Constrained Data-Fitters

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homo economicus flawlessly

- forms Bayesian updates
- evaluates likelihood

machine learning: the two tasks can only be approximated

we: halfway between machine learning and economics

- we relax constraints enough

constrained-optimal models are often simple

Literature

variational Bayes methods: Jordan et al.'99, Kingma&Welling'13, Aridor, da Silveira&Woodford'24

- approximate Bayes' law and maximum-likelihood estimation

misspecified learning: Berk'66, White'82, Esponda&Pouzo'16

- arises as a special case

causal networks: Pearl'09, Spiegler'16

- description of cognitive constraints

information design: Aumann&Maschler'95, Kamenica&Gentzkow'11, Caplin&Dean'13

- posterior approach

Generative Model

an agent holds a model $p(x, z) \in \Delta(X \times Z)$ of

- observable x
- latent z

economics:

- x is the signal (education level)
- z is the state (applicant's type)

machine learning:

- x is high-dimensional data input (job interview)
- z is a compressed representation of x (classification of the applicant)

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Recognition Model

the agent

- observes a draw x from true process $q_0(x) \neq p(x)$
- forms update $q(z \mid x) \neq p(z \mid x)$

recognition model:

$$q(x, z) = q_0(x)q(z \mid x)$$

Constrained Updating

variational Bayes methods, Jordan et al.'99

updates solve

$$\max_{\left(\tilde{q}(z|x)\right)_x} \quad \mathrm{E}_{\tilde{q}(x,z)} \ln p(\hat{x},\hat{z}) + \mathrm{H}\left(\tilde{q}(x,z)\right)$$

$$\text{s.t.} \quad \tilde{q}(x,z) \in \mathcal{Q}$$

the maximizer: constrained updates

the value: constrained likelihood

Constrained Updating

variational Bayes methods, Jordan et al.'99

recognition model solves

$$\max_{\tilde{q}(x,z)} \quad \mathrm{E}_{\tilde{q}(x,z)} \ln p(\hat{x}, \hat{z}) + \mathrm{H}\left(\tilde{q}(x,z)\right)$$

$$\text{s.t.} \quad \tilde{q}(x,z) \in \mathcal{Q}$$

$$\tilde{q}(x) = q_0(x)$$

empirical constraint

Constrained Updating

variational Bayes methods, Jordan et al.'99

recognition model solves

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updating constraint

Constrained Updating

variational Bayes methods, Jordan et al.'99

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reconstruction term

Constrained Updating

variational Bayes methods, Jordan et al.'99

recognition model solves

$$\max_{\tilde{q}(x,z)} \quad \mathrm{E}_{\tilde{q}(x,z)} \ln p(\hat{x}, \hat{z}) + \mathrm{H}\left(\tilde{q}(x,z)\right)$$

$$\text{s.t.} \quad \tilde{q}(x,z) \in \mathcal{Q}$$

$$\tilde{q}(x) = q_0(x)$$

regularization term

Constrained Updating

variational Bayes methods, Jordan et al.'99

recognition model solves

$$\min_{\tilde{q}(x,z)} \quad \mathrm{KL}\left(\tilde{q}(x,z) \,\|\, p(x,z)\right)$$

$$\text{s.t.} \quad \tilde{q}(x,z) \in \mathcal{Q}$$

$$\tilde{q}(x) = q_0(x)$$

Some Updating Constraints

no constraint: $\mathcal{Q} = \Delta(X \times Z)$ details

- Bayesian updates, $q(z \mid x) = p(z \mid x)$
- unconstrained likelihood, $\mathsf{E}_{q_0(x)} \ln p(\hat{x}) + \text{const.}$

analogy-based constraint: $q(z \mid x)$ measurable w.r.to a partition of X

causal constraint; e.g.:

- $z = (z_1, z_2)$
- q must comply with directed acyclical graph $z_1 \leftarrow x \rightarrow z_2$
- $\Leftrightarrow$ factorization constraint $q(x, z_1, z_2) = q(x)q(z_1 \mid x)q(z_2 \mid x)$

Likelihood Evaluation

sample $(x_1, \ldots, x_n)$

via marginalization:

- $p(x) = \sum_z p(x, z)$
- $\ell = \prod_i p(x_i)$

via sample extension:

- extended sample $(x_i, z_i)_{i=1}^n$
- frequencies of (x, z): frequencies of x observed & $z \mid x \sim p(z \mid x)$
- $\ell = \prod_i p(x_i, z_i) \times$ no. of distinct permutations

updating and fit evaluation are related

Constrained Updating

the agent estimates frequencies $q(x, z)$ of the extended sample

$$\begin{array}{ll} \max\limits_{\tilde{q}(x,z)} & p\text{-likelihood} \\ \text{s.t.} & \tilde{q}(x) = q_0(x) \\ & \tilde{q}(x, z) \in \mathcal{Q} \end{array}$$

