The Importance of Unlikely Events

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Long-Run Investment

how should you choose a pension fund?

- enormous number of possible scenarios

popular advice: assume the typical shock distribution

- based on the Law of Large Numbers
- but incorrect for generic EU maximizers
- rare contingencies can matter enormously

we:

- only one (or two) atypical shock distributions matter
- all other contingencies are too rare/inconsequential

Our Contribution

a systematic study of investment problems with long horizons

the most standard DM: Bayes-rational expected-utility maximizer

generically, choice is driven by rare-event considerations

tools from large deviations theory

information has no value in many settings

a simple representation via a fear-of-ruin constraint

Literature

Kelly'56 vs Samuelson'71,'79

Rietz'88, Barro'06

Robson, Samuelson & Steiner'23, Samuelson & Steiner'25, Millner'25

Weitzman'98 Gollier & Weitzman'10

Hansen & Sargent'01, Maccheroni, Marinacci & Rustichini'06, Strzalecki'11

Decision Problem

$t = 0, \ldots, T$

$a \in A$ chosen at $t = 0$; A compact

a known stochastic process generates $(\theta_1, \ldots, \theta_T)$ on finite Θ

- $q \in \Delta(\Theta)$ an empirical distribution of the sequence (a r.v.)

$$w_T(a) = \prod_{t=1}^{T} R(a, \theta_t) = \exp\left[\sum_{t=1}^{T} r(a, \theta_t)\right] = \exp\left[\mathrm{E}_q\, r(a, \theta)\, T\right] = \exp\left[r(a, q)\, T\right]$$

the DM solves

$$\max_{a \in A} \mathrm{E}\, u\big(w_T(a)\big)$$

$u(w)$ increasing, continuous; $r(a, \theta)$ continuous and concave in a

Examples

finance

- set of assets J
- $a \in A \equiv \Delta(J)$ a stationary rebalanced portfolio
- $r(a, \theta)$ the portfolio's log-return

net-present values

- $R(a, \theta)$ is a stochastic discount factor

demographics

- w_T is the population size

Rate Function

assumption (rate function)

The sequence $(q_T)_{T=1}^{\infty}$ satisfies a large-deviation principle with a rate function $I(q)$.

informally,

$$\Pr(q_T \approx q) \approx e^{-I(q)T}$$

rate functions exist for many processes: iid, Markov, ...

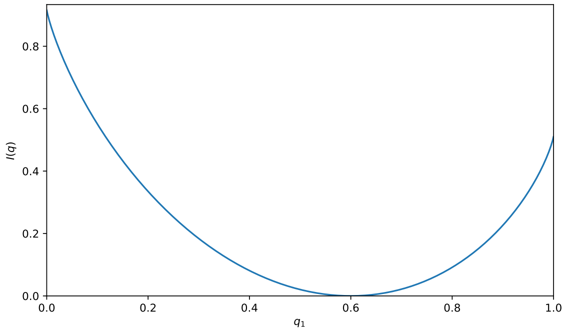
regularity condition $I(q)$ continuous, convex

details

IID Example

θ_t iid from p

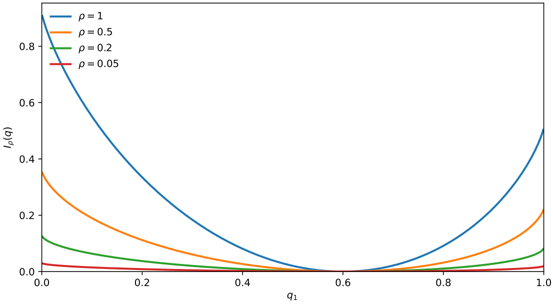
Sanov: $I(q) = \mathrm{KL}(q \| p)$



Persistence Example

Markov transition matrix

$$\begin{bmatrix} 1-\rho p_2 & \rho p_2 \\ \rho p_1 & 1-\rho p_1 \end{bmatrix}$$



Certainty Equivalents

definition

Certainty equivalent $C_T(a)$ is the deterministic growth rate that makes the DM indifferent to a:

$$\operatorname{E} u\big(w_T(a)\big) = u(\exp[C_T(a)\, T])\,.$$

expected-utility maximization is equivalent to

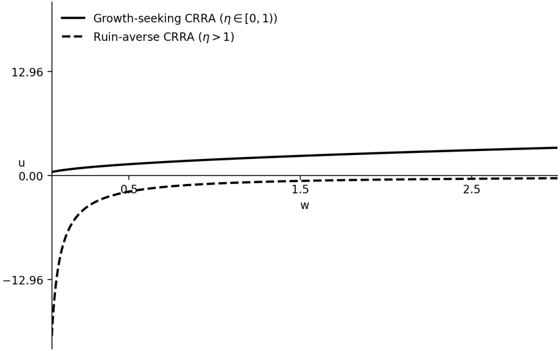
$$\max_{a \in A} C_T(a)$$

asymptotic certainty equivalent:

