 2 sends a confirmation message, and so on. Communication terminates at each step with some small fixed probability. On the one hand, communication enhances knowledge acquisition; without it, agent 2 never learns the value of the parameter. Furthermore, as discussed by Rubinstein, if communication is restricted to a fixed number of messages, beliefs approach common knowledge with high probability as the likelihood of delivery failure vanishes. On the other hand, when the number of messages is unbounded, approximate common knowledge of the parameter is never acquired. Our example differs from the original email game in two significant respects. First, common knowledge is attained without communication, and thus communication only hinders common learning. Second, in our example, mutual knowledge of the parameter—in fact, *every* finite order of interactive knowledge—is eventually acquired with probability 1, whereas in the email game individual learning may fail.

Because both agents directly learn the value of the parameter, our example fits into the framework of CEMS when there is no communication. CEMS assume that each agent learns about the underlying parameter through an infinite sequence of signals that are i.i.d. across time conditional on the parameter. They prove that if the signal spaces are finite then individual learning of the parameter implies common learning. In our example, this result implies that common learning occurs in the absence of communication. The addition of communication can be viewed as a relaxation of the i.i.d. assumption. Communication naturally generates dependence in signal profiles across time since any informative message received by an agent depends on the information possessed by the sender at the time the message was sent. This relaxation leads to a failure of common learning even with finite signal and message spaces.

In our main example, communication occurs exogenously. If instead communication is voluntary, then the incentives to communicate depend on what game the agents play (after the learning process ends) for which knowledge of the parameter is relevant. We show that agents may have a strict incentive to communicate even if communication prevents common learning and approximate common knowledge of the parameter would increase the expected payoffs of both agents. Thus even voluntary communication can destroy common learning.

2 The Example

Two agents, 1 and 2, learn about a parameter θ in periods $t = 0, 1, \ldots$. The parameter θ is drawn before period 0 from the set $\Theta = \{\theta_1, \theta_2\}$ according to the prior distribution $p(\theta_1) = p(\theta_2) = 1/2$, and remains fixed over time. In each period t, each agent i receives a signal $z_t^i \in \mathcal{Z}_i = \{\theta_1, \theta_2, u\}$. Conditional on θ, these signals are i.i.d. across time. Signals for agent 1 are generated with probabilities $\Pr(z_t^1 = \theta_k \mid \theta_k) = \lambda$ and $\Pr(z_t^1 = u \mid \theta_k) = 1 - \lambda$ for each $k = 1, 2$ and some fixed $\lambda \in (0, 1)$. Note that after receiving signal $z_t^1 = \theta_k$, agent 1 knows that the parameter is $\theta = \theta_k$. If $z_s^1 = \theta$ for some $s \leq t$, we will say that agent 1 has observed θ by t. Also note that the signal u carries no information about the value of θ, and hence, absent communication, agent 1's belief about θ remains equal to her prior beliefs until she observes θ. Agent 2 receives the signal θ in each period and hence she observes θ at time 0.[1]

Our main purpose is to understand whether approximate common knowledge of the parameter θ will eventually be acquired by the two agents. Accordingly, following CEMS, we say that Θ is *commonly learned* if for each $\theta \in \Theta$ and $q \in (0, 1)$, there exists some T such that for all $t > T$,

$$\Pr\left(\theta \text{ is common } q\text{-belief at } t \mid \theta\right) > q, \tag{1}$$

where common q-belief is as defined by Monderer and Samet (1989) (see Appendix for definitions of p-beliefs, common p-beliefs, and p-evident events).

It easy to show that in the absence of communication, Θ is commonly learned in our setting.[2] Consider the event F_k^t that $\theta = \theta_k$ and agent 1 has observed θ by time t. At any state in F_k^t, each agent assigns probability at least $1 - (1 - \lambda)^{t+1}$ at time t to the other agent having observed θ (in fact, agent 1 knows that agent 2 has observed θ). Hence whenever $q < 1 - (1 - \lambda)^{t+1}$, F_k^t is q-evident at t and θ_k is common q-belief at t. Conditional on θ_k, the event F_k^t occurs with probability $1 - (1 - \lambda)^{t+1}$. Choosing T large enough so that $q < 1 - (1 - \lambda)^{T+1}$, inequality (1) is satisfied for all $t \geq T$.

