Optimal test allocation

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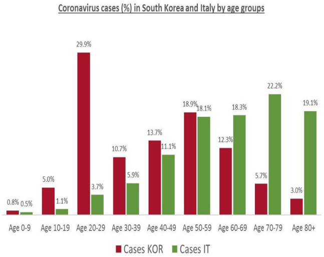
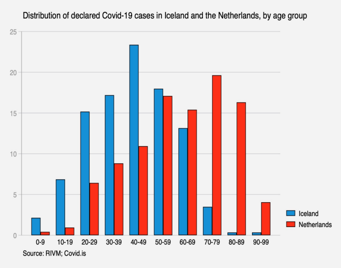
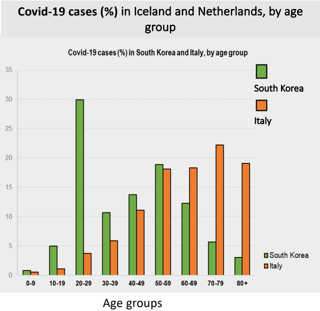
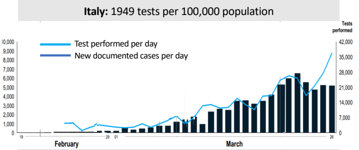
Testing

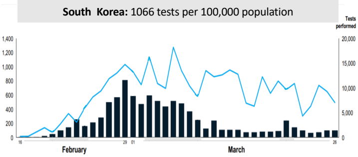

Coronavirus cases (%) in South Korea and Italy by age groups
29.9%
0.8% 0.5%
5.0% 1.1%
3.7%
10.7% 5.9%
13.7% 11.1%
18.9% 18.1%
12.3% 18.3%
5.7% 22.2%
3.0% 19.1%
Age 0-9 Age 10-19 Age 20-29 Age 30-39 Age 40-49 Age 50-59 Age 60-69 Age 70-79 Age 80+
Cases KOR Cases IT


Distribution of declared Covid-19 cases in Iceland and the Netherlands, by age group
25
20
15
10
5
0
0-9 10-19 20-29 30-39 40-49 50-59 60-69 70-79 80-89 90-99
Iceland
Netherlands
Source: RIVM; Covid.is

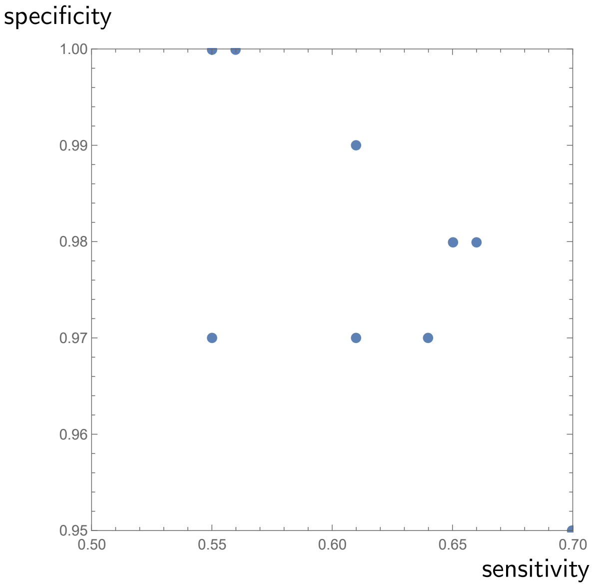
Private and social value of testing



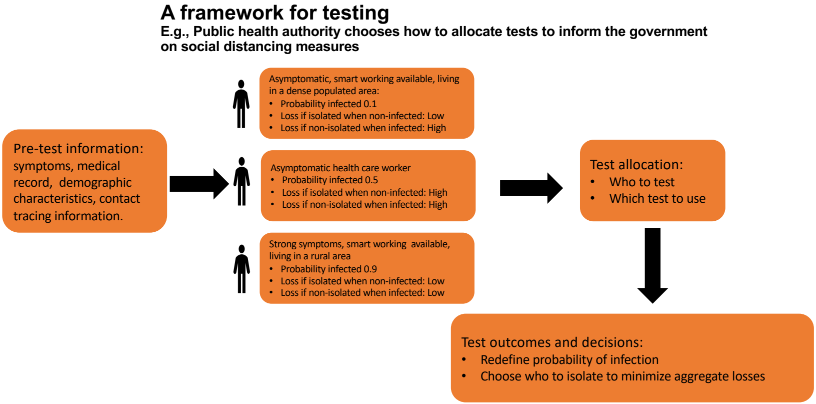




Eight serological tests as validated in Adams et al. (2020).