$$C(a) := \lim_{T \to \infty} C_T(a)$$

CRRA Utility Function

$$u(w) = \frac{w^{1-\eta}}{1-\eta} \qquad (\eta \neq 1).$$



Limit Certainty Equivalents

lemma

growth-seeking DM

$$C_{\text{CRRA}}(a) = \sup_q \left\{ r(a,q) - \frac{1}{1-\eta} I(q) \right\}$$

ruin-averse DM

$$C_{\text{CRRA}}(a) = \inf_q \left\{ r(a,q) + \frac{1}{\eta-1} I(q) \right\}$$

let

$$\mathbb{C}_{\text{CRRA}} := \max_{a \in A} C(a)$$

assume a unique maximizer a^*_{CRRA}

proposition

$a^*_T \to a^*_{\text{CRRA}}$ and $\max_a C_T(a) \to \mathbb{C}_{\text{CRRA}}$.

Intuition

consider a risk-neutral DM

$$
\begin{aligned}
EU_T &= \int_\Delta \exp\left[r(a,q)T\right] d\pi_T(q) \\
&\approx \int_\Delta \exp\left[\left(r(a,q) - I(q)\right)T\right] dq \\
&\approx \exp\Big[\underbrace{\max_{q\in\Delta}\left\{r(a,q) - I(q)\right\}}_{C_{\text{CRRA}}(a;0)}\ T\Big]
\end{aligned}
$$

- growth-seeking: selection on the growth of utility
- ruin-aversion: selection on the growth of disutility

proof via Varadhan's lemma

Example: Welfare Cost of Business Cycles

let's revisit the binary Markov chain

- stationary distribution is independent of persistence
- persistence doesn't affect welfare on the typical path

assume, realistically, $\eta > 1$

- welfare decreases with persistence
- the job of the adversarial Nature is getting easier

Perfect Foresight

the DM with perfect foresight knows q_T before choosing a

asymptotic perfect-foresight certainty equivalent

$$\mathbb{C}_{\text{pf}} = \lim_{T\to\infty} \sup_{a(\cdot)} C_{\text{pf},T}\big(a(\cdot)\big);$$

the deterministic growth rate as good as perfect foresight

Proposition

At the exponential rate, the uninformed CRRA DM achieves the same payoff as with perfect foresight:

$$\mathbb{C}_{\text{CRRA}} = \mathbb{C}_{\text{pf}}.$$

a CRRA DM won't pay any fraction of return for perfect foresight

Proof (ruin aversion)

zero-sum game between the DM and Nature with payoff

$$r(a, q) + \frac{1}{\eta - 1} I(q)$$

uninformed sequential game:

- DM chooses a
- Nature chooses q

perfect-foresight sequential game:

- Nature chooses q
- DM chooses a

Minimax theorem: order of moves doesn't matter

Intuition

the perfect-foresight DM achieves growth rate

$$v(q) = \max_a r(a, q) \text{ in each contingency } q$$

but only one such contingency, q^*, dominates the EU aggregation

uninformed DM best responds q^* and matches the perfect-foresight DM

Example: Kelly Meets Samuelson

Kelly maximizes the typical growth rate

$$a^*_{\text{Kelly}}(p) \in \arg\max_{a \in \Delta(J)} r(a, p), \text{ achieving } v(p)$$

a CRRA investor chooses a Kelly portfolio

$$a^*_{\text{CRRA}} = a^*_{\text{Kelly}}(q^*),$$

for a distorted belief

$$\arg\max_q \left\{ v(q) - \frac{1}{1-\eta} I(q) \right\} \text{ (growth-seeking)}$$

$$\arg\min_q \left\{ v(q) + \frac{1}{\eta-1} I(q) \right\} \text{ (ruin-aversion)}$$

Value of Information

for each T, the DM observes x generated by $\mu_T(x \mid q_T)$ and chooses $a(x)$

Value of Information

$$\text{VoI} = \lim_{T\to\infty} \left(\sup_{a(\cdot)} C_{\text{info},T}\left(a(\cdot)\right) - \sup_{a\in A} C_T(a) \right)$$