We now enrich the example by adding communication according to the following protocol. In each period t, each agent i privately observes a message $m_t^i \in \mathcal{M}_i = \{c, s\}$, representing "confir-

[1] The distribution of agent 2's signals is not important for the results; it matters only that agent 2 eventually learns the value of the parameter.

[2] This result also follows immediately from either Proposition 2 or Proposition 3 of CEMS.

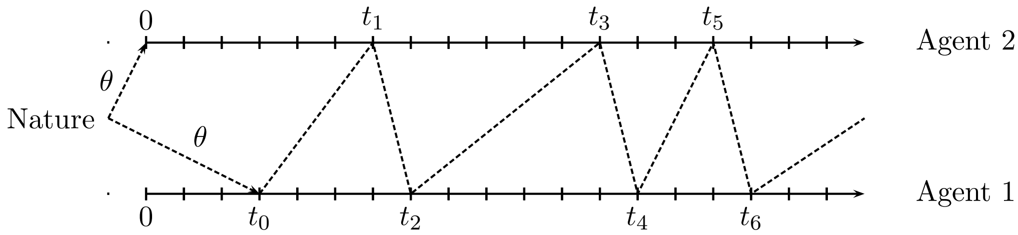

Figure 1: Possible realization of informative signals and message delivery times if agent 1's messages are delayed.

mation" and "silence" respectively. The messages m_t^i are determined by the following stochastic process. As soon as agent 1 first observes θ in some period t_0, she sends the message c to agent 2. This message is received by agent 2 at some date $t_1 > t_0$ according to the distribution described below. At time t_1, agent 2 sends a message c which is received by agent 1 at some time $t_2 > t_1$. At time t_2, agent 1 again sends a message c received by agent 2 at time $t_3 > t_2$, and so on. In every period $t \neq t_n$ for n odd, agent 2 receives the message s, and similarly agent 1 receives the message s in every period $t \neq t_n$ for $n \geq 2$ even.

The message delivery times are stochastic. Messages from one agent are always delivered within one period, whereas the other agent's messages are stochastically delayed. In one case, the messages of agent 1 are delayed, and the distribution of delivery times is as follows. Each message c from agent 2 is received by agent 1 exactly one period later; that is, $t_{n+1} - t_n = 1$ for all odd n. Each message c from agent 1 is delayed according to a geometric distribution with parameter $\delta \in (\lambda, 1)$; that is, given t_n with n even, $t_{n+1} - t_n$ is distributed on the set $\{1, 2, \ldots\}$ according to $\Pr(t_{n+1} - t_n = d) = \delta(1-\delta)^{d-1}$. Delay is independent across messages. In the other case, messages from agent 2 are delayed, and the timing of delay is as above except with odd and even k reversed. Each of these two cases occurs with probability 1/2. Agents do not observe which case occurs; their beliefs about whose messages are delayed are determined by updating their prior based on the times at which they receive messages. Figure 1 depicts the timing of messages.

Letting $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ and $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2$, the set of states is given by $\Theta \times \mathcal{Z}^\infty \times \mathcal{M}^\infty$. The information of agent i at time t is captured by the natural projection of $\Theta \times \mathcal{Z}^\infty \times \mathcal{M}^\infty$ onto $\mathcal{Z}_i^{t+1} \times \mathcal{M}_i^{t+1}$. We will write $h_t^i(\omega) \in \mathcal{Z}_i^{t+1} \times \mathcal{M}_i^{t+1}$ for the private history of agent i at time t in state ω. We abuse notation by writing θ for the event $\{\theta\} \times \mathcal{Z}^\infty \times \mathcal{M}^\infty$.

