Finance 400

A. Penati - G. Pennacchi

"Prospect Theory and Asset Prices"

These notes consider the asset pricing implications of investor behavior that incorporates "Prospect Theory." It summarizes an article by N. Barberis, M. Huang, and T. Santos (2000) "Prospect Theory and Asset Prices," *Quarterly Journal of Economics* (forthcoming). Prospect Theory deviates from von Neumann-Morgenstern expected utility maximization because investor utility is a function of recent changes in, rather than simply the current level of, financial wealth. In particular, investor utility characterized by Prospect Theory may be more sensitive to recent losses than recent gains in financial wealth. This effect is referred to as *loss aversion*. Moreover, losses following previous losses create more disutility than losses following previous gains. After a run-up in asset prices, the investor is less risk-averse because subsequent losses would be "cushioned" by the previous gains. This is the so-called *house money* effect.

An implication of this intertemporal variation in risk-aversion is that after a substantial rise in asset prices, lower investor risk aversion can drive prices even higher. Hence, asset prices display volatility that is greater than that predicted by observed changes in fundamentals, such as changes in dividends. This also generates predictability in asset returns. A substantial recent fall (rise) in asset prices increases (decreases) risk aversion and expected asset returns. It can also imply a high equity risk premium because the "excess" volatility in stock prices leads loss-averse investors to demand a relatively high average rate of return on stocks.

Prospect theory assumes that investors are overly concerned with changes in financial wealth, that is, they care about wealth changes more than would be justified by how these changes affect consumption. The idea was advanced by D. Kahneman and A. Tversky (1979) "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica* 47, p. 263-291. This psychological notion is based on experimental evidence. For example, R. Thaler and E. Johnson (1990) "Gambling with the House Money and Trying to Break Even," *Management Science* 36, p.643-660 find that individuals faced with a sequence of gambles are more willing to take risk if they have made gains from previous gambles.

The Barberis, Huang, and Santos model assumptions are as follows.

Assumptions:

A.1 Technology:

A discrete-time endowment economy is assumed. The risky asset (or portfolio of all risky assets) pays a dividend of perishable output of D_t at date t. The paper presents an "Economy I" model, where aggregate consumption equals dividends. This is the standard Lucas (1978) economy assumption. However, the paper focuses on its "Economy II" model which allows the risky asset's dividends to be distinct from aggregate consumption because there is assumed to be additional (non-traded) non-financial assets, such as labor income. In equilibrium, aggregate consumption, $\overline{C}_t$, then equals dividends, D_t, plus nonfinancial income, Y_t, because both dividends and nonfinancial income are assumed to be perishable. Aggregate consumption and dividends are assumed to follow the joint lognormal process

$$\begin{aligned} \ln\left(\overline{C}_{t+1}/\overline{C}_t\right) &= g_C + \sigma_C \eta_{t+1} \\ \ln\left(D_{t+1}/D_t\right) &= g_D + \sigma_D \varepsilon_{t+1} \end{aligned} \tag{1}$$

where the error terms are serially uncorrelated and distributed

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \tilde{\ } N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \omega \\ \omega & 1 \end{pmatrix} \right)$$

The return on the risky asset from date t to date $t+1$ is denoted R_{t+1}. A one-period risk-free investment is assumed to be in zero-net supply, and its return from date t to date $t+1$ is denoted $R_{f,t}$.[1]

A.2 Preferences:

Representative, infinitely-lived individuals maximize lifetime utility of the form

$$E_0\left[\sum_{t=0}^{\infty}\left(\rho^t \frac{C_t^{1-\gamma}}{1-\gamma} + b_t \rho^{t+1} v\left(X_{t+1}, S_t, z_t\right)\right)\right] \tag{2}$$

where C_t is the individual's consumption at date t, $\gamma > 0$, and ρ is a time discount factor. X_{t+1}

[1] Since the risk-free asset is in zero net supply, the representative individual's equilibrium holding of this asset is zero. $R_{f,t}$ is interpreted as the shadow riskless return.

is the gain in (change in value of) the individual's risky asset position between date t and date $t+1$. S_t is the date t value of the individual's risky asset holdings, and z_t is a measure of the individual's prior gains as a fraction of S_t. $z_t < (>)\, 1$ denotes a situation in which the investor has earned prior gains (losses) on the risky asset.