Corollary

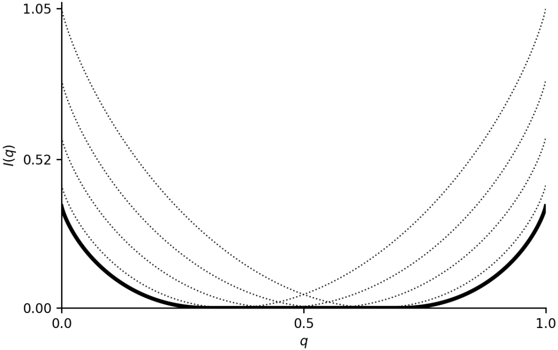
Information has no value for a CRRA DM at the exponential rate

$$\text{VoI} = 0.$$

Example: Hierarchical Process

x drawn from $X \subset \Delta(\Theta)$

θ_t iid from x



Hedging

the DM splits initial wealth into subfunds $k = 1, \dots, K$ with shares λ_k

$$w_T(\mathbf{a}, q_T) = \sum_k \lambda_k w_T(a_k, q_T)$$

hedging is powerful:

- asymptotically, it achieves the best response to q within $\{a_1, \dots, a_K\}$

Value of Hedging

Value of Hedging

$$\text{VoH} = \lim_{T\to\infty}\left(\sup_{\mathbf{a}} C_{\text{hedge},T}(\mathbf{a}) - \max_{a\in A} C_T(a)\right);$$

Corollary

Hedging has no value for a CRRA DM at the exponential rate

$$\text{VoH} = 0.$$

Two-Tailed Utilities

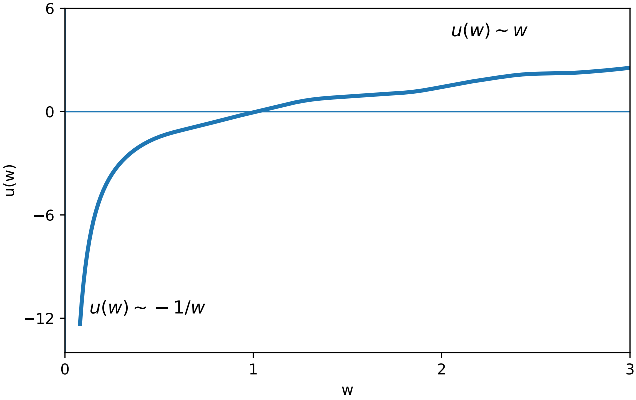
$u : \mathbb{R}_{++} \to \mathbb{R}$ increasing, continuous, and

$$\lim_{w\to\infty} \frac{\ln u(w)}{\ln w} = 1 - \eta^+, \qquad \lim_{w\downarrow 0} \frac{\ln(-u(w))}{\ln w} = 1 - \eta^-.$$

with $\eta^+ \in [0, 1)$ and $\eta^- > 1$

both growth-seeking and ruin aversion

In This Presentation



Certainty Equivalents

recall from the CRRA setting:

$$C^+(a) = \max_q \{r(a,q) - I(q)\}$$

$$C^-(a) = \min_q \{r(a,q) + I(q)\}$$

Lemma

The asymptotic certainty equivalent for the two-tailed utility is

$$C(a) = \begin{cases} C^+(a) & \text{if } C^+(a) + C^-(a) > 0, \\ C^-(a) & \text{if } C^+(a) + C^-(a) < 0. \end{cases}$$

intuition:

- $EU \approx e^{C^+(a)T} - e^{-C^-(a)T}$
- the larger growth coefficient prevails

Solution

definition

Ruin-robust growth program is

$$\begin{aligned} \max_{a \in A} \quad & C^+(a) \\ \text{s.t.} \quad & C^+(a) + C^-(a) \geq 0, \end{aligned}$$

with a convention that if no a satisfies the constraint, then the DM solves

$$\max_{a \in A} C^-(a).$$

Let $\mathbb{C}_{\text{tt}}$ be the value and assume a unique maximizer a^*_{tt}.

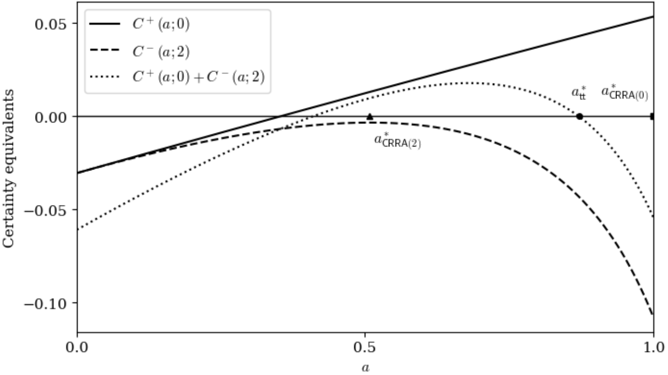
regularity condition: any a that generates a tie can be approximated by actions at which the upper tail is selected.