As above, we will write $t_0(\omega)$, or simply t_0 when the state is clear, for the time at which agent 1 first observes the parameter. For $n \geq 1$, we will write $t_n(\omega)$, or simply t_n, for the time at which the nth confirmation message is received. Formally, let $t_0 = \min\{t \mid z_t^1 = \theta\}$ and for $n \geq 1$, define t_n recursively by $t_n = \min\{t > t_{n-1} \mid m_t^i = c \text{ for } i = 1 \text{ or } 2\}$.

The following result indicates that communication can destroy common learning.

Proposition 1. *In the example with communication, there exists some $q \in (0,1)$ such that, for each $\theta \in \Theta$, θ is not common q-belief at any t in any state of the world. In particular, common* *learning does not occur.*

Before proving the proposition, we introduce some additional notation and a key lemma. The intuition behind the lemma is as follows. Even though all messages are eventually delivered, after the first message is sent, there is always one agent who is uncertain whether her last message has been delivered. For example, suppose that agent i sent the nth message at time t and many periods later has not received a confirmation. The unconditional probability that agent $-i$ has received the nth message is large. However, it is unlikely that if agent $-i$ indeed received the message agent i would not have received the confirmation. Since she has not received a confirmation, agent i knows that either her own message or her opponent's confirmation has been significantly delayed. Both possibilities have the same order of likelihood. Hence conditional on not receiving the confirmation, agent i assigns the same order of belief to each possibility.

For integers $n, t \geq 0$, let

$$M_t^n = \{\omega : t_n(\omega) \leq t \text{ and } t_{n+1}(\omega) > t\}.$$

Thus M_t^n consists of those states in which, by time t, agent 1 has observed θ and exactly n confirmation messages have been received. Similarly, let M_t^{-1} denote the event that, by time t, agent 1 has not observed θ; formally,

$$M_t^{-1} = \{\omega : t_0(\omega) > t\}.$$

Lemma 1. *There exists some $\overline{p} > 0$ such that, for each $t \geq 0$ and $n = 0, \ldots, t$, given any $\omega \in M_t^n$, $\Pr\left(M_t^{n-1} \mid h_t^i(\omega)\right) \geq \overline{p}$ for some $i \in \{1, 2\}$.*

The proof of Lemma 1 is in the appendix.

For $n \geq 1$, Lemma 1 states that if exactly n confirmation messages have been received by t then one of the agents assigns probability at least $\overline{p}$ to only $n-1$ confirmation messages having been received. For $n = 0$, the lemma states that if agent 2 has not received any confirmation message by t then she assigns probability at least $\overline{p}$ to agent 1 not having observed θ. Therefore, in any state of the world and at any time t, at least one agent is uncertain about the order of knowledge about θ acquired by the other agent.

Proof of Proposition 1. Choose any $q \in \left(\frac{1}{2}, 1\right)$ such that $q > 1 - \overline{p}$, with $\overline{p}$ as in Lemma 1. Suppose for contradiction that θ is common q-belief at time t in some state ω. By the characterization of Monderer and Samet (1989), there exists an event F containing ω such that, at time t, F is q-evident and both agents q-believe θ on F.

We will show that F contains a state in M_t^{-1}, i.e. one in which agent 1 has not observed θ by time t. In such a state, agent 1 assigns probability 1/2 to the event θ' for $\theta' \neq \theta$. Since $q > 1/2$, these beliefs violate the hypothesis that both agents q-believe θ on F at time t, giving the desired contradiction.