Estimation

p-likelihood of a single extended sample with frequencies $q(x, z)$

$$\prod_{i=1}^{n} p(x_i, z_i) = \prod_{x,z} p(x, z)^{q(x,z)n}$$

p-likelihood of all such extended samples

$$\ell_n(q) := \prod_{x,z} p(x, z)^{q(x,z)n} \times \mathcal{N}_n(q)$$

the estimate:

$$q_n(x, z) \in \arg\max_{\tilde{q} \in \mathcal{Q}_n} \ell_n(q)$$

Permutations

Illustration

$x \in \{r, b\}$

$z \in \{0, 1\}$

$x^4 = rbrb$

consider $q(x, z)$ uniform on $\{r, b\} \times \{0, 1\}$

four possible extended samples:

x^4	r	b	r	b
z^4	0	0	1	1
z^4	1	0	0	1
z^4	0	1	1	0
z^4	1	1	0	0

Limit

let $\mathcal{Q}_n$ approximate $\mathcal{Q}$ details

proposition

Let the updating problem have a unique optimizer $q(x, z)$. Then,

$$q_n(x, z) \quad \to \quad \text{recognition model } q(x, z)$$

$$\frac{1}{n} \ln \ell_n(q_n) \quad \to \quad \text{constrained likelihood} + \text{const.}$$

because

$$\frac{1}{n} \ln \quad \prod_{x,z} p(x, z)^{q(x,z)n} \quad \times \quad \mathcal{N}_n(q)$$

$$\to \quad \mathrm{E}_{q(x,z)} \ln p(\hat{x}, \hat{z}) \quad + \quad \mathrm{H}\left(q(x, z)\right) - \mathrm{H}\left(q_0(x)\right)$$

Approximate Maximum Likelihood Estimation

variational autoencoder, Kingma&Welling'13

principle

Choose the generative model that maximizes constrained likelihood.

$$\min_{\tilde{p}(x,z),\tilde{q}(x,z)} \quad \mathrm{KL}\left(\tilde{q}(x,z) \parallel \tilde{p}(x,z)\right)$$

$$\text{s.t.} \quad \tilde{p}(x,z) \in \mathcal{P}$$

$$\tilde{q}(x,z) \in \mathcal{Q}$$

$$\tilde{q}(x) = q_0(x)$$

Example

def.: $\mathcal{P}$ has unconstrained margin

- all $p(z)$ are feasible
- a constraint on $\left(p(x \mid z)\right)_z$ independent of $p(z)$

$z = (z_1, z_2)$

recognition model restricted to a chain: $x \to z_1 \to z_2$

deterministic collapse

The agent forms a partially deterministic model:

$$z_2 = d(z_1)$$

a.s. under both p and q, for some deterministic function d.

Comparison

Spiegler:

- true process $q(x, z_1, z_2)$
- DAG; e.g.: $x \to z_1 \to z_2$
- an agent chooses model $p(x, z_1, z_2)$ by projecting q on the DAG

$$p(x) = q(x) \quad p(z_1 \mid x) = q(z_1 \mid x) \quad p(z_2 \mid z_1) = q(z_2 \mid z_1)$$

- generically, $p(z_2 \mid z_1)$ is stochastic

Rational Expectations
definition

def.: the agent has rational expectations if

$$p(z) = \mathrm{E}_{q_0(x)}\, q(z \mid \hat{x}) \equiv q(z)$$

in general, our agent won't have RE because she

- isn't Bayes' rational
- is misspecified

Rational Expectations

result

proposition

If $\mathcal{P}$ has unconstrained margin, then the agent has rational expectations.

proof: optimize over $\tilde{p}(z)$,

$$\mathrm{KL}\left(q(x,z) \,\|\, \tilde{p}(x,z)\right) = \mathrm{KL}\left(q(z) \,\|\, \tilde{p}(z)\right) + \sum_z q(z)\,\mathrm{KL}\left(q(x \mid z) \,\|\, p(x \mid z)\right)$$

Discussion

standard Bayes' plausibility is forced by the Bayes' law

- it can fail in our framework, but holds at the optimum

a popular non-Bayesian intuition in support of RE:

- systematically surprised agent should adjust her prior
- indeed, our agent chooses $p(z)$ to match $q(z)$

Posterior Approach

information design

posterior representation: $q(z)$, $\left(q(x \mid z)\right)_z$, and $\left(p(x \mid z)\right)_z$

- specifies both models $p(x, z)$ and $q(x, z)$

lemma: posterior-separable objective

If $\mathcal{P}$ has unconstrained margin, then the model-fitting problem becomes

$$\max_{\tilde{q}(z),(\tilde{q}(x|z))_z,(\tilde{p}(x|z))_z} \quad \mathrm{E}_{\tilde{q}(z)}\left[\mathrm{E}_{\tilde{q}(x|\hat{z})} \ln \tilde{p}(\hat{x} \mid \hat{z}) + \mathrm{H}\left(\tilde{q}(x \mid \hat{z})\right)\right]$$

$$\text{s.t.} \quad \left(\tilde{p}(x \mid z)\right)_z \in \mathcal{P}'$$

$$\tilde{q}(z)\tilde{q}(x \mid z) \equiv \tilde{q}(x, z) \in \mathcal{Q}$$

$$\mathrm{E}_{\tilde{q}(z)}\, \tilde{q}(x \mid z) = q_0(x).$$

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Deterministic Collapse

proof

recall the chain constraint $\mathcal{Q}$: $x \to z_1 \to z_2$

- equivalent to $q(x \mid z_1, z_2) = q(x \mid z_1)$

consider a solution p and q

for each z_1, re-optimize $q(z_2 \mid z_1)$, fixing $q(z_1)$, $q(x \mid z)$ and $p(x \mid z)$