A simple framework



A basic example

The decision maker has to choose whether to isolate or not Andrea

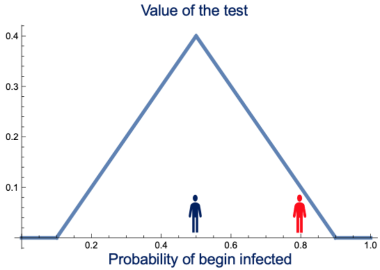
State\Action	Isolate	Not isolate
Infected	Society gains 1	Society loses -1
Not infected	Society loses -1	Society gains 1

Scenario 1: Andrea is asymptomatic and the probability of being infected is estimated to be 0.5

Scenario 2: Andrea is symptomatic and the probability of being infected is 0.8

You have a test that has 2% of false-positive (specificity) and 2% of false-negative (sensitivity)

In which scenario is the test more valuable?



This paper

- The health authority has some pre-test probabilities of infections for each individual
- It has a portfolio of tests, possibly scarce
- Tests are allocated, result is observed, and the health authority takes an action for each individual
- There are losses to take the wrong action given the true individual state
- **Question:** What is the shape of the optimal allocation?
- **Question:** What is the value of a new test?

Model

- Set $\mathcal{I} = \{1, 2, ..., I\}$ of individuals in two possible states $\theta_i \in \{0, 1\}$ and $p_i = \Pr[\theta_i = 1]$

- Health authority chooses an action $a_i \in \{0, 1\}$ for each i and wishes to match i's state

- Wrong action leads to losses:

$$l_i^\theta = u_i(\theta, \theta) - u_i(1 - \theta, \theta) > 0.$$

- Set of available tests $\mathcal{T}$. $t(1|1)$ is the test t sensitivity and $t(0|0)$ is the test t specificity.

Problem

- Define

$$v_i(q) = \max_{a \in \{0,1\}} \{q u_i(a,1) + (1-q) u_i(a,0)\}$$

The value of a test t to individual i is

$$V(p_i, t) = \underbrace{\mathbb{E}_x[v_i(q_{t,p_i}(x))]}_{\text{Expected losses after testing}} - \underbrace{v_i(p_i)}_{\text{Expected losses without testing}}$$

- Find a one-to-one test allocation rule $\tau : I \to \mathcal{T}$ that solves

$$\max_{\tau} \sum_{i \in \mathcal{I}} V(p_i, \tau(i))$$

Quite standard problem

- Allocation problem in which the DM has complete information about the surplus from each matched pair of the individual and the test
- See Koopmans and Beckmann (1957) for the linear-programming solution for a general class of these problems
- Our contributions: provide conditions on the shape of optimal allocation that follows from the specific structure of the testing problem

Analysis: Simple transformation

Solving the optimization problem is equivalent to solve the following modified problem.

$$\max_{\tau} W(p_i, \ell_i, \tau(i)),$$

where

- $W(p, \ell, t) = p\ell^1 t(1|1) + (1-p) l^0 t(0|0)$
- $\hat{\mathcal{T}}$ be like $\mathcal{T}$ but with two types of uninformative tests: $\emptyset^x$ is the test that always returns you $x \in \{0, 1\}$
- $\tau : \hat{\mathcal{T}} \to \mathcal{I}$

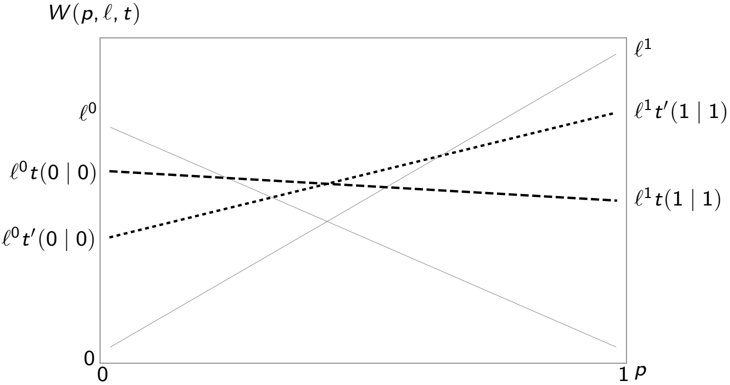


Figure: Modified values $W(p, \ell, t)$ for a population with individual-independent losses $\ell_i = \ell = (\ell^0, \ell^1)$. The two full lines: tests $\emptyset^0$ and $\emptyset^1$