Let $n^* = \min \{n \mid F \cap M_t^n \neq \emptyset\}$ and choose some $\omega' \in F \cap M_t^{n^*}$. We will show that $n^* = -1$. Suppose for contradiction that $n^* \geq 0$. By Lemma 1, for some i, agent i assigns probability at least $\overline{p}$ to the event $M_t^{n^*-1}$ at the private history $h_t^i(\omega')$. Writing $B_p^i(E)$ for the event that agent i p-believes the event E at time t, we have

$$\omega' \in B_{\overline{p}}^i \left(M_t^{n^*-1} \right).$$

Since F is q-evident at time t, we also have

$$\omega' \in B_q^i(F).$$

By the choice of q, $\overline{p}+q > 1$ and hence $M_t^{n^*-1} \cap F \neq \emptyset$, contradicting the definition of n^*. Therefore, $n^* = -1$. □

3 Incentives to Communicate

In the preceding example, communication between agents is exogenous. Since communication causes common learning to fail in that setting, one might think that if communication was voluntary then agents would choose not to communicate. In this section, we provide an example to show that this is not true in general; whether or not agents communicate voluntarily depends on what game is played after the agents learn about the parameter. For some games, there is a tension between the acquisition of common knowledge and the speed of acquisition of finite orders of belief; the latter can be facilitated by communication at the expense of the former. This tension is particularly stark in the example below: no matter what order of belief has been attained, agents always strictly prefer that higher orders be attained as quickly as possible. This feature ensures that both agents have a strict incentive to send messages even though doing so leads to the failure of common learning.

The example is based on an infection argument.[3] Suppose that communication ends at some stochastic time $t \in \mathbb{N}$, at which point agents play a game. Each agent can choose a safe action or a risky action that may require coordinating with the other agent. In favorable states of the world, a risky action is dominant for agent i whenever she has achieved *individual* approximate knowledge of θ. But then agent $-i$ prefers to play a risky action whenever she believes that the state of the world is favorable for i and that i approximately knows θ. Proceeding in this fashion, in any state of the world each agent plays a risky action whenever she has achieved a sufficient finite order of interactive knowledge.

The details of the example are as follows. Consider the following Bayesian game Γ played after the learning phase terminates. Each agent i chooses an action $a^i \in \{\theta_1, \theta_2, S\}$. The payoff for action S is 0 regardless of the state and the action of the other agent. The payoff $u^i\left(a, \theta, x^i\right)$ for action θ_j depends on the action profile a, the parameter θ, and a private signal x^i. Each agent i observes $x^i \in \{0, 1, 2, \dots\}$ before chosing an action. Thus the type of agent i consists of the pair

[3]See Morris, Rob, and Shin (1995) for a general elucidation of infection arguments.

$\left(x^i, h_t^i(\omega)\right)$, where $h_t^i(\omega)$ is the private history from the learning phase. Payoffs for actions $a^i \neq S$ are as follows. For $x^i = 0$,

$$u^i(a,\theta,0) = \begin{cases} 1-p & \text{if } a^i = \theta, \\ -p & \text{if } a^i \neq \theta, \end{cases}$$

regardless of a^{-i}, where $p \in (1/2, 1)$. For $x^i \geq 1$,

$$u^i\left(a,\theta,x^i\right) = \begin{cases} 2-p & \text{if } a^i = a^{-i} = \theta \text{ and } x^{-i} = x^i - 1, \\ -p & \text{otherwise.} \end{cases}$$

The signals x^i are independent of the state ω. Conditional on each $x^i \geq 1$, agent i assigns probability 1/2 to $x^{-i} = x^i - 1$ and 1/2 to $x^{-i} = x^i$. These posterior beliefs are consistent with a class of heterogeneous priors. Heterogeneity of the priors is not important for the result; it is easy to construct a similar, but more complicated example with a common prior. In addition to generating the given posterior beliefs, agent i's prior assigns positive probability to each x^i.

