

## Finance 400

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# The Capital Asset Pricing Model

Let us revisit the problem of an investor who maximizes expected utility that depends only on the expected return and variance (or standard deviation) of end-of-period wealth. Such an investor, no matter what her risk-aversion, would be interested only in portfolios on the efficient frontier. This mean-variance efficient frontier was the solution to the problem of computing portfolio weights that would maximize a portfolio's expected return for a given portfolio standard deviation or, alternatively, minimizing a portfolio's standard deviation for a given expected portfolio return. The point on this efficient frontier ultimately selected by a given investor was that combination of expected portfolio return and portfolio standard deviation that maximized the particular investor's expected utility.

Recall that for the case of  $N$  risky assets and a riskfree asset, the problem of finding mean-variance efficient portfolios could be stated as

$$\min_w L = \frac{1}{2}w'Vw + \lambda \left\{ \bar{R}_p - [R_f + w'(\bar{R} - R_f e)] \right\} \quad (1)$$

The solution to this quadratic optimization problem was that an efficient portfolio requires proportions invested in the  $N$  risky assets of

$$w^* = kV^{-1}(\bar{R} - R_f e) \quad (2)$$

where  $k \equiv \frac{\bar{R}_p - R_f}{\varsigma - 2\alpha R_f + \delta R_f^2}$  and  $\alpha \equiv \bar{R}'V^{-1}e = e'V^{-1}\bar{R}$ ,  $\varsigma \equiv \bar{R}'V^{-1}\bar{R}$ , and  $\delta \equiv e'V^{-1}e$ . The amount invested in the riskfree asset is then  $1 - e'w^*$ . Note that  $k$  in equation (2) is a scalar that is linear in  $\bar{R}_p$ .  $\bar{R}_p$ , which is a given investor's equilibrium portfolio expected return, differs across investors and depends on where the investor's indifference curve is tangent to the efficient frontier. Thus, because differences in  $\bar{R}_p$  just affect the scalar,  $k$ , we see that all investors, no matter what their degree of risk aversion, choose to hold the risky assets in the same *relative* proportions.

Mathematically, we showed that the efficient frontier is given by

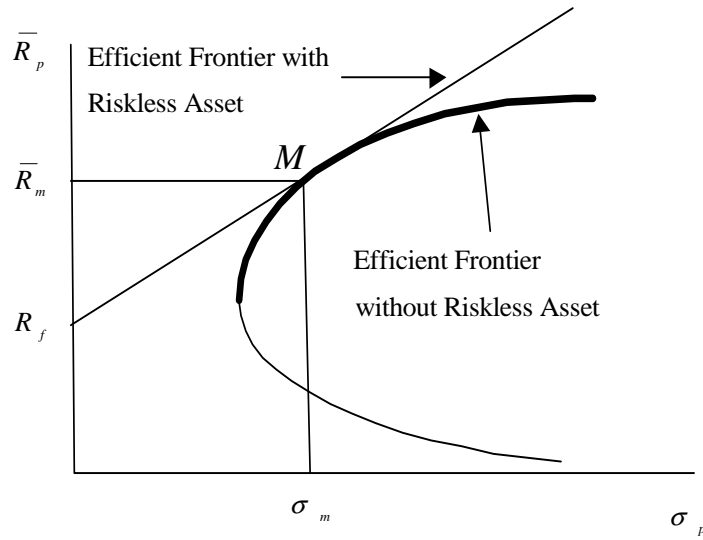


Figure 0-1: Capital Market Equilibrium

$$\sigma_p^2 = w^{*'} V w^* = \frac{(\bar{R}_p - R_f)^2}{\varsigma - 2\alpha R_f + \delta R_f^2} \quad (3)$$

or

$$\sigma_p = \frac{\bar{R}_p - R_f}{\left(\varsigma - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}}} \quad (4)$$

In  $\sigma_p, \bar{R}_p$  space, the efficient frontier is *linear*. This is illustrated in Figure 1.

The efficient frontier, given by the line through  $R_f$  and  $M$ , implies that investors optimally choose to hold combinations of the riskfree asset and the efficient frontier portfolio of risky assets given by point  $M$ . We can easily solve for this unique “tangency” portfolio of risky assets since it is the point where an investor would have a zero position in the riskfree asset, that is,  $e'w^* = 1$ , or  $\bar{R}_p = \bar{R}'w^*$ . Substituting in  $\bar{R}_p = \bar{R}'w^*$  in (2) and solving for  $w$ , we obtain

$$w_m = mV^{-1}(\bar{R} - R_f e) \quad (5)$$

where  $m \equiv [\alpha - \delta R_f]^{-1}$ . Let us now investigate the relationship between this tangency port-

