Finance 400

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Notes on

On the Efficiency of Competitive Stock Markets Where Traders Have Diverse Information

by Sanford Grossman

This model shows how the heterogeneous information of different investors can be reflected in the equilibrium price of a security. We look at two equilibria: a "competitive," but not fully rational, equilibrium and a fully-revealing rational expectations equilibrium.

Assumptions:

A.1 This is a single-period portfolio choice problem. At the beginning of the period, traders can choose between a risk-free asset, which pays an end-of-period return of $1+r$, or a risky asset that has a beginning-of-period price of P_0 per share and has an end-of-period random payoff (price) of P_1 per share. The unconditional distribution of P_1 is assumed to be $N(m, \sigma^2)$. The aggregate supply of shares of the risky asset is fixed at $\bar{X}$, but the risk-free asset is in perfectly elastic supply.

A.2 There are n different traders. The i^{th} trader has beginning-of-period wealth W_{0i} and is assumed to maximize expected utility over end-of-period wealth, $\tilde{W}_{1i}$. The form of the i^{th} trader's utility function is

$$U_i(\tilde{W}_{1i}) = -e^{-a_i \tilde{W}_{1i}}, \quad a_i > 0. \tag{1}$$

A.3 At the beginning of the period, the i^{th} trader observes y_i, which is a realized value from the noisy signal of the risky asset end-of-period value

$$\tilde{y}_i = \tilde{P}_1 + \tilde{\epsilon}_i \tag{2}$$

where $\tilde{\epsilon}_i \sim N(0, \sigma_i^2)$ and is independent of $\tilde{P}_1$.

Analysis:

Let X_{fi} be the amount invested in the risk-free asset and X_i be the number of shares of the risky asset chosen by the i^{th} trader at the beginning of the period. Thus,

$$W_{0i} \;=\; X_{fi} \;+\; P_0\, X_i. \tag{3}$$

The i^{th} trader's wealth accumulation equation can be written as

$$\tilde{W}_{1i} \;=\; (1+r)W_{0i} \;+\; \left[\,\tilde{P}_i - (1+r)P_0\,\right] X_i. \tag{4}$$

Denote I_i as the information available to the i^{th} trader at the beginning of the period. The trader's maximization problem is then

$$\max_{X_i} E\left[\,U_i(\tilde{W}_{1i}) \mid I_i\,\right] \;=\; \max_{X_i} E\left[\,-e^{-a_i\left((1+r)W_{0i} \;+\; \left[\tilde{P}_i-(1+r)P_0\right]X_i\right)} \mid I_i\,\right]. \tag{5}$$

If $\tilde{W}_{1i}$, which depends on $\tilde{P}_1$, is assumed to be normally distributed, then because of the exponential form of the utility function, (5) is the moment generating function of a normal random variable. Therefore, the maximization problem is equivalent to

$$\max_{X_i} \left\{ E\left[\,\tilde{W}_{1i} \mid I_i\,\right] \;-\; \frac{1}{2}\, a_i \,\mathrm{Var}\left[\,\tilde{W}_{1i} \mid I_i\,\right] \right\} \tag{6}$$

or

$$\max_{X_i} \left\{ X_i \left(E\left[\,\tilde{P}_1 \mid I_i\,\right] - (1+r)P_0 \right) \;-\; \frac{1}{2}\, a_i\, X_i^2 \,\mathrm{Var}\left[\,\tilde{P}_1 \mid I_i\,\right] \right\}.$$

The first-order condition with respect to X_i then gives us the optimal number of shares held in the risky asset:

$$X_i \;=\; \frac{E\left[\,\tilde{P}_1 \mid I_i\,\right] - (1+r)\,P_0}{a_i\,\mathrm{Var}\left[\,\tilde{P}_1 \mid I_i\,\right]}. \tag{7}$$