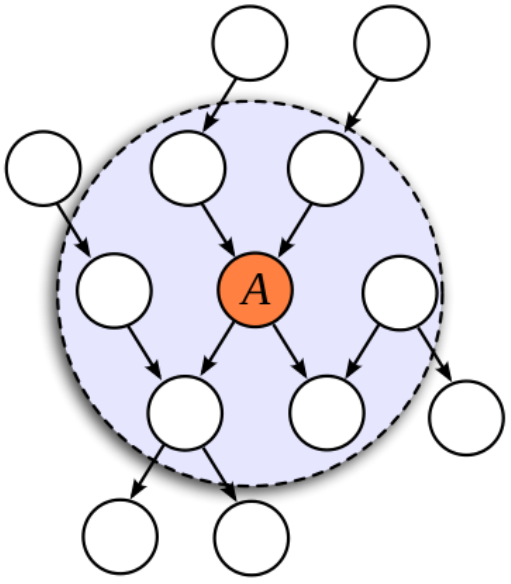
each (z_1, z_2) has posterior value $\mathrm{KL}\left(q(x \mid z_1) \,\|\, p(x \mid z_1, z_2)\right)$

- z_2 affects it only via p

deterministically pick the maximizer $z_2^* = d(z_1)$

Markov Boundary

Pearl '88



Markov boundary of A: minimal set that contains all information about A

e.g. in $x \to z_1 \to z_2$

- z_1 is in the Markov boundary of x
- z_2 isn't

General Simplicity Result

generic DAG

z^B – the latent variables from the Markov boundary of x

- DAG-compatible $q(x \mid z)$ depends only on z^B

say q' is simpler than q if

- $q'(x, z^B) = q(x, z^B)$, and
- $z^{-B} \mid z^B$ is deterministic under q'

$\mathcal{Q}$: q compatible with the DAG and any q' simpler than q

deterministic collapse

A solution exists such that z^{-B} is a deterministic function of z^B.

Two Frictions

		$\mathcal{P}$	
		Well-specified	**Miss-specified**
$\mathcal{Q}$	**Bayes' Rationality**	Wald'49	Berk'66
	Updating Friction	model-fitting	model-fitting

Information vs Moment Projection

moment projection: sample $\rightarrow$ model

White'82/Berk'66:

agent observes sample and chooses model $\tilde{p}(y) \in \mathcal{P}$

$$\min_{\tilde{p} \in \mathcal{P}} \mathrm{KL}\left(q_0(y) \parallel \tilde{p}(y)\right)$$

information projection: model $\rightarrow$ sample

Sanov's Theorem:

agent holds model $p(y)$ and reasons about sample

$$\min_{\tilde{q} \in \mathcal{Q}} \mathrm{KL}\left(\tilde{q}(y) \parallel p(y)\right)$$

Example: Analogy-Based Reasoning

a measurability constraint on conditional distributions

moment projection $\Rightarrow$ arithmetic mean (Jehiel'05)

- data to model

information projection $\Rightarrow$ geometric mean

- model to data

White/Berk As a Special Case

what model $p(x)$ of the observable variable the agent chooses?

proposition

If updating is unconstrained, then $p(x)$ is the moment projection

$$p(x) \in \operatorname*{arg\,min}_{\tilde{p}(x) \in \mathcal{P}'} \mathrm{KL}\left(q_0(x) \parallel \tilde{p}(x)\right).$$

follows from the chain rule

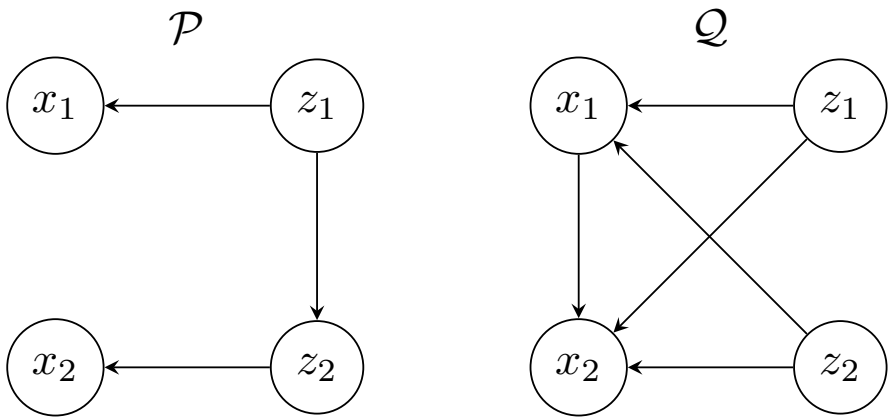
$$\mathrm{KL}\left(\tilde{q}(x,z) \parallel \tilde{p}(x,z)\right) = \mathrm{KL}\left(q_0(x) \parallel \tilde{p}(x)\right) + \sum_x q_0(x)\, \mathrm{KL}\left(\tilde{q}(z \mid x) \parallel \tilde{p}(z \mid x)\right)$$