Homogenous losses

Define the *slope* of the test t as

$$\sigma_t = t(1 \mid 1)\ell^1 - t(0 \mid 0)\ell^0.$$

That is, the slope of test t is the loss-weighted difference between its sensitivity and specificity.

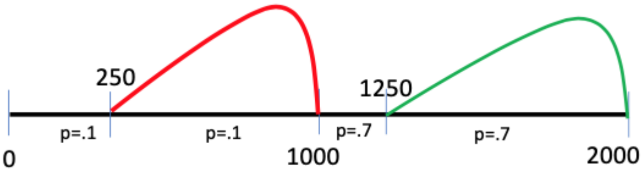
Corollary

Suppose $\ell_i^0 = \ell_j^0$ *and* $\ell_i^1 = \ell_j^1$ *for all* $i, j \in \mathcal{I}$.

1. *Slopes of the optimally allocated tests are nondecreasing in the individuals' pre-test probabilities. That is, if* $p_i > p_j$*, then* $\sigma_{\tau(i)} \geq \sigma_{\tau(j)}$.
2. *Individuals with sufficiently low or high pre-test probabilities are not tested. That is, there exists* $\underline{p} < \overline{p}$ *such that, if* $p_i < \underline{p}$*, then the DM chooses* $a_i = 0$ *without testing individual* i*. If* $p_i > \overline{p}$*, then the DM chooses* $a_i = 1$ *without testing* i*. If* $\underline{p} < p_i < \overline{p}$*, then the DM applies a non-trivial test to* i *and chooses* a_i *equal to the test's result.*

Example

Test	Specificity	Sensitivity	Capacity
Specific	99%	61%	750
Sensitive	95%	70	750



No testing: 400 errors
Random allocation: 334 errors
Random among the 1500 riskier: 331 errors
Optimal allocation: 304 errors

Homogenous pre-test probabilities

Corollary

Suppose $p_i = p_j$ *for all* $i, j \in \mathcal{I}$.

1. *If* $\ell_i^1 = \ell_j^1$ *for all* $i, j \in \mathcal{I}$ *then specificities of the optimally allocated tests are nondecreasing in the individuals' false-positive losses. That is, if* $\ell_i^0 >= \ell_j^0$ *then* $\tau(i)(0|0) \geq \tau(j)(0|0)$.
2. *If* $\ell_i^0 = \ell_j^0$ *for all* $i, j \in \mathcal{I}$ *then sensitivities of the optimally allocated tests are nondecreasing in the individuals' false-negaive losses. That is, if* $\ell_i^1 >= \ell_j^1$ *then* $\tau(i)(1|1) \geq \tau(j)(1|1)$.

Marginal benefit

- We characterise the marginal benefit of a test that becomes newly available relative to the current DM's portfolio
- This is the difference between: The value of the new optimal allocation (when the new test is added) and the value of the optimal allocation to start with
- In principle optimal allocation can change dramatically
- Contribution: use of monotonicity to provide simple algorithms to compute marginal benefit

Marginal benefit: Example

	Sensitivity	Specificity	Marginal benefit
Wondfo	69%	99.1%	0.13
Livzon	78.7%	99.7%	0.17
“sensitive” test	70%	95%	0.097
“specific” test	61%	99%	0.11

Table 1: Four serological tests. See footnote 4 for the validation studies for the first two tests and Adams et al. (2020) for validation of the last two tests. Since we are unable to verify the details of the validation studies, these tests' precision parameters are illustrative.

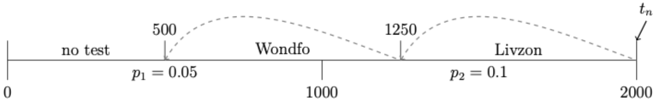


Figure 3: Optimal allocation and optimal replacement chain caused by replacement of one Livzon test with the “sensitive” test t_n.