Now consider an equilibrium in which each trader uses his knowledge of the unconditional distribution of $\tilde{P}_1$ along with the conditioning information from his private signal, y_i, so that $I_i = \{y_i\}$. Then using Bayes rule and the fact that $\tilde{P}_1$ and $\tilde{y}_i$ are jointly normally distributed with a squared correlation $\rho_i \equiv \dfrac{\sigma^2}{\sigma^2 + \sigma_i^2}$, the i^{th} trader's conditional expected value and variance of $\tilde{P}_1$ are

$$\begin{aligned} E\left[\tilde{P}_1 \mid I_i\right] &= m + \rho_i\,(y_i - m) \\ \operatorname{Var}\left[\tilde{P}_1 \mid I_i\right] &= \sigma^2\,(1 - \rho_i). \end{aligned} \tag{8}$$

Note that ρ_i is not the correlation *coefficient* $\frac{\operatorname{cov}(\tilde{P}_1, \tilde{y}_i)}{\sigma_{\tilde{P}_1}\sigma_{\tilde{y}_i}} = \frac{\sigma^2}{\sigma\sqrt{\sigma^2+\sigma_i^2}} = \sqrt{\rho_i}$. Substituting these into (7), we have

$$X_i = \frac{m + \rho_i\,(y_i - m) - (1+r)\,P_0}{a_i\,\sigma^2\,(1 - \rho_i)}. \tag{9}$$

From the denominator of (9) one sees that the individual's demand for the risky asset is greater the lower is his risk aversion, a_i, and the greater is the precision of his signal (the closer is ρ_i to 1, that is, the lower is σ_i). Now by aggregating the individual traders' risky asset demands for shares and setting the sum equal to the fixed supply of shares, we can solve for the equilibrium risky asset price, P_0, that equates supply and demand:

$$\bar{X} = \sum_{i=1}^{n}\left[\frac{m + \rho_i\,(y_i - m) - (1+r)\,P_0}{a_i\,\sigma^2\,(1-\rho_i)}\right] = \sum_{i=1}^{n}\left[\frac{m + \rho_i\,(y_i - m)}{a_i\,\sigma^2\,(1-\rho_i)}\right] - \sum_{i=1}^{n}\left[\frac{(1+r)\,P_0}{a_i\,\sigma^2\,(1-\rho_i)}\right] \tag{10}$$

or

$$P_0 = \frac{1}{1+r}\left[\sum_{i=1}^{n}\frac{m + \rho_i\,(y_i - m)}{a_i\,\sigma^2\,(1-\rho_i)} - \bar{X}\right] \Big/ \left[\sum_{i=1}^{n}\frac{1}{a_i\,\sigma^2\,(1-\rho_i)}\right]. \tag{11}$$

From (11) we see that the price reflects a weighted average of the traders' conditional expectation of the payoff of the risky asset. For example the weight on the i^{th} trader's conditional expectation, $m + \rho_i(y_i - m)$, is

$$\frac{1}{a_i\,\sigma^2\,(1-\rho_i)} \;\Big/\; \left[\sum_{i=1}^{n} \frac{1}{a_i\,\sigma^2\,(1-\rho_i)}\right]. \tag{12}$$

The more precise (higher ρ_i) is trader i's signal or the lower is his risk aversion (more aggressively he trades), the more that the equilibrium price reflects his expectations.

The solution for the price, P_0, in equation (11) can be viewed as a competitive equilibrium: each trader uses information on his own signal and, in equilibrium, takes the price of the risky asset as given in deciding on how much to demand of the risky asset. However, this equilibrium neglects the possibility of individual traders obtaining information about other traders' signals from the equilibrium price itself, what practitioners call "price discovery." In this sense, the previous equilibrium is not a rational expectations equilibrium. To see this, note that if traders initially formulate their demands according to equation (9) (using only information about their own signals), and the equilibrium price in (11) then results, an individual trader could then infer information about the other traders' signals from the formula for P_0 in (11). Hence, this trader would have the incentive to change his or her demand from that initially formulated in (9). This implies that equation (11) would not be the rational expectations equilibrium price.

Therefore, to derive a fully rational expectations equilibrium, we need to allow traders' information sets to depend not only on their individual signals, but on the equilibrium price itself: $I_i = \{y_i, P_0^*(y)\}$ where $y \equiv (y_1 \;\; y_2 \;\; ... \;\; y_n)$ is a vector of the traders' individual signals.