For any type with signal $x^i = 0$, action θ_k is strictly dominant whenever i assigns probability greater than p to the parameter being θ_k. Deleting dominated strategies for types with signal 0, action θ_k is strictly dominant for any type with signal $x^i = 1$ whenever i p-believes both that $\theta = \theta_k$ and that $-i$ p-believes $\theta = \theta_k$. Proceeding inductively, we see that action θ_k is serially dominant for any type with signal x^i that attains p-belief of θ_k to order $x^i + 1$.[4] In fact, this game has an essentially unique Bayesian Nash equilibrium in which each agent with signal x^i plays θ_k if she has attained p-belief of θ_k to order $x^i + 1$, and plays S otherwise. In equilibrium, higher orders of belief about the parameter θ lead to higher expected payoffs.

Under the communication protocol of Section 2, if p is sufficiently close to 1 then the order of p-belief attained at any time is determined by the number of delivered messages. Once exactly $n \geq 1$ confirmation messages have been delivered, agent 1 attains p-belief of θ_k to order $\lfloor n/2 \rfloor + 2$ and agent 2 to order $\lceil n/2 \rceil + 1$ (and to no higher order).

Consider a dynamic game in which the learning phase from Section 2 terminates at a stochastic time t, determined according to a distribution with unbounded support. When the learning phase terminates, the two agents play the game Γ. Agents communicate during the learning phase according to the same protocol as in Section 2 except that communication is voluntary. At each finite history in which an agent sends a message in the protocol, that agent may now choose whether to (costlessly) send the message or not. The other agent does not observe this decision directly, and hence cannot distinguish an unsent message from one that she has not yet received. Agents have no other opportunities to communicate. In particular, once an agent chooses not to send a message, there is no further communication.

When p is sufficiently close to 1, the strategy profile in which both agents always send messages

[4]Formally, p-belief of θ_k to order n refers to the event $C_n^{it}(\theta_k)$, defined recursively as follows: $C_1^{it}(\theta_k) = B_p^{it}(\theta_k)$, and for $n > 1$, $C_n^{it}(\theta_k) = B_p^{it}\left(\theta_k \cap C_{n-1}^{jt}(\theta_k)\right)$, where $j \neq i$ and $B_p^{it}(E)$ is the set of states ω at which i p-believes E at time t.

and play according to the Bayesian Nash equilibrium of Γ constitutes a weak perfect Bayesian equilibrium of this dynamic game. In equilibrium, when given the option to send a message, each agent strictly prefers to send it because the expected payoff in Γ increases with the order of belief about θ, which in turn increases with the number of delivered messages. If one agent deviates to not sending a message after p-belief of θ has been attained to order n, then the other agent believes that the message is delayed, and by Lemma 1, p-belief to order $n+1$ is never attained. Moreover, as long as communication does not affect agents' beliefs about the private parameters x^i, one can show that there is no equilibrium in which agents do not communicate when p is close to 1.[5] Since each received message increases the order of p-belief, agent 1 has a strict incentive to send a message after observing θ so that agent 2 acquires second order belief more quickly.

The equilibrium with communication may be inefficient. Common learning occurs if agents do not communicate (and believe that the other chooses not to communicate). If the learning phase terminates only after a large number of periods, then non-communicating agents successfully coordinate on the risky action with high probability for all realizations of x^1 and x^2. On the other hand, if the delay in messages tends to be large, then the probability of efficient coordination in the equilibrium with communication may be small. Thus agents may end up in a "communication trap" where both would be better off without the option to communicate.

4 Discussion

Section 2 provides an example in which common learning fails because of communication. The key to the example is that the possibility of delay generates persistent higher order uncertainty regarding whether agent 1 has observed the parameter even though the unconditional probability of this event tends to 1 over time. This persistent uncertainty does not arise if, instead of delay, each message fails to be delivered with some positive probability (as in the original email game). In this case, common learning turns out to occur because if agent 2 does not receive the first message from agent 1, then after many periods agent 2 assigns high probability to the event that agent 1 observed the parameter but her message was not delivered.

For common learning to fail in our example, it was necessary to assume that $\delta > \lambda$, so that delays in communication tend to be shorter than delays in agents' observation of θ. Otherwise, if after many periods agent 2 has not received the first message from agent 1, then she assigns high probability to agent 1 having observed θ but the message having been delayed. Paradoxically, lowering δ below λ can rescue common learning even though doing so makes communication worse in the sense that message delays tend to be longer.