Simple Model Preferred for Constrained Updating

example

$x = (x_1, x_2)$ and $z = (z_1, z_2)$

the true process $q_0(x_1, x_2)$ exhibits correlation



agent is well-specified

- $\Rightarrow$ learns the true process q_0 if updates are unconstrained

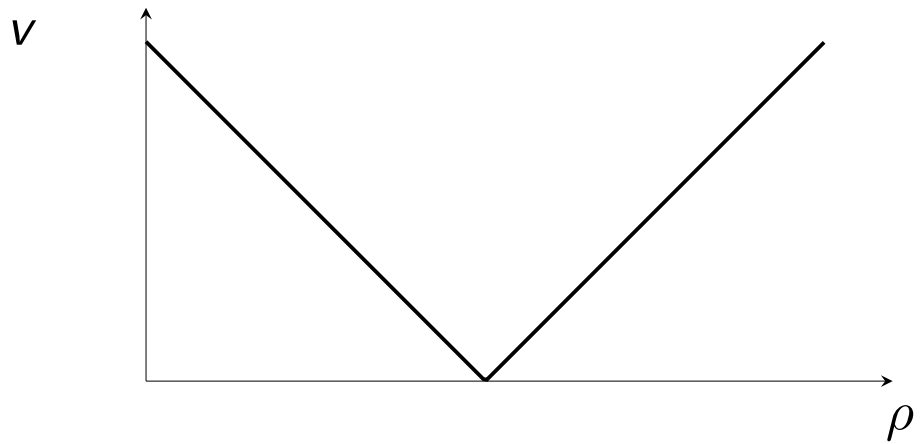
the updating constraint

- $\Rightarrow$ optimal correlation neglect $p(x_1, x_2) = p(x_1)p(x_2)$ proof

Rational Inattention

- payoff state $x \sim q_0(x)$
- agent chooses experiment $q(z \mid x)$
- maps the observed signal to action a
- maximizes $\mathsf{E}\, u(a, x) + \mathsf{E}\, \mathsf{H}\left(q(x \mid z)\right)$

plot $\rho \mapsto \max_a \mathsf{E}_{\rho(x)}\, u(a, \hat{x})$



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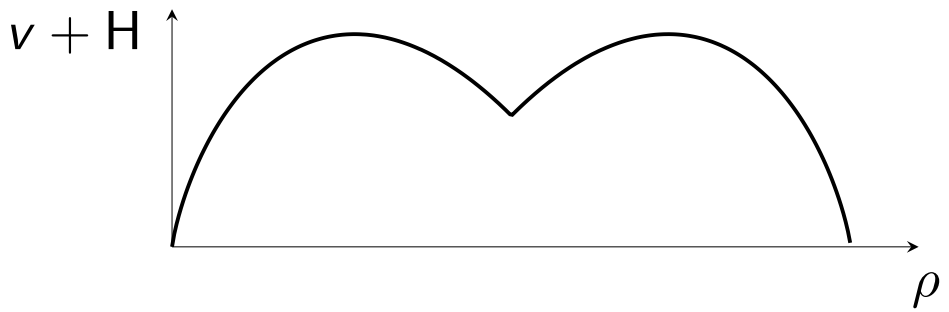
plot $\rho \mapsto \max_a \mathsf{E}_{\rho(x)}\, u(a, \hat{x}) + \mathsf{H}\left(\rho\right)$



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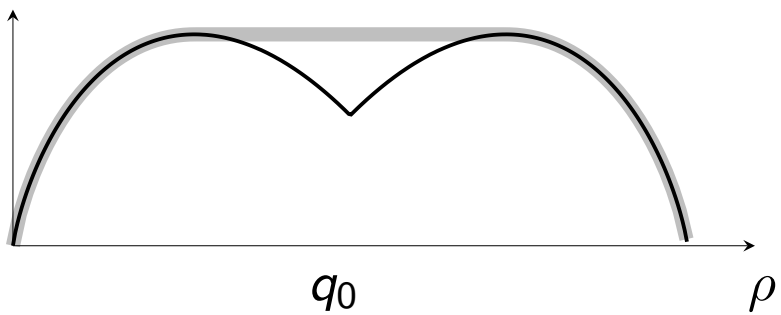
find the optimal posteriors



Posterior Separable Constraints

$\mathcal{P}$ and $\mathcal{Q}$ are posterior separable if

- p and q are feasible iff $p(x \mid z) \in \bar{\mathcal{P}}$ and $q(x \mid z) \in \bar{\mathcal{Q}}$

posterior separable $\mathcal{P}$:

- agent models the process generating x
- she is endowed with a set of primitive distributions $\bar{\mathcal{P}}$
- can build any mixture distribution from the convex hull of $\bar{\mathcal{P}}$

posterior separable $\mathcal{Q}$:

- agent partitions the observed dataset x^n
- assigns distinct z to each cell
- each cell must have an empirical distribution in $\bar{\mathcal{Q}}$

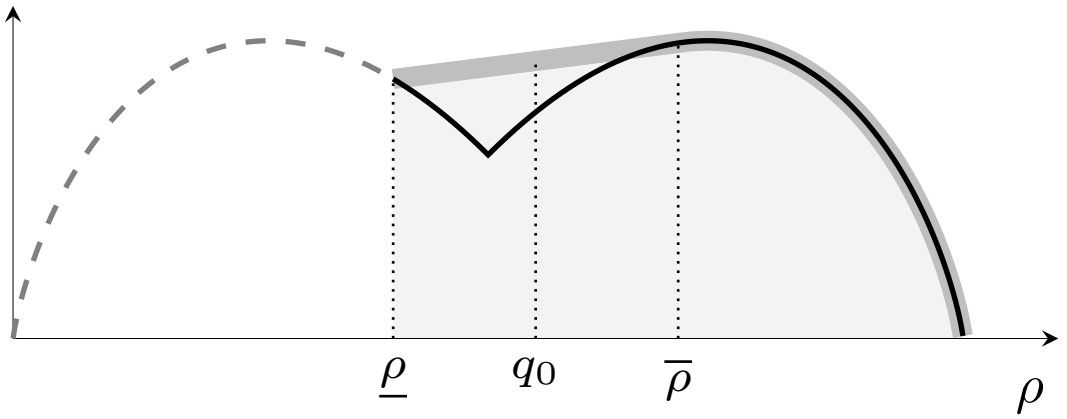
Connection

primitive distributions are "actions" $\bar{\mathcal{P}} = \{p_a(x)\}_a$

writing $\ln p_a(x) = u(a, x)$, our problem becomes the RI problem:

$$\begin{aligned} \max \quad & \mathrm{E}\left[u(\hat{a}, \hat{x}) + \mathrm{H}\left(\tilde{q}(x \mid \hat{z})\right)\right] \\ \text{s.t.:} \quad & \mathrm{E}_{\tilde{q}(z)}\, \tilde{q}(x \mid \hat{z}) = q_0(x) \\ & \tilde{q}(x \mid z) \in \bar{\mathcal{Q}} \end{aligned}$$

