Finance 400

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The Cox-Ross-Rubinstein Option Pricing Model

The previous notes showed that the absence of arbitrage restricts the price of an option in terms of its underlying asset. However, the no-arbitrage assumption alone cannot determine an exact option price as a function of the underlying asset price. To do so, one needs to make additional assumptions regarding the distribution of returns earned by the underlying asset. Certain distributional assumptions can imply a complete market for the underlying asset's risk that allows us to determine a unique option price.

The model in these notes makes the assumption that the underlying asset, hereafter referred to as a stock, takes on one of only two possible values each period. While this may seem unrealistic, the assumption leads to a formula that can accurately price options. This "binomial" option pricing technique is often applied by Wall Street practitioners to numerically compute the prices of complex options. Here, we start by considering the pricing of a simple European option written on a non-dividend-paying stock.

In addition to assuming the absence of arbitrage opportunities, the binomial model assumes that the current stock price, S, either moves up, by a proportion u, or down, by a proportion d, each period. The probability of an up move is q, so that the probability of a down move is $1-q$. This can be illustrated as

$$\begin{array}{ccl} & & uS \quad \text{with probability } q \\ & \nearrow & \\ S & & \\ & \searrow & \\ & & dS \quad \text{with probability } 1-q \end{array} \tag{1}$$

Denote R_f as one plus the risk-free interest rate for the period. This risk-free return is assumed to be the constant over time. To avoid arbitrage between the stock and the risk-free investment, we must have $d < R_f < u$.

Let C equal the value of a European call option written on the stock and having a strike price of X. At expiry, $C = \max[0, S_T - X]$. Thus:

One period prior to expiry:

$$
C \begin{array}{l} \nearrow \\ \\ \searrow \end{array}
\begin{array}{l} C_u \equiv \max\left[0, uS - X\right] \text{ with probability } q \\ \\ \\ C_d \equiv \max\left[0, dS - X\right] \quad \text{with probability } 1-q \end{array}
\tag{2}
$$

What is C one period before expiry? Consider a portfolio containing Δ shares of stock and $\$B$ of bonds. It has current value equal to $\Delta S + B$. Then the value of this portfolio evolves over the period as

$$
\Delta S + B \begin{array}{l} \nearrow \\ \\ \searrow \end{array}
\begin{array}{l} \Delta uS + R_f B \text{ with probability } q \\ \\ \\ \Delta dS + R_f B \quad \text{with probability } 1-q \end{array}
\tag{3}
$$

With two securities (the bond and stock) and two states of the world (up or down), Δ and B can be chosen to replicate the payoff of the call option:

$$\Delta uS + R_f B = C_u \tag{4a}$$

$$\Delta dS + R_f B = C_d \tag{4b}$$

Solving for Δ and B that satisfy these two equations, we have

$$\Delta^* = \frac{C_u - C_d}{(u-d)\, S} \tag{5a}$$

$$B^* = \frac{uC_d - dC_u}{(u-d)\, R_f} \tag{5b}$$

Hence, a portfolio of Δ^* shares of stock and $\$B^*$ of bonds produces the same cashflow as the call option. This is possible because the market is complete. Trading in the stock and bond produces payoffs that span the two states. Now since the portfolio's return replicates that of the option, the absence of arbitrage implies

$$C = \Delta^* S + B^* \tag{6}$$

Example: If $S = \$50$, $u = 2$, $d = .5$, $R_f = 1.25$, $X = \$50$, then

$$uS = \$100,\ dS = \$25,\ C_u = \$50,\ C_d = \$0.$$

Therefore:

$$\Delta^* = \frac{50 - 0}{(2 - .5)\, 50} = \frac{2}{3}$$

$$B^* = \frac{0 - 25}{(2 - .5)\, 1.25} = -\frac{40}{3}$$

so that

$$C = \Delta^* S + B^* = \frac{2}{3}(50) - \frac{40}{3} = \frac{60}{3} = \$20$$

If $C < \Delta^* S + B^*$, then an arbitrage is to short sell Δ^* shares of stock, invest $\$-B^*$ in bonds, and buy the call option. Conversely, if $C > \Delta^* S + B^*$, then an arbitrage is to write the call option, buy Δ^* shares of stock, and borrow $\$-B^*$.