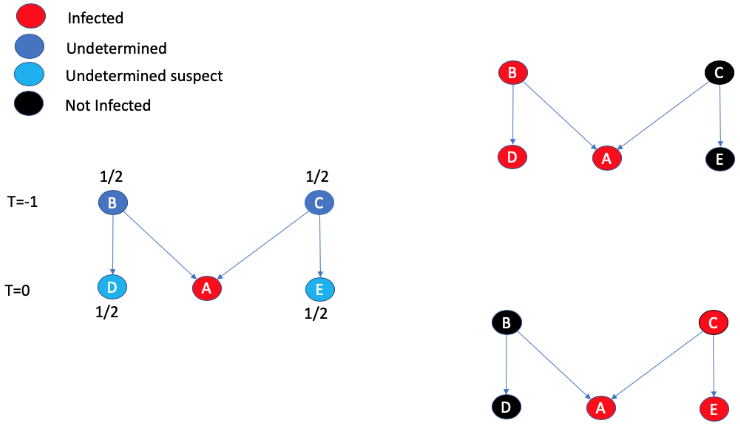
Open questions 1: Test and tracing

- Work in progress with Ely and Steiner
- Basic idea: Once you find out that Ann is infected, tracing and testing must take into considerations the correlations in infection across traced individuals.
- Think about tracing as a test: If I trace you from Ann that is a signal about your state of infection
- Traced individuals that highly correlated with the state of Ann have no value of being tested
- Traced individuals that very mildly correlate with the state of Ann have no value of being tested
- The others should be tested

Open questions 1: A simple example of Test and tracing

- Two interactions per day (sampled from a large/continuum population)
- Very small infection rates (this is when targeted contact tracing is feasible). Covid prevalence is p and we think $p \to 0$
- If infected A meets B, then B becomes infected
- Infected A recovers after one day

Open questions 1: A simple example of Test and tracing



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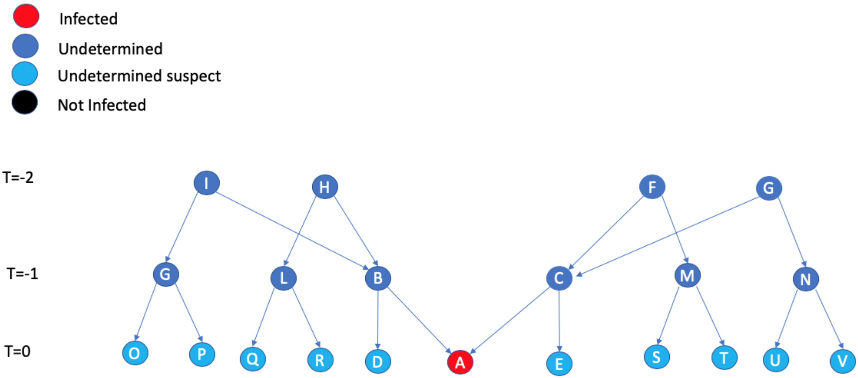
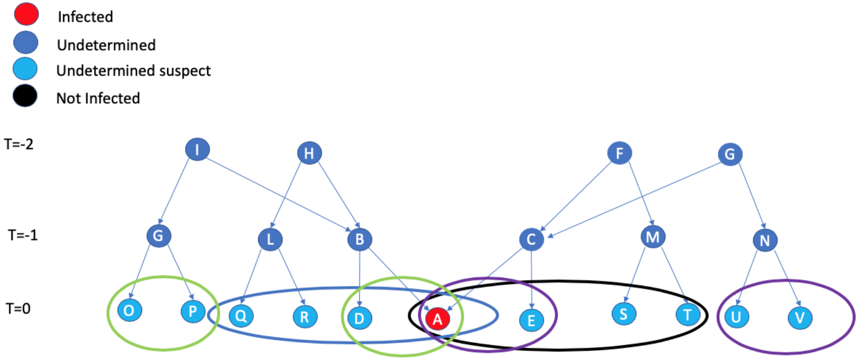


Open questions 1: A simple example of Test and tracing



11 Undetermined suspects classified in four 4 equivalence classes. Hence, four representative suspects

Conclusion

THANKS