Concavification of the Augmented Value Function



Base-Rate Neglect

comparative statics w.r.to the true process

local invariance

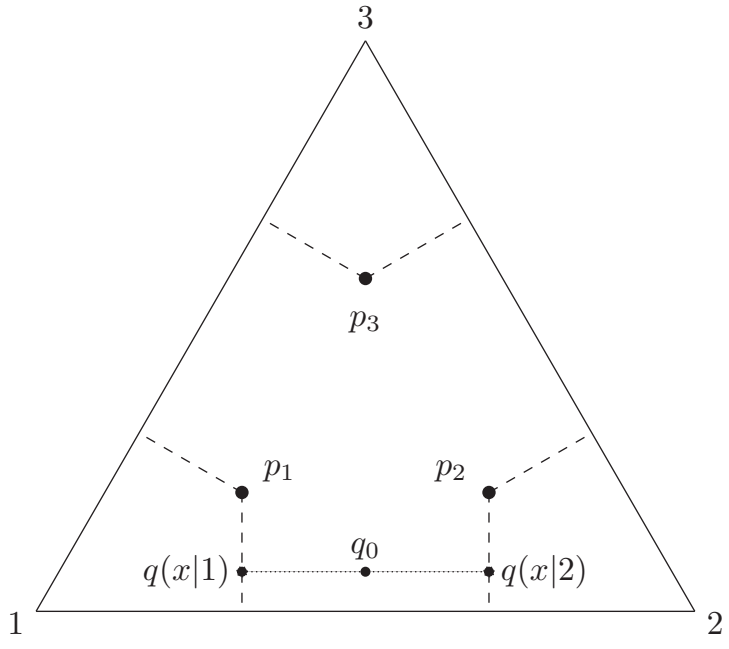
Let true process $q_0^*(x)$ induce posteriors by $p^*(x \mid z)$ and $q^*(x \mid z)$.
For all processes $q_0(x)$ in the convex hull of $\left(q^*(x \mid z)\right)_z$:

$$\begin{aligned} p(x \mid z) &= p^*(x \mid z) \\ q(x \mid z) &= q^*(x \mid z). \end{aligned}$$

Illustration

Matysková&Montes'23

no updating constraint



generative model employs 1, 2, or 3 primitive distributions

- accompanied by a recognition model of the same complexity

Hallucination

optimal recognition model may hallucinate:

- there may exist z and z' such that

$$\begin{aligned} p(x \mid z) &= p(x \mid z') \\ q(x \mid z) &\neq q(x \mid z') \end{aligned}$$

this cannot happen when $\bar{\mathcal{Q}}$ is convex

- akin to the recommendation lemma in RI
- beneficial randomization over $q(x \mid z)$

More Literature

non-Bayesian updating: Dominiak, Kovach & Tserenjigmid '21; Jakobsen '21; Ortoleva '12; Zhao '22

machine learning: Caplin, Martin & Marx '23; Zhao, Ke, Wang & Hsieh '20; Aridor, da Silveira, & Woodford '24

Bayesian networks: Spiegler '16,'20; Sloman '05; Pearl '88; Ambuehl & Thysen '24; Andre, Haaland, Roth & Wohlfart '23

inconsistent updating: Aina, Amelio & Brütt '23; Bohren & Hauser '23

misspecified learning: Esponda & Pouzo '16; Fudenberg, Lanzani & Strack '21; Frick, Iijima, & Ishii '23

Conclusion

machine learning $\rightarrow$ economics:

- updating and likelihood evaluation are hard
- two distinct statistical models are handy
- tractable constrained updating and model-fitting problems

economics $\rightarrow$ machine learning:

- relaxed constraints may generate solutions with interesting structure
- optimal models are often simple