The resulting option pricing formula has an interesting implication. It can be re-written as

$$\begin{aligned} C &= \Delta^* S + B^* = \frac{C_u - C_d}{(u-d)} + \frac{uC_d - dC_u}{(u-d)\, R_f} \\ &= \frac{\left[\frac{R_f - d}{u-d} \max\left[0, uS - X\right] + \frac{u - R_f}{u-d} \max\left[0, dS - X\right]\right]}{R_f} \end{aligned} \tag{7}$$

which *does not* depend on the probability of an up or down move of the stock, q. Thus, given S, investors will agree on the no-arbitrage value of the call option even if they do not agree on q.

Since q determines the stock's expected rate of return, $uq + d(1-q) - 1$, this does not need to be known or estimated in order to solve for the no-arbitrage value of the option, C. However, we do need to know u and d, the size of movements per period, which determine the stock's *volatility*.

But note that the call option value, C, does not *directly* depend on investors' attitudes toward risk. It is a *relative* (to the stock) pricing formula.

Note also that we can re-write C as

$$C = \frac{1}{R_f}\left[pC_u + (1-p)\,C_d\right] \tag{8}$$

where $p \equiv \frac{R_f - d}{u-d}$.

Since $0 < p < 1$, p has the properties of a probability. In fact, this "pseudo-probability" p would equal the true probability q if investors were *risk-neutral*, since then the expected return on the stock would equal R_f:

$$\left[uq + d\,(1-q)\right] S = R_f S \tag{9}$$

or

$$q = \frac{R_f - d}{u-d} = p. \tag{10}$$

Perhaps this is not surprising, since the expression

$$C = \frac{1}{R_f}\left[pC_u + (1-p)\,C_d\right]$$

does not depend on risk-preferences, and so it must be consistent with all possible risk preferences, including risk-neutrality.

Next, consider the option's value with:

Two periods prior to expiration:

The stock price process is

$$
\begin{array}{ccccc}
 & & & & u^2 S \\
 & & & \nearrow & \\
 & & uS & & \\
 & \nearrow & & \searrow & \\
S & & & & duS \\
 & \searrow & & \nearrow & \\
 & & dS & & \\
 & & & \searrow & \\
 & & & & d^2 S
\end{array}
\tag{11}
$$

so that the option price process is

$$
\begin{array}{ccccc}
 & & & & C_{uu} \equiv \max\left[0, u^2 S - X\right] \\
 & & & \nearrow & \\
 & & C_u & & \\
 & \nearrow & & \searrow & \\
C & & & & C_{du} \equiv \max\left[0, duS - X\right] \\
 & \searrow & & \nearrow & \\
 & & C_d & & \\
 & & & \searrow & \\
 & & & & C_{dd} \equiv \max\left[0, d^2 S - X\right]
\end{array}
\tag{12}
$$

Using the results from our analysis when there was only one period to expiry, we know that

$$
C_u = \frac{pC_{uu} + (1-p)\, C_{du}}{R_f} \tag{13a}
$$

$$
C_d = \frac{pC_{du} + (1-p)\, C_{dd}}{R_f} \tag{13b}
$$

With two periods to expiry, the one period to go cashflows of C_u and C_d can be replicated once again by the stock and bond portfolio composed of $\Delta^* = \frac{C_u - C_d}{(u-d)S}$ shares of stock and $B^* = \frac{uC_d - dC_u}{(u-d)R_f}$ of bonds. No-arbitrage implies

$$
C = \Delta^* S + B^* = \frac{1}{R_f}\left[pC_u + (1-p)\, C_d\right] \tag{14}
$$