folio and individual assets. Consider the covariance between the tangency portfolio and the individual risky assets. Define  $\sigma_M$  as the  $N \times 1$  vector of covariances of the tangency portfolio with each of the  $N$  risky assets. Then using (5) we see that

$$\sigma_M = Vw_m = m(\bar{R} - R_f e) \quad (6)$$

Note that the variance of the tangency portfolio is simply  $\sigma_m^2 = w_m' V w_m$ . So that if we then pre-multiply equation (6) by  $w_m'$  we obtain

$$\begin{aligned} \sigma_m^2 &= w_m' \sigma_M = m w_m' (\bar{R} - R_f e) \\ &= m(\bar{R}_m - R_f) \end{aligned} \quad (7)$$

where  $\bar{R}_m \equiv w_m' \bar{R}$  is the expected return on the tangency portfolio.<sup>1</sup> Re-arranging (6) and substituting in for  $m$  from (7), we have

$$(\bar{R} - R_f e) = \frac{1}{m} \sigma_M = \frac{\sigma_M}{\sigma_m^2} (\bar{R}_m - R_f) = \beta (\bar{R}_m - R_f) \quad (8)$$

where  $\beta \equiv \frac{\sigma_M}{\sigma_m^2}$  is the  $N \times 1$  vector whose  $i^{th}$  element is  $\frac{cov(\bar{R}_m, \tilde{R}_i)}{var(\bar{R}_m)}$ . Equation (8) shows that a simple relationship links the “excess expected return” on the tangency portfolio,  $(\bar{R}_m - R_f)$ , to the “excess expected returns” on the individual risky assets,  $(\bar{R} - R_f e)$ .

Now suppose that individual investors, each taking the set of individual assets’ expected returns and covariances as fixed (exogenous), all optimally choose mean-variance efficient portfolios. Thus, each of them decides to allocate his or her wealth between the riskfree asset and a unique tangency portfolio. Because individual investors demand the risky assets in the same relative proportions, we know that the *aggregate demands* for the risky assets will have the same relative proportions, namely those of the tangency portfolio. Recall that our derivation of this result *does not* assume a “representative” individual in the sense of requiring all individuals to have identical utility functions or beginning-of-period wealth. Implicitly, it *did* assume that

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<sup>1</sup>Note that the elements of  $w_m$  sum to 1 since the tangency portfolio has zero weight in the riskfree asset.

investors have identical beliefs regarding the probability distribution of asset returns, that all risky assets can be traded, that there are no indivisibilities in asset holdings, and that there are no limits on borrowing or lending at the riskfree rate.

We can now define an equilibrium as a situation where the investors' demands for the assets equal their supplies. The manner in which these assets are supplied has been left unmodeled. One way to model asset supplies is to assume they are fixed. For example, the economy could be characterized by a fixed *quantity* of physical assets that produce random output at the end of the period. Such an economy is often referred to as an "endowment" or "fruit tree" economy. There can also be an asset that produces a non-random, riskfree amount of output. In this case, equilibrium occurs by adjustment of the date 0 asset prices so that demand conforms to the inelastic asset supply. The change in the assets' date 0 prices effectively *adjusts the assets' returns* to that which makes the "tangency portfolio" and the net demand for the riskfree asset equal the fixed supply of these assets.

Alternatively, one could model the economy's asset return distributions as fixed but allow the quantities of these assets to be elastically supplied. This type of economy is known as a "production" economy. This can be done by assuming there are  $N$  risky, constant-returns-to-scale "technologies." These technologies require date 0 investments of physical capital and produce end-of-period physical investment returns having a distribution with mean  $\bar{R}$  and covariance matrix of  $V$  at the end of the period. Also, there can be a riskfree technology that generates a one-period return on physical capital of  $R_f$ . In this case of a fixed return distribution, supplies of the assets adjust to the demands for the tangency portfolio and riskfree asset determined by the technological return distribution.

Whichever method is used to model asset supplies does not affect the results that we can now derive regarding the equilibrium relationship between asset returns. We simply note that the tangency portfolio having weights  $w_m$  must be the equilibrium portfolio of risky assets supplied in the market. Thus, equation (8) can be interpreted as an equilibrium relationship between the excess expected return on any asset and the excess expected return on the *market* portfolio. In other words, in equilibrium, the tangency portfolio chosen by all investors must be the market portfolio of all risky assets. The market portfolio can be likened to an "indexed" mutual fund of all risky assets.