Approximation

correspondence $\mathcal{Q}(\theta)$, $\theta \in [0,1]$

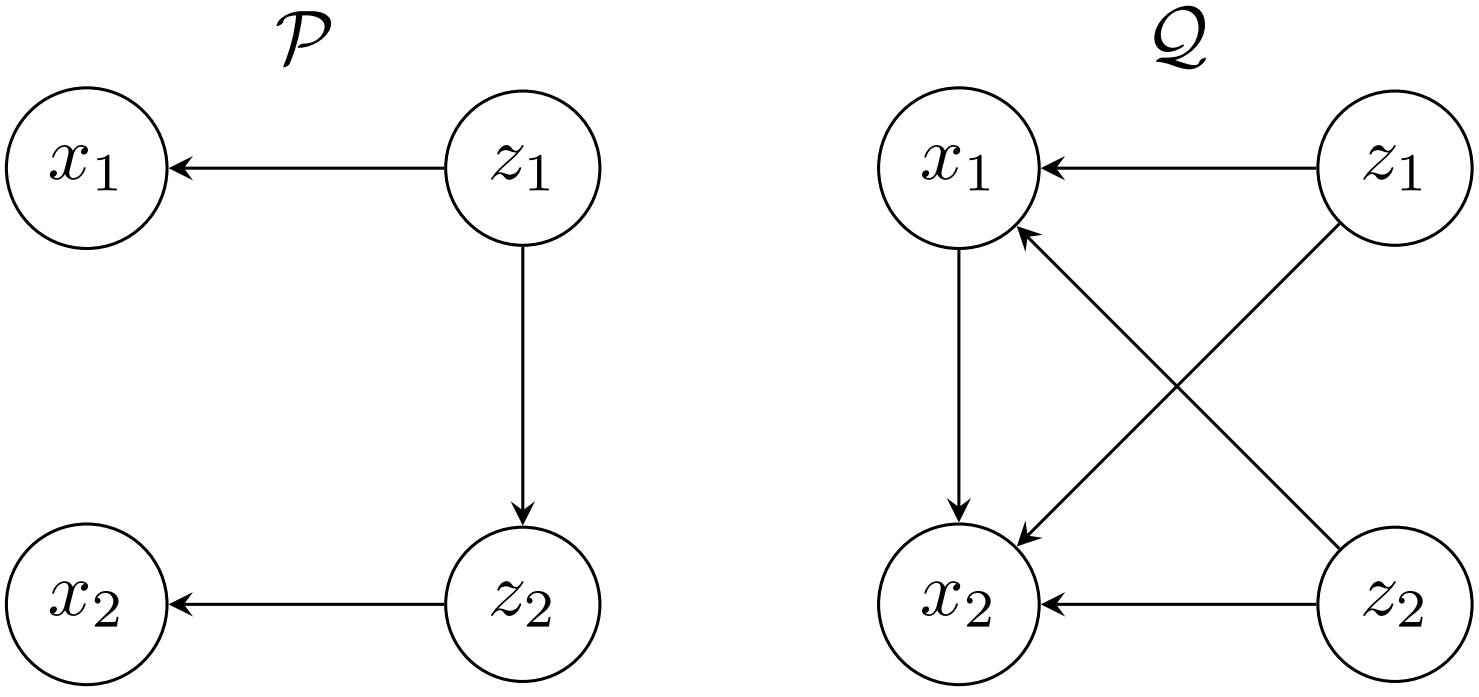
$\mathcal{Q}(0) = \mathcal{Q} \cap \{\tilde{q}(x,z) : \tilde{q}(x) = q_0(x)\}$

$\mathcal{Q}(\theta) = \mathcal{Q}^{\lfloor \frac{1}{\theta} \rfloor}$ for $\theta > 0$

continuity at $\theta = 0$

back

Proof: Correlation Neglect



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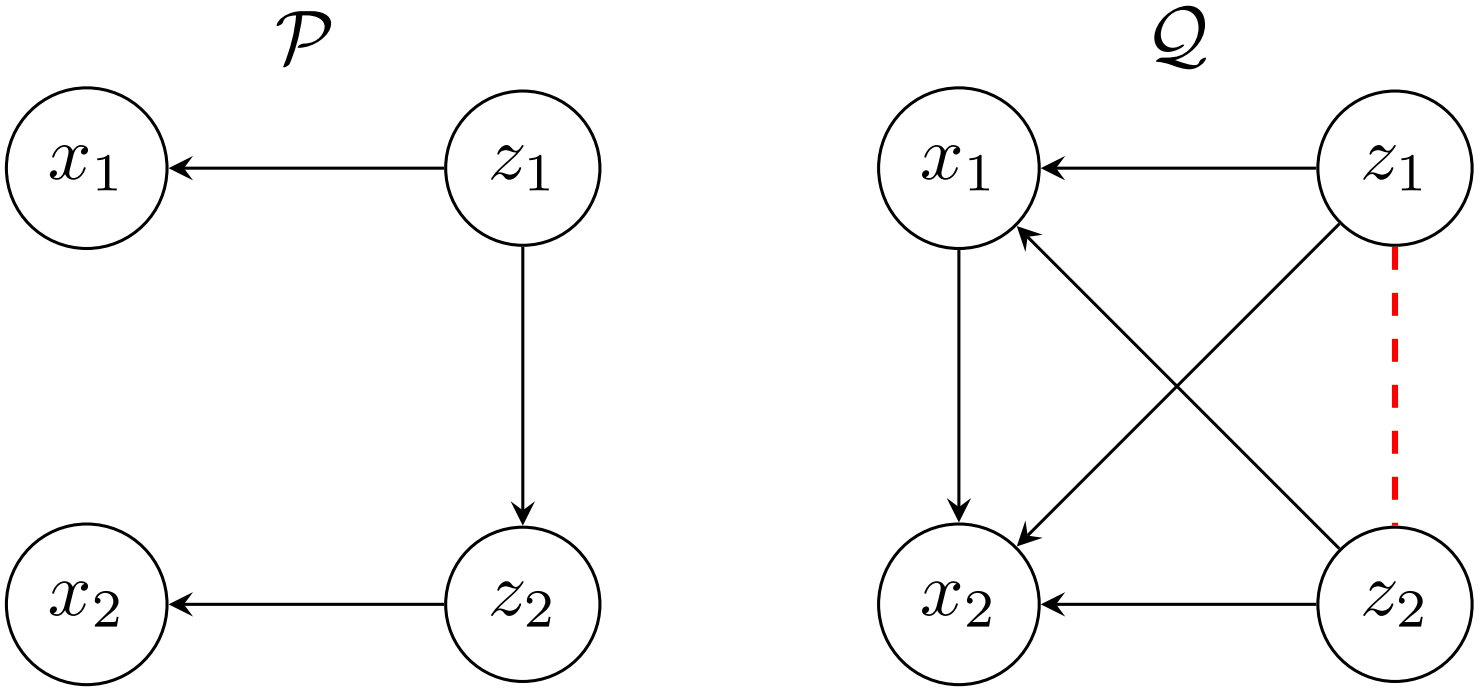

