Finance 400

A. Penati - G. Pennacchi

"Market Micro-Structure: Notes on the Kyle Model"

These notes consider the single-period model in Kyle (1985) "Continuous Auctions and Insider Trading," *Econometrica* 15, p.1315-1335. This model derives equilibrium security prices when traders have asymmetric information. Equilibrium prices partially reflect inside information. The model assumes three types of agents in a market for a particular security: a market maker (such as a stock exchange specialist or security dealer); a "noise" trader; and an insider. It provides a theoretical framework for determining bid-ask spreads and the "market impact" of trades.

I. Assumptions:

A.1) The model is a single period model. At the beginning of the period, agents trade in an asset that has a random end of period liquidation value of $\tilde{\nu} \sim N(p_0, \Sigma_0)$.

A.2) *Noise traders* have needs to trade that are exogenous to the model. It is simply assumed that they, as a group, submit a "market" order to buy $\tilde{u}$ shares of the asset, where $\tilde{u} \sim N(0, \sigma_u^2)$. $\tilde{u}$ and $\tilde{\nu}$ are assumed to be independently distributed.[1]

A.3) The single risk-neutral *insider* is assumed have better information than the other agents. He knows with perfect certainty the realized end of period value of the risky security $\tilde{\nu}$ (but not $\tilde{u}$) and chooses to submit a market order of size x that maximizes his expected end of period profits.[2]

A.4) The single risk-neutral *market maker* (for example, the specialist) receives the market orders submitted by the noise traders and the insider, which in total equal $u + x$. Importantly, the market maker cannot distinguish what part of this total order consists of orders made by noise traders and what part consists of the order of the insider. (The traders are anonymous.)

[1] Why rational noise traders submit these orders has been modeled by assuming they have exogenous "shocks" to their wealth and need to rebalance their portfolio (M. Spiegel and A. Subrahmanyam (1992) "Informed Speculation and Hedging in a Noncompetitive Securities Market," *Review of Financial Studies* 5, p.307-329) or by assuming that they have uncertainty regarding the timing of their consumption (G. Gorton and G. Pennacchi (1993) "Security Baskets and Index-Linked Securities," *Journal of Business* 66, p.1-27).

[2] This assumption can be weakened to the case of the insider having uncertainty over $\tilde{\nu}$ but having more information on $\tilde{\nu}$ than the other traders. One can also allow the insider to submit "limit" orders, that is, orders that are a function of the equilibrium market price (a demand schedule), as in Kyle (1989) "Informed Speculation with Imperfect Competition," *Review of Economic Studies* 56, p.317-56.

The market maker sets the market price, p, and then takes the position $-(u+x)$ to clear the market. It is assumed that market making is a perfectly competitive profession, so that the market maker sets the price p such that, given the total order submitted, his profit at the end of the period is expected to be zero.

II. Analysis:

Since the noise traders' order is exogenous, we need only to consider the optimal actions of the market maker and the insider.

The market maker observes only the total order flow, $u+x$. Given this information, he must then set the equilibrium market price p that gives him zero expected profits. Since his end of period profits are $-(\tilde{\nu}-p)(u+x)$, this implies that the price set by the market maker satisfies

$$p = E[\tilde{\nu} \mid u+x] \tag{1}$$

The information on the total order size is important to the market maker. The greater the total order size, the more likely it is that x is large because the insider knows ν is greater than p_0. Thus, the market maker would tend to set p higher than otherwise. Similarly, if $u+x$ is low, the more likely it is that x is low because the insider knows ν is below p_0 and is submitting a sell order. In this case, the market maker would tend to set p lower than otherwise. Thus, the *pricing rule* of the market maker is a function of $x+u$, that is, $P(x+u)$.

Since the insider sets x, it is an endogenous variable that depends on $\tilde{\nu}$. The insider chooses x to maximize his expected end of period profits, $\tilde{\pi}$, given knowledge of ν and the way that the market maker behaves in setting the equilibrium price:

$$\max_x E[\tilde{\pi} \mid \nu] = \max_x E[(\nu - P(x+\tilde{u}))\, x \mid \nu] \tag{2}$$

An *equilibrium* in this model is a pricing rule chosen by the market maker and a trading strategy chosen by the insider such that: the insider maximizes expected profits, given the market maker's pricing rule; the market maker sets the price to earn zero expected profits, given the trading strategy of the insider; the insider and market maker have rational expectations, that is, the equilibrium is a fixed-point where an agent's actual behavior is that expected by the other.

