Finance 400

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"Recursive Utility"

A class of non-time-separable utility known as *recursive utility* has been studied by a number of authors. D. Kreps and E. Porteus (1978) "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica* 46, p.185-200 and L. Epstein and S. Zin (1989) "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica* 57, p.937-69 analyze this type of utility in a discrete-time setting, while D. Duffie and L. Epstein (1992) "Asset Pricing with Stochastic Differential Utility," *Review of Financial Studies* 5, p.411-36 study the continuous-time limit. In continuous-time, recall that standard, time-separable utility can be written as:

$$V_t = E_t\left[\int_t^T U[C(s),\, s]\, ds\right]$$

where $U[C(s),\, s]$ is often taken to be of the form $U[C(s),\, s] = e^{-\rho(s-t)}u[c(s)]$. Recursive utility, however, is specified as

$$V_t \;=\; E_t\left[\int_t^T f[C(s),\, V_s]\, ds\right]$$

where f is known as an "aggregator" function. The specification is recursive in nature because current utility, V_t, depends on expected values of *future* utility, V_s, $s > t$. When f has appropriate properties, D. Duffie and L. Epstein (1992) "Stochastic Differential Utility," *Econometrica* 60, p.353-94 show that a Bellman-type equation can be derived which characterizes the optimal consumption and portfolio choice policies for utility of this type. For particular functional forms, they have been able to work out a number of asset pricing models.

These notes consider general equilibrium in an economy where representative consumer-investors have recursive utility. The specific model that we will consider is that of Maurice Obstfeld (1994) "Risk-Taking, Global Diversification, and Growth" *American Economic Review* 84, p.1310-1329.

<u>Assumptions</u>:

A.1 Technology:

A single capital-consumption good can be invested in up to two different technologies. The first is a risk-free technology whose output, $V^B(t)$, follows the process

$$dV^B/V^B = idt \tag{1}$$

The second is a risky technology whose output, $V^K(t)$, follows the process

$$dV^K/V^K = \alpha dt + \sigma dz \tag{2}$$

Comment:

Note that in this "production" economy the specification of technologies fixes the expected rates of return and variances of the safe and risky investments. Individuals' asset demands will determine equilibrium quantities of the assets supplied rather than asset prices. Since i, α, and σ are assumed to be constants, there is a constant investment opportunity set.

A.2 Preferences:

Representative, infinitely-lived households must choose between consuming (at rate C_s at date s) and investing in the single capital-consumption good. The lifetime utility function at date t faced by each of these households, denoted U_t, is

$$U_t = E_t \int_t^{\infty} f(C_s, U_s)\, ds \tag{3}$$

where f is given by

$$f(C_s, U_s) = \delta \frac{\left\{ C_s^{1-\left(\frac{1}{\epsilon}\right)} - \left[(1-R)\, U_s\right]^{\frac{\epsilon-1}{\epsilon(1-R)}} \right\}}{\left\{ \left(1-\frac{1}{\epsilon}\right) \left[(1-R)\, U_s\right]^{\frac{\epsilon-1}{\epsilon(1-R)}-1} \right\}} \tag{4}$$

Comment:

Note this specification's recursive nature in that current lifetime utility, U_t, depends on expected values of *future* lifetime utility, U_s, $s > t$. $f(C, U)$ is known as the "aggregator" function. The form of equation (4) is ordinally equivalent to the continuous-time limit of the discrete-time utility function specified in Obstfeld (1994). Recall that utility functions are ordinally equivalent, that is, they result in the same consumer choices, if the utility functions evaluated at equivalent sets of decisions produce values that are linear transformations of each other. It can be shown (see Epstein and Zin (1989) and Duffie and Epstein (1992) that $\delta > 0$ is the continuously compounded subjective rate of time preference, $\epsilon > 0$ is the household's elasticity of intertemporal substitution, and $R > 0$ is the household's coefficient of relative risk aversion. For the special case of $R = 1/\epsilon$, the utility function given in (3) and (4) is (ordinally) equivalent to the time-separable, constant relative risk-aversion case:

$$U_t = E_t \int_t^{\infty} e^{-\delta s} \frac{C_s^{1-R}}{1-R} ds \tag{5}$$

Let $\omega(t)$ be the proportion of each household's wealth invested in the risky asset (technology). The the intertemporal budget constraint is given by

$$dW = [\omega(\alpha - i)W + iW - C]\, dt + \omega\sigma W dz \tag{6}$$

When the aggregator function, f, is put in a particular form by an ordinally equivalent change in variables, what Duffie and Epstein (1992) *Econometrica* refer to as a "normalization" then a Bellman equation can be used to solve the problem. The aggregator in (4) is in "normalized" form.

As before, let us define $J(W_t)$ as the maximized lifetime utility at date t

$$\begin{aligned} J(W_t) &= \max_{\{C,w\}} E_t \int_t^{\infty} f(C_s, U_s)\, ds \\ &= \max_{\{C,w\}} E_t \int_t^{\infty} f(C_s, J(W_s))\, ds \end{aligned} \tag{7}$$

Since this is an infinite horizon problem with constant investment opportunities, and the aggregator function, $f(C, U)$, is not an explicit function of calendar time, the only state variable is W.