In equilibrium, the aggregate demand for the shares of the risky asset must equal the aggregate supply, implying

$$\bar{X} \;=\; \sum_{i=1}^{n} \left[\frac{E\left[\tilde{P}_1 \mid y_i,\, P_0^*(y)\right] - (1+r)\, P_0^*(y)}{a_i \operatorname{Var}\left[\tilde{P}_1 \mid y_i,\, P_0^*(y)\right]}\right]. \tag{13}$$

Now one can show that a rational expectations equilibrium exists for the case of the ϵ_i's being independent and having the same variance, that is, $\sigma_i^2 = s^2$, for $i = 1, ..., n$.

Theorem: There exists a rational expectations equilibrium with $P_0^*(y)$ given by

$$P_0^*(y) \;=\; \frac{m \;+\rho\,(\bar{y}-m)}{1+r} \;-\; \frac{\sigma^2\,(1-\rho)\,\bar{X}}{1+r} \Bigg/ \left[\sum_{i=1}^{n}\frac{1}{a_i}\right] \tag{14}$$

where $\bar{y} \equiv \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\rho \equiv \frac{\sigma^2}{\sigma^2 + \frac{s^2}{n}}$.

Proof: For details, see the Grossman article. An intuitive outline of the proof is as follows. Note that in (14) $P_0^*(y)$ is a linear function of $\bar{y}$ with a fixed coefficient of $\rho/(1+r)$. Therefore, if a trader observes $P_0^*(y)$ (and knows the structure of the model, that is, the other parameters), then he can "invert" to infer the value of $\bar{y}$. Now because all traders' signals were assumed to have equal precision (same s^2), the average signal, $\bar{y}$, is a sufficient statistic for the information contained in all of the other signals. Further, because of the assumed independence of the signals, the precision of this average of signals is proportional to the number of traders, n. Hence, the average signal would have the same precision as a single signal with variance $\frac{s^2}{n}$.

Now if individual traders' demands are given by equation (9) but where y_i is replaced with $\bar{y}$ and ρ_i is replaced with ρ, then by aggregating these demands and setting them equal to $\bar{X}$ as in equation (10), we end up with the solution in equation (14), which is consistent with our initial assumption that traders can invert $P_0^*(y)$ to find $\bar{y}$. Hence, $P_0^*(y)$ in equation (14) is the rational expectations equilibrium price of the risky asset.

Note that the information, $\bar{y}$, reflected in the equilibrium price is superior to any single trader's private signal, y_i. In fact, since $\bar{y}$ is a sufficient statistic for all traders' information, it makes knowledge of any single signal, y_i, redundant. The equilibrium would be the same if all traders received the same signal, $\bar{y} \sim N(0, \frac{s^2}{n})$ or if they all decided to share information on their private signals among one another before trading commenced.

Therefore, the above equilibrium is a *fully-revealing* rational expectations equilibrium. The equilibrium price fully reveals all private information. This result has some interesting features in that it shows that prices can aggregate relevant information to help agents make more efficient investment decisions than would be the case if they relied solely on their private information and did not attempt to obtain information from the equilibrium price itself.

However, this fully revealing equilibrium is not robust to some small changes in assumptions.

For example, suppose each trader needed to pay a tiny cost, c, to obtain his private signal, y_i. With any finite cost of obtaining information, the equilibrium would not exist because each individual receives no additional benefit from knowing y_i given that they can observe $\bar{y}$ from the price. In other words, a given individual does not personally benefit from having private (inside) information in a fully-revealing equilibrium. In order for individuals to benefit from obtaining (costly) information, we need an equilibrium where the price is only partially revealing. For this to happen, there needs to be one or more additional sources of uncertainty that add "noise" to individuals' signals, so that other agents cannot infer it perfectly. An example of this type of model is D. Diamond and R. Verrecchia (1981) "Information Aggregation in a Noisy Rational Expectations Economy," *Journal of Financial Economics* 9, p.221-236.