Suppose the market maker chooses a market price that is a linear function of the total order flow, $P(y) = \mu + \lambda y$. We will later argue that a linear pricing rule is optimal. If this is so, what is the insider's choice of x? From (2) we have

$$\begin{aligned} \max_x E\left[(\nu - P(x+\tilde{u}))\, x \mid \nu\right] &= \max_x E\left[(\nu - \mu - \lambda(x+\tilde{u}))\, x \mid \nu\right] \\ &= \max_x\, (\nu - \mu - \lambda x)\, x, \qquad \text{since } E[\tilde{u}] = 0 \end{aligned} \tag{3}$$

Thus, the solution to the insider's problem in (3) is

$$x = \alpha + \beta\nu \tag{4}$$

where $\alpha = -\frac{\mu}{2\lambda}$ and $\beta = \frac{1}{2\lambda}$. Therefore, if the market maker uses a linear pricing-setting rule, the optimal trading strategy for the insider is a linear trading rule.

Next, let us return to the market maker's problem of choosing the market price that, conditional on knowing the total order flow, results in a competitive (zero) expected profit. Given the assumption that market making is a perfectly competitive profession, a market maker needs to choose the "best" possible estimate of $E[\tilde{\nu} \mid u+x]$ in setting the price $p = E[\tilde{\nu} \mid u+x]$. What estimate of the mean of ν is best? The maximum likelihood estimate of $E[\tilde{\nu} \mid u+x]$ is best in the sense that it attains maximum efficiency and is also the minimum variance unbiased estimate.

Note that if the insider follows the optimal trading strategy, which according to equation (4) is $x = \alpha + \beta\tilde{\nu}$, then from the point of view of the market maker, $\tilde{\nu}$ and $y \equiv \tilde{u} + x = \tilde{u} + \alpha + \beta\tilde{\nu}$ are jointly normally distributed. Because ν and y are jointly normal, the maximum likelihood estimate of the mean of ν conditional on y is linear in y, that is, $E[\tilde{\nu} \mid y]$ is *linear* in y when they are jointly normally distributed. Hence, the previously assumed linear pricing rule is, in fact, optimal in equilibrium. Therefore, the market maker should use the maximum likelihood estimator, which in the case of ν and y being normally distributed is equivalent to the "least squares" estimator. This is the one that minimizes

$$
\begin{aligned}
E\left[(\tilde{\nu} - P(y))^2\right] &= E\left[(\tilde{\nu} - \mu - \lambda y)^2\right] \qquad (5) \\
&= E\left[(\tilde{\nu} - \mu - \lambda(\tilde{u} + \alpha + \beta\tilde{\nu}))^2\right]
\end{aligned}
$$

Thus, the optimal pricing rule equals $\mu + \lambda y$ where μ and λ minimize

$$
\min_{\mu,\lambda} E\left[(\tilde{\nu}(1 - \lambda\beta) - \lambda\tilde{u} - \mu - \lambda\alpha)^2\right] \qquad (6)
$$

Recalling the assumptions $E[\nu] = p_0$, $E\left[(\nu - p_0)^2\right] = \Sigma_0$, $E[u] = 0$, $E\left[u^2\right] = \sigma_u^2$, and $E[u\nu] = 0$, the above objective function can be written as

$$
\min_{\mu,\lambda} (1 - \lambda\beta)^2 \left(\Sigma_0 + p_0^2\right) + (\mu + \lambda\alpha)^2 + \lambda^2\sigma_u^2 - 2(\mu + \lambda\alpha)(1 - \lambda\beta)p_0 \qquad (7)
$$

The first order conditions with respect to μ and λ are

$$
\mu = -\lambda\alpha + p_0(1 - \lambda\beta) \qquad (8a)
$$

$$
-2\beta(1 - \lambda\beta)\left(\Sigma_0 + p_0^2\right) + 2\alpha(\mu + \lambda\alpha) + 2\lambda\sigma_u^2 - 2p_0\left[-\beta(\mu + \lambda\alpha) + \alpha(1 - \lambda\beta)\right] = 0 \qquad (8b)
$$

Substituting $\mu + \lambda\alpha = p_0(1 - \lambda\beta)$ from (8a) into (8b), we see that (8b) simplifies to

$$
\lambda = \frac{\beta\Sigma_0}{\beta^2\Sigma_0 + \sigma_u^2} \qquad (8b)
$$

Substituting in for the definitions $\alpha = -\frac{\mu}{2\lambda}$ and $\beta = \frac{1}{2\lambda}$ in (8a) and (8b), we have[3]

$$
\mu = p_0 \qquad (9a)
$$

$$
\lambda = \tfrac{1}{2}\frac{\sqrt{\Sigma_0}}{\sigma_u} \qquad (9b)
$$

[3]Note the typo in the solution for in Theorem 1 of the Kyle paper.