Risky asset gains are assumed to be measured relative to the alternative of holding wealth in the risk-free asset:

$$X_{t+1} = S_t \left(R_{t+1} - R_{f,t} \right) \tag{3}$$

The prior gain factor, z_t, is assumed to follow the process

$$z_t = 1 + \eta \left(z_{t-1} \frac{\overline{R}}{R_t} - 1 \right) \tag{4}$$

where $0 \leq \eta \leq 1$. If $\eta = 0$, $z_t = 1$ for all t. However, if $\eta = 1$, z_t is smaller (larger) than $z_{t-1}$when risky asset returns were relatively high last period, $R_t > \overline{R}$. In this case, the benchmark rate, z_t adjusts slowly to prior asset returns. In general, the greater is η, the longer is the investor's memory in measuring prior gains from the risky asset.

$v(\cdot)$ is a function characterizing the prospect theory effect of risky asset gains on utility.[2] For the case of $z_t = 1$ (no prior gains or losses), this function displays pure loss aversion:

$$v\left(X_{t+1}, S_t, 1\right) = \begin{cases} X_{t+1} & \text{if } X_{t+1} \geq 0 \\ \lambda X_{t+1} & \text{if } X_{t+1} < 0 \end{cases}$$

where $\lambda > 1$. Hence, ceteris paribus, losses have a disproportionately bigger impact on utility. When $z_t \neq 1$, the function $v(\cdot)$ reflects Propect Theory's house money effect. In the case of prior gains ($z_t \leq 1$), the function takes the form

$$v\left(X_{t+1}, S_t, z_t\right) = \begin{cases} X_{t+1} & \text{if } R_{t+1} \geq z_t R_{f,t} \\ X_{t+1} + (\lambda - 1)\, S_t \left(R_{t+1} - z_t R_{f,t}\right) & \text{if } R_{t+1} < z_t R_{f,t} \end{cases} \tag{5}$$

The interpretation of this function is that when a return exceeds the cushion built by prior

[2] Since $v(\cdot)$ depends only on the risky asset's returns, it is assumed that the individual is not subject to loss aversion on nonfinancial assets.

gains, that is, $R_{t+1} \geq z_t R_{f,t}$, it affects utility one-for-one. However, when the gain is less than the amount of prior gains, $R_{t+1} < z_t R_{f,t}$, it has a greater than one-for-one impact on disutility. In the case of prior losses ($z_t > 1$), the function becomes

$$v\left(X_{t+1}, S_t, z_t\right) = \begin{cases} X_{t+1} & \text{if } X_{t+1} \geq 0 \\ \lambda\left(z_t\right) X_{t+1} & \text{if } X_{t+1} < 0 \end{cases} \tag{6}$$

where $\lambda\left(z_t\right) = \lambda + k\left(z_t - 1\right)$, $k > 0$. Here we see that losses that follow previous losses are penalized at the rate, $\lambda\left(z_t\right)$, which exceeds λ and grows larger as prior losses become larger (z_t exceeds unity).

Finally, the prospect theory term in the utility function is scaled to make the risky asset price-dividend ratio and the risky asset risk premium be stationary variables as aggregate wealth increases over time.[3] The form of this scaling factor is chosen to be

$$b_t = b_0 \overline{C}_t^{-\gamma} \tag{7}$$

where $b_0 > 0$ and $\overline{C}_t$ is aggregate consumption at date t.[4]

Solution to the Model:

The state variables for the individual's consumption - portfolio choice problem are wealth, W_t, and z_t. Intuitively, since the aggregate consumption - dividend growth process in (1) is an independent, identical distribution, the dividend level is not a state variable. We start by assuming that the ratio of the risky asset price to its dividend is a function of only the state variable z_t, that is $f_t \equiv P_t/D_t = f_t\left(z_t\right)$, and then show that an equilibrium exists in which this is true.[5] Given this assumption, the return on the risky asset can be written as

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{1 + f\left(z_{t+1}\right)}{f\left(z_t\right)} \frac{D_{t+1}}{D_t} \tag{8}$$

[3]Without the scaling factor, as wealth (output) grows at rate g_D, the prospect theory term would dominate the conventional constant relative risk aversion term.

[4]Because $\overline{C}_t$ is assumed to be aggregate consumption, the individual views b_t as an exogeneous variable.