Then the solution to the consumer's consumption - portfolio choice problem is given by the continuous-time stochastic Bellman equation

$$0 = \max_{\{C,w\}} f[C_t, J(W_t)] + L[J(W_t)] \tag{8}$$

or

$$
\begin{aligned}
0 &= \max_{\{C,w\}} f\left[C, J(W)\right] + J_W\left[\omega(\alpha - i)W + iW - C\right] + \frac{1}{2}J_{WW}\omega^2\sigma^2W^2 \qquad (9)\\
&= \max_{\{C,w\}} \delta\frac{\left\{C^{1-\left(\frac{1}{\epsilon}\right)} - \left[(1-R)J\right]^{\frac{\epsilon-1}{\epsilon(1-R)}}\right\}}{\left\{\left(1-\frac{1}{\epsilon}\right)\left[(1-R)J\right]^{\frac{\epsilon-1}{\epsilon(1-R)}-1}\right\}} + J_W\left[\omega(\alpha - i)W + iW - C\right] + \frac{1}{2}J_{WW}\omega^2\sigma^2W^2
\end{aligned}
$$

Taking the first order condition with respect to C,

$$
\delta\frac{C^{-\left(\frac{1}{\epsilon}\right)}}{\left[(1-R)J\right]^{\frac{\epsilon-1}{\epsilon(1-R)}-1}} - J_W = 0 \qquad (10)
$$

or

$$
C = \left(\frac{J_W}{\delta}\right)^{-\epsilon}\left[(1-R)J\right]^{\frac{1-\epsilon}{1-R}+\epsilon} \qquad (10')
$$

Taking the first order condition with respect to ω,

$$
J_W(\alpha - i)W + J_{WW}\omega\sigma^2W^2 = 0 \qquad (11)
$$

or

$$
\omega = -\frac{J_W}{J_{WW}W}\frac{(\alpha - i)}{\sigma^2} \qquad (11')
$$

Substituting the optimal values for C and ω given by (10) and (11) into the Bellman equation (9) we obtain the differential equation:

$$
\delta\left\{\frac{\left(\frac{J_W}{\delta}\right)^{1-\epsilon}\left[(1-R)J\right]^{(\epsilon-1)\left[1-\frac{\epsilon-1}{\epsilon(1-R)}\right]} - \left[(1-R)J\right]^{\frac{1-\epsilon}{\epsilon(1-R)}}}{\left(1-\frac{1}{\epsilon}\right)\left[(1-R)J\right]^{\frac{\epsilon-1}{\epsilon(1-R)}-1}}\right\} \qquad (12)
$$

$$
+J_W\left[-\frac{J_W}{J_{WW}}\frac{(\alpha - i)^2}{\sigma^2} + iW - \left(\frac{J_W}{\delta}\right)^{-\epsilon}\left[(1-R)J\right]^{\frac{1-\epsilon}{1-R}+\epsilon}\right] + \frac{1}{2}\frac{J_W^2}{J_{WW}}\frac{(\alpha - i)^2}{\sigma^2} = 0
$$

or

$$\frac{\epsilon\delta}{\epsilon-1}\left\{\left(\frac{J_W}{\delta}\right)^{-\epsilon}\left[(1-R)\,J\right]^{\frac{1-\epsilon}{1-R}+\epsilon}-\left[(1-R)\,J\right]\right\} \tag{12'}$$

$$+J_W\left[-\frac{J_W}{J_{WW}}\frac{(\alpha-i)^2}{\sigma^2}+iW-\left(\frac{J_W}{\delta}\right)^{-\epsilon}\left[(1-R)\,J\right]^{\frac{1-\epsilon}{1-R}+\epsilon}\right]+\frac{1}{2}\frac{J_W^2}{J_{WW}}\frac{(\alpha-i)^2}{\sigma^2}=0$$

If one "guesses" that the solution is of the form $J(W)=(aW)^{1-R}/(1-R)$ and substitutes this into ($12'$), one finds that $a=\mu^{1/(1-\epsilon)}$ where

$$\mu=\epsilon\delta^{1-\epsilon}-(\epsilon-1)\,\delta^{-\epsilon}\left[i+\frac{(\alpha-i)^2}{2R\sigma^2}\right] \tag{13}$$

Thus, substituting this value for J into ($10'$) we find that optimal consumption is a fixed proportion of wealth

$$C=\mu W \tag{14}$$

and the optimal portfolio weight of the risky asset is

$$\omega=\frac{\alpha-i}{R\sigma^2} \tag{15}$$

which is the same as for an individual with standard constant relative risk-aversion and time-seperatble utility.

The results show that the portfolio choice decision depends only on risk-aversion, R, but that the consumption-saving decision depends on both risk-aversion and the elasticity of intertemporal substitution, ϵ.

We can study how the growth rate of the economy depends on the model's parameters. Assuming $0<\omega<1$ and substituting (14) and (15) into (6) we have that wealth follows the process

$$dW/W \quad = \quad \left[\omega^*(\alpha-i)+i-\mu\right]dt+\omega^*\sigma dz \tag{16}$$

$$
\begin{aligned}
&= \left[\frac{(\alpha-i)^2}{2R\sigma^2} + i - \epsilon\delta + (\epsilon-1)\left[i + \frac{(\alpha-i)^2}{2R\sigma^2}\right]\right] dt + \frac{(\alpha-i)}{R\sigma} dz \\
&= \left[\epsilon\,(i-\delta) + (1+\epsilon)\,\frac{(\alpha-i)^2}{2R\sigma^2}\right] dt + \frac{(\alpha-i)}{R\sigma} dz
\end{aligned}
$$

Thus, the expected growth rate of wealth, as well as consumption (since $C = \mu W$), is

$$
g = \epsilon\,(i-\delta) + (1+\epsilon)\,\frac{(\alpha-i)^2}{2R\sigma^2} \tag{17}
$$

We see that the growth rate is increasing in α, but decreasing in R and σ. Obstfeld points out that the integration of global financial markets that allows residents to hold risky foreign, as well as domestic, investments increases diversification and effectively reduces individuals' risky portfolio variance, σ^2. This will lead to an increase in nations' expected growth rates because individuals will allocate a greater proportion of their wealth in the (higher-yielding) risky assets.