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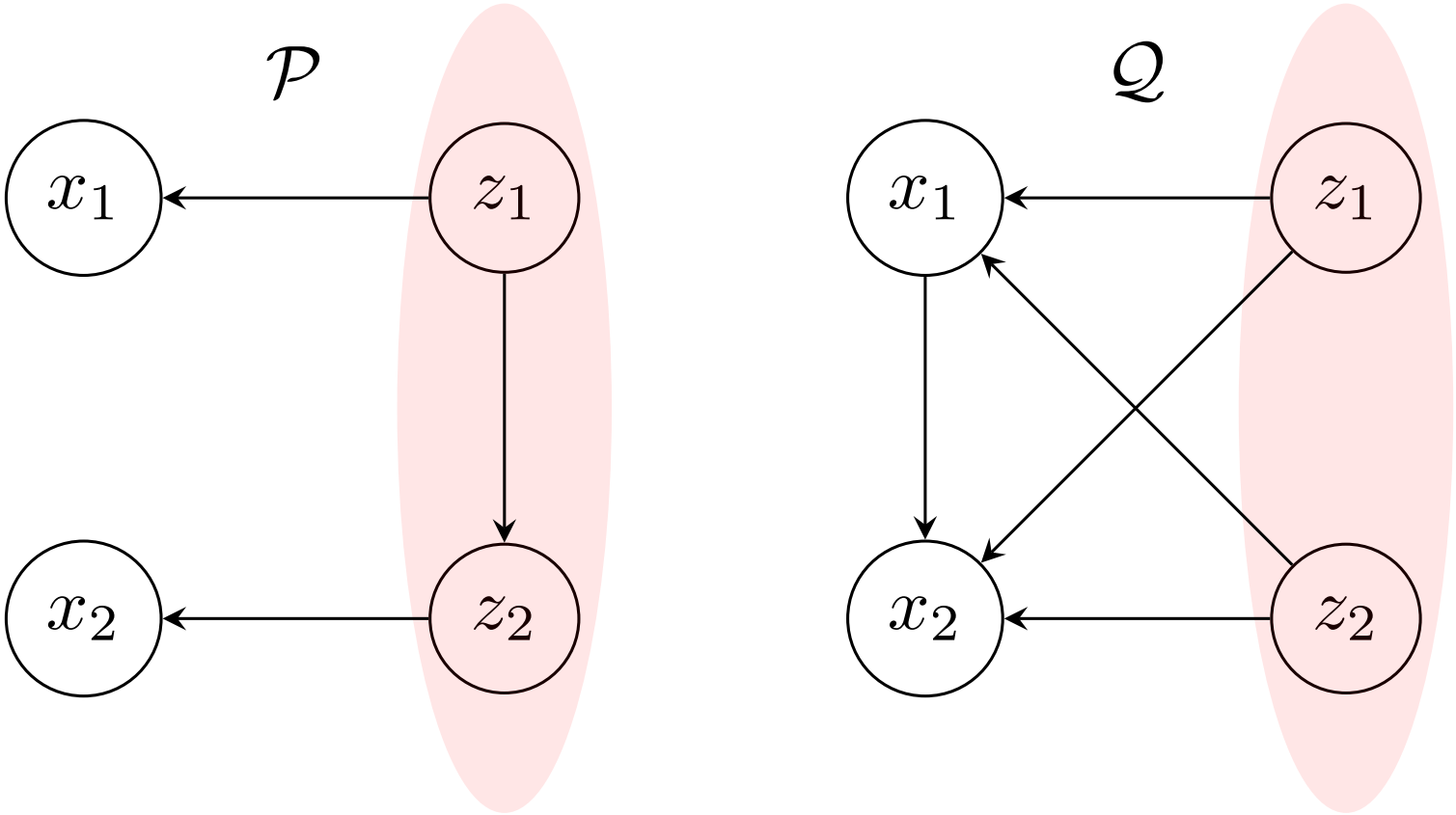