[5]This is plausible because the standard part of the utility function displays constant relative risk aversion. With this type of utility, optimal portfolio proportions would not be a function of wealth.

$$= \frac{1+f(z_{t+1})}{f(z_t)} e^{g_D + \sigma_D \varepsilon_{t+1}}$$

It is also assumed that an equilibrium exists in which the risk-free return is constant, that is, $R_{f,t} = R_f$. This will be verified by the solution to the agent's first order conditions. Making this assumption simplifies the form of the function v. From (5) and (6) it can be verified that v is proportional to S_t. Hence, $v(X_{t+1}, S_t, z_t)$ can be written as $v(X_{t+1}, S_t, z_t) = S_t \widehat{v}(R_{t+1}, z_t)$ where for $z_t < 1$

$$\widehat{v}(R_{t+1}, z_t) = \begin{cases} R_{t+1} - R_f & \text{if } R_{t+1} \geq R_f \\ R_{t+1} - R_f + (\lambda - 1)(R_{t+1} - z_t R_f) & \text{if } R_{t+1} < R_f \end{cases} \tag{9}$$

and for $z_t > 1$

$$\widehat{v}(R_{t+1}, z_t) = \begin{cases} R_{t+1} - R_f & \text{if } R_{t+1} \geq R_f \\ \lambda(z_t)(R_{t+1} - R_f) & \text{if } R_{t+1} < R_f \end{cases} \tag{10}$$

The individual's maximization problem is then

$$\max_{\{C_t, S_t\}} E_0 \left[\sum_{t=0}^{\infty} \left(\rho^t \frac{C_t^{1-\gamma}}{1-\gamma} + b_0 \rho^{t+1} \overline{C}_t^{-\gamma} S_t \widehat{v}(R_{t+1}, z_t) \right) \right] \tag{11}$$

subject to the budget constraint

$$W_{t+1} = (W_t + Y_t - C_t) R_f + S_t (R_{t+1} - R_f) \tag{12}$$

and the dynamics for z_t given in (4). Define $\rho^t J(W_t, z_t)$ as the derived utility of wealth function. Then the Bellman equation for this problem is

$$J(W_t, z_t) = \max_{\{C_t, S_t\}} \frac{C_t^{1-\gamma}}{1-\gamma} + E_t \left[b_0 \rho \overline{C}_t^{-\gamma} S_t \widehat{v}(R_{t+1}, z_t) + \rho J(W_{t+1}, z_{t+1}) \right] \tag{13}$$

Taking the first order conditions with respect to C_t and S_t one obtains

$$0 = C_t^{-\gamma} - \rho R_f E_t \left[J_W (W_{t+1}, z_{t+1}) \right] \tag{14}$$

$$
\begin{aligned}
0 &= E_t\left[b_0\overline{C}_t^{-\gamma}\widehat{v}\left(R_{t+1}, z_t\right) + J_W\left(W_{t+1}, z_{t+1}\right)\left(R_{t+1} - R_f\right)\right] \\
&= b_0\overline{C}_t^{-\gamma}E_t\left[\widehat{v}\left(R_{t+1}, z_t\right)\right] + E_t\left[J_W\left(W_{t+1}, z_{t+1}\right)R_{t+1}\right] - R_f E_t\left[J_W\left(W_{t+1}, z_{t+1}\right)\right]
\end{aligned} \tag{15}
$$

It is straightforward to show that (14) and (15) imply the standard envelope condition

$$
C_t^{-\gamma} = J_W\left(W_t, z_t\right) \tag{16}
$$

Substituting this into (14), one obtains the Euler equation

$$
1 = \rho R_f E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right] \tag{17}
$$

Using (16) and (17) in (15) implies

$$
\begin{aligned}
0 &= b_0\overline{C}_t^{-\gamma}E_t\left[\widehat{v}\left(R_{t+1}, z_t\right)\right] + E_t\left[C_{t+1}^{-\gamma}R_{t+1}\right] - R_f E_t\left[C_{t+1}^{-\gamma}\right] \\
&= b_0\overline{C}_t^{-\gamma}E_t\left[\widehat{v}\left(R_{t+1}, z_t\right)\right] + E_t\left[C_{t+1}^{-\gamma}R_{t+1}\right] - C_t^{-\gamma}/\rho
\end{aligned} \tag{18}
$$

or

$$
1 = b_0\rho E_t\left[\widehat{v}\left(R_{t+1}, z_t\right)\right] + \rho E_t\left[R_{t+1}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right] \tag{19}
$$