Questions about the influence of communication on common knowledge acquisition are related to a larger literature on the emergence of consensus with communication. A consensus is said to emerge about an event E if all agents eventually have the same belief about E. Heifetz (1996) shows that, as suggested by Parikh and Krasucki (1990), consensus can emerge in dynamic settings

[5] More precisely, given all other parameters, there exists $\overline{p} < 1$ such that whenever $p > \overline{p}$, there is no equilibrium in which, with probability 1, no messages are sent along the equilibrium path.

without ever becoming common knowledge. Koessler (2001) proves that, although consensus may emerge, full common knowledge of an event is never attained under any noisy and non-public communication protocol unless the event was common knowledge initially.[6] We diverge from this literature by combining communication with the individual learning of CEMS. Consensus about θ almost surely emerges in our example with or without communication. Unlike the previous literature, however, common learning of θ fails only *with* communication.

It is easy to construct examples in which communication enables common learning, that is, in which common learning occurs with communication but fails without it. This would be the case, for instance, if only one agent privately learns the parameter, and communication consists of that agent publicly announcing each of her signals. That communication can also cause common learning to fail raises interesting questions about the role of communication in common knowledge acquisition and the conditions under which it enhances or hinders common learning. We plan to pursue these questions in future research.

A Appendix

A.1 Definitions

For convenience, we review the concepts of p-belief and p-evidence due to Monderer and Samet (1989).

Consider a probability space (Ω, Σ, μ), agents 1 and 2, and for each $i = 1, 2$, a measurable information partition Π^i of Ω for agent i. For any $\omega \in \Omega$, let $\Pi^i(\omega)$ denote the element of Π^i containing ω. For $E \in \Sigma$ and $p \in [0, 1]$, let

$$B_p^i(E) = \left\{\omega \mid \mu\left(E \mid \Pi^i(\omega)\right) \geq p\right\}.$$

If $\omega \in B_p^i(E)$ then we say that agent i p-believes E in state ω.

For $E \in \Sigma$ and $p \in [0, 1]$, *common p-belief of E* refers to the event

$$C_p(E) = \bigcap_{n=1}^{\infty} C^{(n)},$$

where $C^{(0)} = E$ and for $n \geq 1$, $C^{(n)} = \bigcap_{i=1,2} B_p^i\left(C^{(n-1)}\right)$. We say that E is common p-belief at ω if $\omega \in C_p(E)$.

An event $E \in \Sigma$ is *p-evident* if $E \subseteq \bigcap_{i=1,2} B_p^i(E)$, that is, if both agents p-believe E in every state in E.

Monderer and Samet (1989) provide the following characterization of common beliefs.

Proposition 2 (Monderer and Samet, 1989)**.** *An event E is common p-belief at ω if and only if there exists a p-evident event F such that $\omega \in F$ and $F \subseteq \bigcap_{i=1,2} B_p^i(E)$.*

[6] See also Halpern and Moses (1990), who obtain a similar result when messages have unbounded delivery times.

A.2 Proof of Lemma 1

Proof. For $i = 1, 2$, let D_i denote the event that the messages of agent i are delayed; that is, let $D_1 = \{\omega \mid t_{n+1}(\omega) - t_n(\omega) = 1 \text{ for all } n \text{ odd}\}$, and $D_2 = \{\omega \mid t_{n+1}(\omega) - t_n(\omega) = 1 \text{ for all } n \text{ even}\}$. We begin by calculating each agent's beliefs over D_1 and D_2 after any finite history, beginning with agent 2.