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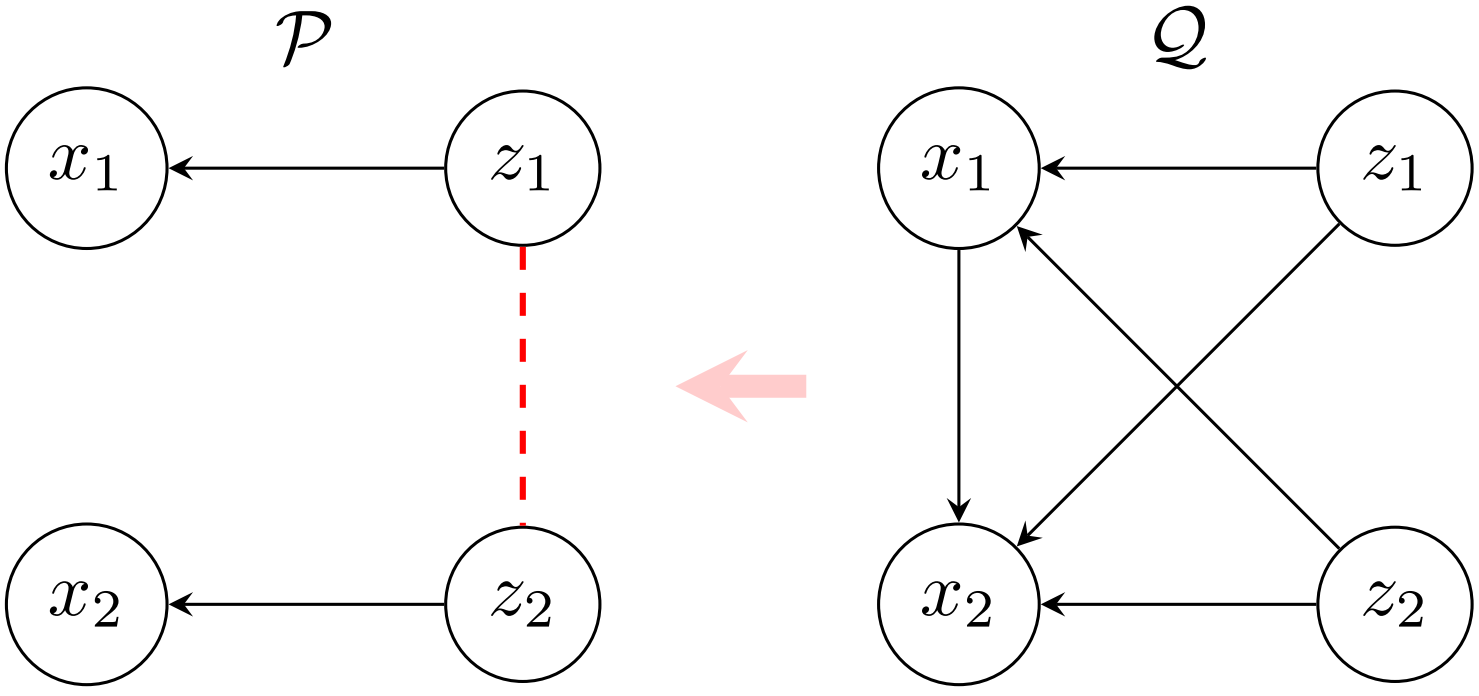

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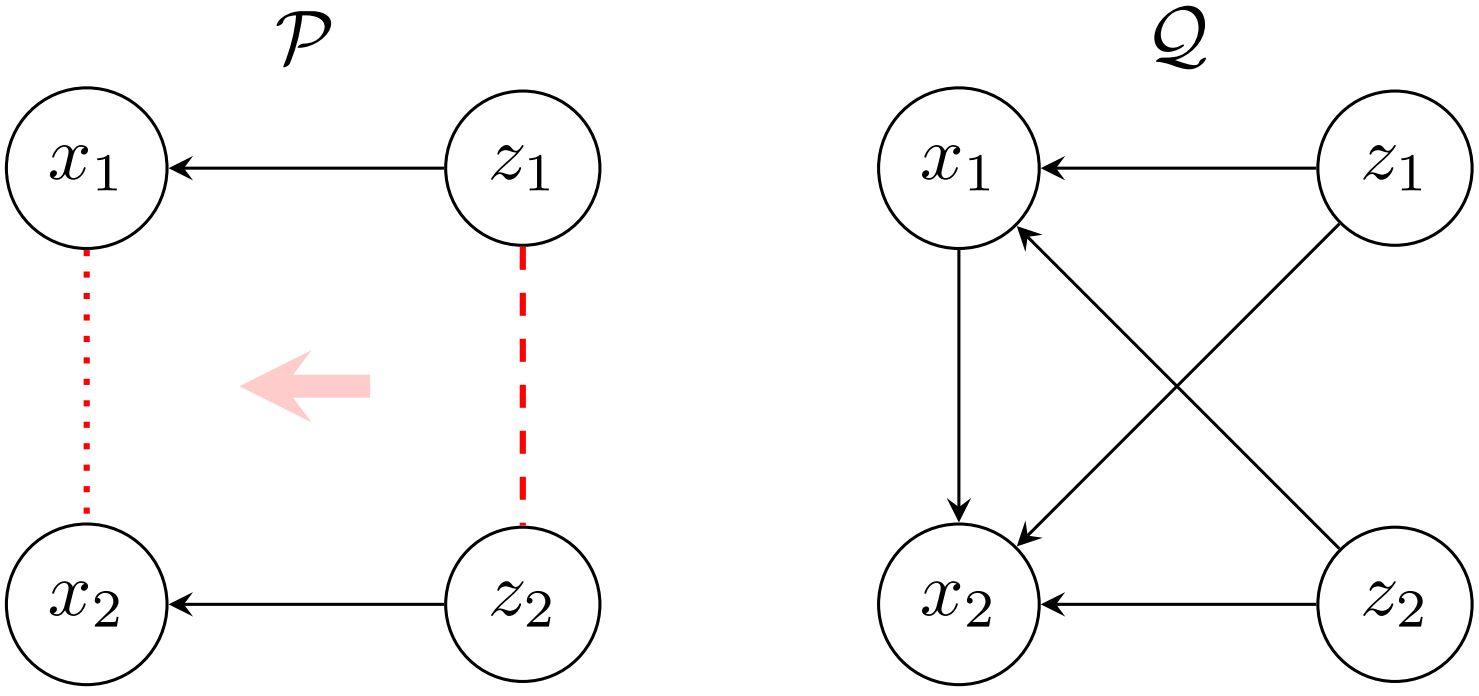

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Details

$$\mathrm{KL}\left(q(x,z) \parallel p(x,z)\right)$$

back

Details

$$\mathrm{KL}\left(q(x) \parallel p(x)\right) + \sum_{x} q(x)\,\mathrm{KL}\left(q(z \mid x) \parallel p(z \mid x)\right)$$

back

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back

Details

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back

Details

$$-\left(\mathrm{E}_{q_0(x)} \ln p(\hat{x}) + \mathrm{H}\left(q_0(x)\right)\right)$$

back