In summary, the equilibrium price is

$$p = p_0 + \tfrac{1}{2}\frac{\sqrt{\Sigma_0}}{\sigma_u}\left(\tilde{u} + \tilde{x}\right) \tag{10}$$

where the equilibrium order submitted by the insider is

$$x = \frac{\sigma_u}{\sqrt{\Sigma_0}}\left(\tilde{\nu} - p_0\right) \tag{11}$$

From (11), we see that the greater is the volatility (amount) of noise trading, σ_u, the larger is the magnitude of the order submitted by the insider for a given deviation of ν from its unconditional mean. Hence, the insider trades more actively on his private information the greater is the "camouflage" provided by noise trading. More noise trading makes it more difficult for the market maker to extract the "signal" of insider trading from the noise. Note that if equation (11) is substituted into (10), one obtains

$$\begin{aligned} p &= p_0 + \tfrac{1}{2}\frac{\sqrt{\Sigma_0}}{\sigma_u}\tilde{u} + \tfrac{1}{2}\left(\tilde{\nu} - p_0\right) \\ &= \tfrac{1}{2}\left(\frac{\sqrt{\Sigma_0}}{\sigma_u}\tilde{u} + p_0 + \tilde{\nu}\right) \end{aligned} \tag{12}$$

Thus we see that only one-half of the insider's private information, $\frac{1}{2}\ \tilde{\nu}$, is reflected in the equilibrium price, so that the price is *not fully revealing.* (A fully-revealing price would be $p = \tilde{\nu}$.) To obtain an equilibrium of incomplete revelation of private information, it was necessary to have a second source of uncertainty, namely, the amount of noise trading.

Using (11) and (12), we can calculate the insider's expected profits:

$$E\left[\tilde{\pi}\right] = E\left[x\left(\nu - p\right)\right] = E\left[\frac{\sigma_u}{\sqrt{\Sigma_0}}\left(\tilde{\nu} - p_0\right)\tfrac{1}{2}\left(\nu - p_0 - \frac{\sqrt{\Sigma_0}}{\sigma_u}\tilde{u}\right)\right] \tag{13}$$

Conditional on knowing ν, that is, after learning the realization of ν at the beginning of the period, the insider expects profits of

$$E\left[\tilde{\pi} \mid \nu\right] = \tfrac{1}{2}\frac{\sigma_u}{\sqrt{\Sigma_0}}\left(\nu - p_0\right)^2 \tag{14}$$

Hence, the larger is ν's deviation from p_0, the larger the expected profit. Unconditional on knowing $\tilde{\nu}$, that is, before the start of the period, the insider expects a profit of

$$E\left[\tilde{\pi}\right] = \frac{1}{2}\frac{\sigma_u}{\sqrt{\Sigma_0}}E\left[\left(\tilde{\nu} - p_0\right)^2\right] = \frac{1}{2}\sigma_u\sqrt{\Sigma_0} \tag{15}$$

which is proportional to the standard deviations of noise traders' order and the end of period value of ν.

Since, by assumption, the market maker sets the security price in a way that gives him zero expected profits, the expected profits of the insider equals the expected losses of the noise traders. In other words, it is the noise traders, not the market maker, that lose, on average, from the presence of the insider.

From equation (10), we see that $\lambda = \frac{1}{2}\frac{\sqrt{\Sigma_0}}{\sigma_u}$ is the amount that the market maker raises the price when the total order flow, $(u + x)$, goes up by 1 unit. Hence, the amount of order flow necessary to raise the price by \$1 equals $1/\lambda = 2\frac{\sigma_u}{\sqrt{\Sigma_0}}$, which is a measure of the "depth" of the market or market "liquidity." The higher is the proportion of noise trading to the value of insider information, $\frac{\sigma_u}{\sqrt{\Sigma_0}}$, the deeper or more liquid is the market. Intuitively, the more noise traders relative to the value of insider information, the less the market maker needs to adjust the price in response to a given order, since the likelihood of the order being that of a noise trader, rather than an insider, is greater. While the greater is the number of noise traders (that is, the greater is σ_u), the greater is the profits of the insider (see equation 15) and the greater is the *total* losses of the noise traders. However, an *individual* noise trader's expected loss is less.[4]

[4] See G. Gorton and G. Pennacchi (1993) "Security Baskets and Index-Linked Securities," *Journal of Business* 66 p.1-27 for an explicit derivation of this result.