Note that since  $\tilde{R}_i$  can be defined as  $\bar{R}_i + \tilde{\nu}_i$  and  $\tilde{R}_m$  can be defined as  $\bar{R}_m + \tilde{\nu}_m$ , substituting these into (8) we have

$$\begin{aligned}
\tilde{R}_i &= R_f + \beta_i(\tilde{R}_m - \tilde{\nu}_m - R_f) + \tilde{\nu}_i \\
&= R_f + \beta_i(\tilde{R}_m - R_f) + \tilde{\nu}_i - \beta_i\tilde{\nu}_m \\
&= R_f + \beta_i(\tilde{R}_m - R_f) + \tilde{\varepsilon}_i
\end{aligned} \tag{9}$$

where  $\tilde{\varepsilon}_i \equiv \tilde{\nu}_i - \beta_i\tilde{\nu}_m$ . Note that

$$\begin{aligned}
cov(\tilde{R}_m, \tilde{\varepsilon}_i) &= cov(\tilde{R}_m, \tilde{\nu}_i) - \beta_i cov(\tilde{R}_m, \tilde{\nu}_m) \\
&= cov(\tilde{R}_m, \tilde{R}_i) - \beta_i cov(\tilde{R}_m, \tilde{R}_m) \\
&= \beta_i var(\tilde{R}_m) - \beta_i var(\tilde{R}_m) = 0
\end{aligned} \tag{10}$$

Thus, by construction,  $\tilde{\varepsilon}_i$  is the part of the return on risky asset  $i$  that is uncorrelated with the return on the market portfolio. An implication of (10) is that equation (9) represents a regression equation. In other words, an estimate of  $\beta_i$  can be obtained by running an Ordinary Least Squares regression of asset  $i$ 's excess return on the market portfolio's excess return. The orthogonal, mean zero residual,  $\tilde{\varepsilon}_i$ , is sometimes referred to as idiosyncratic, unsystematic, or diversifiable risk. This is the particular asset's risk that is eliminated or diversified away when the market portfolio is held. Since this portion of the asset's risk can be eliminated by the individual who invests optimally, there is no "price" or "risk premium" attached to it in the sense that the asset's equilibrium expected return is not altered by it.

To make clear what risk is priced, let us denote the covariance between the return on the  $i^{th}$  asset and the return on the market portfolio as  $\sigma_{Mi} = cov(\tilde{R}_m, \tilde{R}_i)$ , which is the  $i^{th}$  element

of  $\sigma_M$ . Also let  $\rho_{im}$  be the correlation between the return on the  $i^{th}$  asset and the return on the market portfolio, and let  $\sigma_i$  be the standard deviation of the return on the  $i^{th}$  asset. Then equation (8) can be re-written as

$$\begin{aligned}\bar{R}_i - R_f &= \frac{\sigma_{Mi}}{\sigma_m} \frac{(\bar{R}_m - R_f)}{\sigma_m} \\ &= \rho_{im} \sigma_i \frac{(\bar{R}_m - R_f)}{\sigma_m} \\ &= \rho_{im} \sigma_i S_e\end{aligned}\tag{11}$$

where  $S_e \equiv \frac{(\bar{R}_m - R_f)}{\sigma_m}$  is the equilibrium excess return on the market portfolio per unit of market risk. Thus, this can be viewed as the market price of systematic or non-diversifiable risk. It is the slope of the efficient frontier, the line connecting points  $R_f$  and  $M$  in the above graph. Now if we define  $w_{mi}$  as the weight of asset  $i$  in the market portfolio and  $V_i$  as the  $i^{th}$  row of covariance matrix  $V$ , then

$$\frac{\partial \sigma_m}{\partial w_{mi}} = \frac{1}{2\sigma_m} \frac{\partial \sigma_m^2}{\partial w_{mi}} = \frac{1}{2\sigma_m} \frac{\partial w'_m V w_m}{\partial w_{mi}} = \frac{1}{2\sigma_m} 2V_i w_m = \frac{1}{\sigma_m} \sum_{j=1}^N w_{mj} \sigma_{ij}\tag{12}$$

where  $\sigma_{ij}$  is the  $i, j^{th}$  element of  $V$ . Since  $\tilde{R}_m = \sum_{j=1}^N w_{mj} \tilde{R}_j$ , then  $cov(\tilde{R}_i, \tilde{R}_m) = cov(\tilde{R}_i, \sum_{j=1}^N w_{mj} \tilde{R}_j) = \sum_{j=1}^N w_{mj} \sigma_{ij}$ . Hence, (12) can be re-written

$$\frac{\partial \sigma_m}{\partial w_{mi}} = \frac{1}{\sigma_m} cov(\tilde{R}_i, \tilde{R}_m) = \rho_{im} \sigma_i\tag{13}$$

Thus,  $\rho_{im} \sigma_i$  can be interpreted the marginal increase in “market risk,”  $\sigma_m$ , from a marginal increase of asset  $i$  in the market portfolio. In this sense,  $\rho_{im} \sigma_i$  is the *quantity* of asset  $i$ 's systematic or non-diversifiable risk. Equation (11) shows that this quantity of systematic risk, multiplied by the “price” of systematic risk,  $S_e$ , determines the asset's required excess expected return or “risk premium.”