Substituting in for C_u and C_d, we have

$$\begin{aligned} C &= \frac{1}{R_f^2}\left[p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}\right] \qquad (15) \\ &= \frac{1}{R_f^2}\left[p^2 \max\left[0, u^2 S - X\right] + 2p(1-p)\max\left[0, duS - X\right] + (1-p)^2 \max\left[0, d^2 S - X\right]\right] \end{aligned}$$

Note that C depends only current S, X, u, d, R_f, and the time until maturity, 2 periods. Repeating this analysis for three, four, five, ..., n periods prior to expiry, we always obtain

$$C = \Delta^* S + B^* = \frac{1}{R_f}\left[pC_u + (1-p)C_d\right]$$

By repeated substitution for C_u, C_d, C_{uu}, C_{ud}, C_{dd}, C_{uuu}, etc., we obtain the formula:

n periods prior to expiration:

$$C = \frac{1}{R_f^n}\left[\sum_{j=0}^{n}\left(\frac{n!}{j!\,(n-j)!}\right)p^j(1-p)^{n-j}\max\left[0, u^j d^{n-j} S - X\right]\right] \qquad (16)$$

This formula can be simplified by defining "a" as the minimum number of upward jumps of S for it to exceed X. Thus a is the smallest non-negative integer such that $u^a d^{n-a} S > X$. Taking the natural logarithm of both sides, a is the minimum integer $> ln(X/Sd^n)/ln(u/d)$.

Therefore for all $j < a$ (the option expires out-of-the money),

$$\max\left[0, u^j d^{n-j} S - X\right] = 0, \qquad (17a)$$

while for all $j > a$ (the option expires in-the-money),

$$\max\left[0, u^j d^{n-j} S - X\right] = u^j d^{n-j} S - X \qquad (17b)$$

Thus, the formula for C can be re-written:

$$C = \frac{1}{R_f^n}\left[\sum_{j=a}^{n}\left(\frac{n!}{j!\,(n-j)!}\right)p^j(1-p)^{n-j}\left[u^j d^{n-j} S - X\right]\right] \qquad (18)$$

Breaking this up into two terms, we have:

$$C = S\left[\sum_{j=a}^{n}\left(\frac{n!}{j!\,(n-j)!}\right)p^j\,(1-p)^{n-j}\left[\frac{u^j d^{n-j}}{R_f^n}\right]\right] \\ -XR_f^{-n}\left[\sum_{j=a}^{n}\left(\frac{n!}{j!\,(n-j)!}\right)p^j\,(1-p)^{n-j}\right] \tag{19}$$

The terms in brackets are complementary binomial distribution functions, so that we can write this as

$$C = S\phi[a;n,p'] - XR_f^{-n}\phi[a;n,p] \tag{20}$$

where $p' \equiv \left(\frac{u}{R_f}\right)p$ and $\phi[a;n,p]$ = the probability that the sum of n random variables which equal 1 with probability p and 0 with probability $1-p$ will be $\geq a$. These formulas imply that C is the discounted expected value of the call's terminal payoff that would occur in a risk-neutral world.

If we define τ as the time until maturity of the call option and σ^2 as the variance per unit time of the stock's rate of return (which depends on u and d), then by taking the limit as the number of periods $n \to \infty$, but the length of each period $\frac{\tau}{n} \to 0$, the Cox-Ross-Rubinstein binomial option pricing formula becomes the well-known Black-Scholes-Merton option pricing formula[1]

$$C = SN\left(z\right) - XR_f^{-\tau}N\left(z - \sigma\sqrt{\tau}\right) \tag{21}$$

where $z \equiv \frac{\left[\ln\left(\frac{S}{XR_f^{-\tau}}\right) + \frac{1}{2}\sigma^2\tau\right]}{\left(\sigma\sqrt{\tau}\right)}$ and $N\left(\cdot\right)$ is that standard normal distribution function.

[1]In the Black-Scholes-Merton formula, R_f is now the risk-free return per unit time rather than the risk-free return for each period. The relationship between σ and u and d will be discussed shortly.