Fix $t > 0$. Since D_1 and D_2 are equally likely *ex ante*,

$$\Pr(D_2 \mid t_1 = t) = \frac{\Pr(t_1 = t \mid D_2)}{\Pr(t_1 = t \mid D_2) + \Pr(t_1 = t \mid D_1)}. \tag{2}$$

We have

$$\Pr(t_1 = t \mid D_2) = \lambda(1-\lambda)^{t-1} \tag{3}$$

and

$$\begin{aligned} \Pr(t_1 = t \mid D_1) &= \sum_{s=0}^{t-1} \lambda(1-\lambda)^s \delta(1-\delta)^{t-s-1} \\ &= \delta\lambda \frac{(1-\lambda)^t - (1-\delta)^t}{\delta - \lambda}. \end{aligned} \tag{4}$$

Substituting equations (3) and (4) into equation (2) gives

$$\Pr(D_2 \mid t_1 = t) = \left(1 + \frac{\delta(1-\lambda)}{\delta - \lambda}\left(1 - \left(\frac{1-\delta}{1-\lambda}\right)^t\right)\right)^{-1}. \tag{5}$$

Since $\delta > \lambda$ by assumption, this last expression is decreasing in t and approaches $\frac{\delta-\lambda}{2\delta-\delta\lambda-\lambda} > 0$ as t tends to infinity.

Note that, agent 2's belief about D_2 never changes after she receives the first message; since the distribution of all subsequent messages received by agent 2 is independent of D_1 or D_2, she assigns probability $\Pr(D_2 \mid t_1(\omega) = t)$ to D_2 in every period $t' \geq t_1(\omega)$. Similarly, agent 1 assigns probability $\frac{1}{2}$ to D_1 after any finite history. Let σ_1 denote agent 1's belief in D_1 and σ_2 denote agent 2's belief in D_2, suppressing from the notation the dependence of σ_2 on the history.

Next we compute agent i's belief that her most recent message has been received. For any $n \geq 1$, consider the event M_t^n that n confirmation messages have been received by time t. Let i be the sender of the nth confirmation message, that is $i = 1$ for n odd and $i = 2$ for n even. Fix any state $\omega \in M_t^n$, and let $d = t - t_{n-1}$ be the length of time that has passed since agent i sent her last confirmation message. Note that $d > 0$. We have (given σ_i)

$$\Pr\left(M_t^{n-1} \mid h_t^i(\omega)\right) = \frac{\sigma_i(1-\delta)^d}{(1-\sigma_i)(1-\delta)^{d-1} + \sigma_i\delta(1-\delta)^{d-1} + \sigma_i(1-\delta)^d} = \sigma_i(1-\delta). \tag{6}$$

Since σ_i is bounded away from 0, there exists some $\overline{p}_1 > 0$ such that

$$\Pr\left(M_t^{n-1} \mid h_t^i(\omega)\right) \geq \overline{p}_1.$$

Finally, consider agent 2's belief of whether agent 1 has observed θ if she has not yet received a confirmation message at time $t \geq 0$. For any $\omega \in M_t^0$, we have

$$\begin{aligned} \Pr\left(M_t^{-1} \mid h_t^2(\omega)\right) &= \frac{(1-\lambda)^{t+1}}{(1-\lambda)^{t+1} + \lambda(1-\lambda)^t + \frac{1}{2}\sum_{s=0}^{t-1}\lambda(1-\lambda)^s(1-\delta)^{t-s}} \\ &= \frac{\delta-\lambda}{1 - \frac{1-\delta}{1-\lambda} + \frac{1}{2}\lambda\frac{1-\delta}{1-\lambda}\left(1-\left(\frac{1-\delta}{1-\lambda}\right)^t\right)}, \end{aligned} \tag{7}$$

which is decreasing in t and approaches $\frac{2(\delta-\lambda)(1-\lambda)}{2\delta-\lambda-\delta\lambda} > 0$ as t tends to infinity.

Taking $\overline{p} = \min\left\{\overline{p}_1, \frac{2(\delta-\lambda)(1-\lambda)}{2\delta-\lambda-\delta\lambda}\right\}$ gives the result. □

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