Proposition

$a^*_T \to a^*_{\text{tt}}$ and $\max_a C_T(a) \to \mathbb{C}_{\text{tt}}$.

Example: Equity Premium Puzzle

$$R(a,\theta) = \begin{array}{c|cc} & H & L \\ \hline s & 1.10 & 0.2 \\ c & 0.97 & 0.97 \end{array}$$



Valuable Information

Lemma

The perfect-foresight certainty equivalent for the two-tailed utility is

$$\mathbb{C}_{\mathsf{pf}} = \begin{cases} \mathbb{C}^{+}_{\mathsf{CRRA}} & \text{if } \mathbb{C}^{+}_{\mathsf{CRRA}} + \mathbb{C}^{-}_{\mathsf{CRRA}} > 0, \\ \mathbb{C}^{-}_{\mathsf{CRRA}} & \text{if } \mathbb{C}^{+}_{\mathsf{CRRA}} + \mathbb{C}^{-}_{\mathsf{CRRA}} < 0. \end{cases}$$

$\Rightarrow$ information can generate value for two-tailed DMs

Hedging Achieves Perfect Foresight

Proposition

The DM with two-tailed utility who can hedge achieves the same payoff at the exponential rate as if she had perfect foresight:

$$\mathbb{C}_{\text{hedge}} = \mathbb{C}_{\text{pf}}.$$

constructive proof: hedge with two subfunds, with $a^*_{\text{CRRA}(+)}$ and $a^*_{\text{CRRA}(-)}$ matches the perfect foresight, asymptotically

Kelly Meets Samuelson Again

A two-tailed investor achieves the upper bound, $\mathbb{C}_{\mathsf{pf}}$, on her long-run performance by dividing her initial wealth into two distinct Kelly portfolios,

$$a^*_{\text{Kelly}}(q^*_+) \text{ and } a^*_{\text{Kelly}}(q^*_-),$$

optimized for Nature's benevolent and malevolent large deviations:

$$q^*_+ \in \arg\max_{q \in \Delta(\Theta)} \left\{ v(q) - I(q) \right\},$$

$$q^*_- \in \arg\min_{q \in \Delta(\Theta)} \left\{ v(q) + I(q) \right\}.$$

Hedging Makes Information Redundant

Define the *marginal value of information (given hedging)*,

$$\mathrm{mVoI} = \lim_{T\to\infty}\left(\sup_{\mathbf{a}(\cdot)} C_{\mathrm{hedge,info},T}(\mathbf{a}(\cdot)) - \sup_{\mathbf{a}} C_{\mathrm{hedge},T}(\mathbf{a})\right),$$

as the asymptotic increase of the certainty equivalent due to information relative to that of the DM who can hedge but has no information.

Corollary

$$\mathrm{mVoI} = 0.$$

Utilitarian Social Planner

continuous homogenous population with private shocks

utilitarian welfare = expected utility

but welfare evaluation is driven by a vanishing fraction of population

Truncated Utilitarian Welfare

assume the SP disregards extreme wealth tails

truncated utilitarian welfare:

$$W_T^{\varepsilon}(a) = \int_{\varepsilon}^{1-\varepsilon} u(w) dP(w; a).$$

certainty equivalent

$$(1-2\varepsilon)\, u\Big(\exp[C_T^{\varepsilon}(a)\, T]\Big) = W_T^{\varepsilon}(a);$$

delivers the same welfare once the ε wealth tails are truncated

Proposition

For any $\varepsilon \in (0, 1/2)$,

$$\lim_{T\to\infty} C_T^{\varepsilon}(a) = r(a, p).$$

Conclusion

EUT for long-horizon growth control is

- driven by rare events
- tractable

Is it normatively appealing?

Details

The family of probability measures $(\pi_T)_{T=1}^{\infty}$ over q_T satisfies the large deviation principle with a rate function I:

For any closed set $F \subseteq \Delta(\Theta)$ and any open set $G \subseteq \Delta(\Theta)$,

$$\lim \sup_{T\to\infty} \quad \frac{1}{T} \ln \pi_T(F) \leq - \inf_{q\in F} I(q),$$

$$\lim \sup_{T\to\infty} \quad \frac{1}{T} \ln \pi_T(G) \geq - \inf_{q\in G} I(q).$$

back