In equilibrium, conditions (17) and (19) hold with the representative agent's consumption, C_t, replaced with aggregate consumption, $\overline{C}_t$. Using the assumption in (1) that aggregate consumption is lognormally distributed, we can compute the expectation in (17) to solve for the risk-free interest rate:

$$
R_f = e^{\gamma g_C - \frac{1}{2}\gamma^2\sigma_C^2}/\rho \tag{20}
$$

Using (1) and (8), condition (19) can also be simplified:

$$
1 = b_0\rho E_t\left[\widehat{v}\left(R_{t+1}, z_t\right)\right] + \rho E_t\left[\frac{1 + f\left(z_{t+1}\right)}{f\left(z_t\right)}e^{g_D + \sigma_D\varepsilon_{t+1}}\left(e^{g_C + \sigma_C\eta_{t+1}}\right)^{-\gamma}\right] \tag{21}
$$

or

$$
\begin{aligned}
1 &= b_0 \rho E_t \left[\widehat{v} \left(\frac{1+f(z_{t+1})}{f(z_t)} e^{g_D + \sigma_D \varepsilon_{t+1}}, z_t \right) \right] \\
&\quad + \rho e^{g_D - \gamma g_C + \frac{1}{2}\gamma^2 \sigma_C^2 (1-\omega^2)} E_t \left[\frac{1+f(z_{t+1})}{f(z_t)} e^{(\sigma_D - \gamma \omega \sigma_C)\varepsilon_{t+1}} \right]
\end{aligned} \tag{22}
$$

The price - dividend ratio, $P_t/D_t = f_t(z_t)$, can be computed numerically from (22). However, because $z_{t+1} = 1 + \eta \left(z_t \frac{\overline{R}}{R_{t+1}} - 1 \right)$ and $R_{t+1} = \frac{1+f(z_{t+1})}{f(z_t)} e^{g_D + \sigma_D \varepsilon_{t+1}}$, z_{t+1} depends upon z_t, $f(z_t)$, $f(z_{t+1})$, and ε_{t+1}, that is

$$
z_{t+1} = 1 + \eta \left(z_t \frac{\overline{R} f(z_t) e^{-g_D - \sigma_D \varepsilon_{t+1}}}{1 + f(z_{t+1})} - 1 \right) \tag{23}
$$

Therefore, (22) and (23) need to be solved jointly. Barberis, Huang, and Santos describe an iterative numerical technique for finding the function $f(\cdot)$. Given all other parameters, they guess an initial function, $f^{(0)}$, and then use it to solve for z_{t+1} in (23) for given z_t and ε_{t+1}. Then, they find a new candidate solution, $f^{(1)}$, using the following recursion that is based on (22):

$$
\begin{aligned}
f^{(i+1)}(z_t) &= \rho e^{g_D - \gamma g_C + \frac{1}{2}\gamma^2 \sigma_C^2 (1-\omega^2)} E_t \left[\left[1 + f^{(i)}(z_{t+1}) \right] e^{(\sigma_D - \gamma \omega \sigma_C)\varepsilon_{t+1}} \right] \\
&\quad + f^{(i)}(z_t) b_0 \rho E_t \left[\widehat{v} \left(\frac{1+f^{(i)}(z_{t+1})}{f^{(i)}(z_t)} e^{g_D + \sigma_D \varepsilon_{t+1}}, z_t \right) \right], \ \forall z_t
\end{aligned} \tag{24}
$$

where the expectations are computed using a Monte Carlo simulation of the ε_{t+1}. Given the new candidate function, $f^{(1)}$, z_{t+1} is again found from (23). The procedure is repeated until the function $f^{(i)}$ converges.

For reasonable parameter values, Barberis, Huang, and Santos find that $P_t/D_t = f_t(z_t)$ is a decreasing function of z_t. The intuition was described earlier: if there were prior gains from holding the risky asset (z_t is low), then investors become less risk averse and bid up the price of the risky asset.

Using their estimate of $f(\cdot)$, the unconditional distribution of stock returns is simulated from a randomly generated sequence of ε_t's. Because dividends and consumption follow separate processes and stock prices have volatility exceeding that of dividend fundamentals, the volatility of stock prices can be made substantially higher than that of consumption. Moreover, because of loss aversion, the model can generate a significant equity risk premium for reasonable values of the consumption risk aversion parameter γ. Because the investor cares about stock volatility, per se, a large premium can exist even though stocks may not have a high correlation with consumption.[6]

The model also generates predictability in stock returns: returns tend to be higher following crashes and smaller following expansions. An implication of this is that stock returns are negatively correlated at long horizons, a feature documented by recent empirical research.

[6]Recall that in standard consumption asset pricing models, an asset's risk premium depends only on its return's covariance with consumption.