Finance 400

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Notes on

"An Intertemporal Capital Asset Pricing Model"

I. Assumptions

These notes are based on the article Robert C. Merton (1973) "An Intertemporal Capital Asset Pricing Model," *Econometrica* 41, p.867-887. It extends the analysis in Merton's earlier Consumption and Portfolio Rules paper to consider the equilibrium relations between asset rates of return. This model is not a fully general equilibrium model, since it takes asset supplies and the *form* of the asset prices processes as given. However, the asset price processes considered by Merton turn out to be of the same form as those derived by J. Cox, J. Ingersoll, and S. Ross (1985) "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica* 53, p.363-84, in their general equilibrium production economy model. So, it turns out that the model's implications hold in more general contexts.

Consider the model

$$\max_{\{C,w\}} E_0 \left[\int_0^T u\left(C\left(t\right), t\right) dt + B\left(W\left(T\right), T\right) \right] \tag{1}$$

subject to

$$dW = \sum_{i=1}^{n} w_i \left(\alpha_i - r\right) W dt + \left(rW - C\right) dt + \sum_{i=1}^{n} w_i W \sigma_i dz_i \tag{2}$$

where W and C are the individual's level of wealth and consumption, and each individual consumer in the economy can choose between one instantaneously risk-free asset and n risky assets whose means and standard deviations can depend, in general, on a state variable, x. w = $(w_1\ w_2\ ...\ w_n)$ is the $nx1$ vector of portfolio weights held in the risky assets. To keep things simple, we assume x is one-dimensional, but in general it could be a vector of state variables.

$$dP_i\left(t\right) / P_i\left(t\right) = \alpha_i\left(x\right) dt + \sigma_i\left(x\right) dz_i, \qquad i = 1, ..., n \tag{3}$$

$$dx = \alpha_x(x)\,dt + \sigma_x(x)\,dz_x \tag{4}$$

We allow correlation between the processes, so that $\sigma_{ij}(x)\,dt = \sigma_i(x)\,dz_i\sigma_j(x)\,dz_j$ and $\sigma_{ix}(x)\,dt = \sigma_i(x)\,dz_i\sigma_x(x)\,dz_x$. Define the value of the instantaneous risk-free investment, that is, the value of a "money market fund" as $P_0(t)$. Then this investment's value satisfies:

$$dP_0(t)/P_0(t) = r(x)\,dt \tag{5}$$

where $r(t)$ can depend on the state variable, x. Equations (3), (4), and (5) determine a vector Itô process. Thus, we have a stochastic investment opportunity set determined by the level of the state variable $x(t)$.

II. Solution to the Individual's Problem

Consider, as before, the optimality condition for each individual in the economy. Defining

$$J(W,x,t) = \max_{\{C,w\}} E_t\left[\int_t^T u(C(s),s)\,ds + B(W(T),T)\right] \tag{6}$$

The optimality condition is

$$0 = \max_{\{C,w\}} \left[u(C(t),t) + L[J]\right] \tag{7}$$

or

$$0 = \max_{\{C,w\}} \left[u(C(t),t) + J_t + J_W\left\{\sum_{i=1}^{n} w_i(\alpha_i - r)W + (rW - C)\right\} + J_x\alpha_x\right. \tag{8}$$

$$\left. +\frac{1}{2}J_{WW}W^2\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}w_iw_j + \frac{1}{2}J_{xx}\sigma_x^2 + J_{Wx}W\sum_{i=1}^{n}w_i\sigma_{ix}\right]$$

The first order conditions with respect to C and w_i are

$$0 = u_C - J_W \tag{9}$$

$$0 = J_W W (\alpha_i - r) + J_{WW} W^2 \sum_{j=1}^{n} w_j \sigma_{ij} + J_{Wx} W \sigma_{ix}, \qquad i = 1, ..., n. \tag{10}$$

Letting υ_{ij} be the i, j^{th} element of the inverse covariance matrix of asset returns, Ω^{-1}, then the system of linear equations in (10) can be solved to obtain:

$$w_i^* = -\frac{J_W}{W J_{WW}} \sum_{j=1}^{n} \upsilon_{ij} (\alpha_j - r) - \frac{J_{Wx}}{W J_{WW}} \sum_{j=1}^{n} \upsilon_{ij} \sigma_{jx} \tag{11}$$

or

$$w_i^* W = A \sum_{j=1}^{n} \upsilon_{ij} (\alpha_j - r) + H \sum_{j=1}^{n} \upsilon_{ij} \sigma_{jx} \tag{12}$$

where $A = -\frac{J_W}{J_{WW}}$ and $H = -\frac{J_{Wx}}{J_{WW}}$. Note that A and H will, in general, differ from one individual to another, depending on the form of the particular individual's utility function and level of wealth. Thus, unlike in the constant investment opportunity set case (where $J_{Wx} = 0$), w_i^*/w_j^* is not the same for all investors, that is, a *Two-Mutual Fund Theorem* does *not* hold. However, with one state variable, x, a *Three-Fund Theorem* holds. Investors will be satisfied choosing between a fund holding only the risk-free asset, a fund of risky assets that provides optimal instantaneous diversification (this is the same one we solved for earlier in the constant investment opportunity set case), and a third fund composed of a portfolio of the risky assets that has the maximum absolute correlation with the state variable, x. A/W and H/W, which depend on the individual's preferences, then determine the relative amounts that the individual invests in these two risky portfolios.

The individual's portfolio holdings can be reinterpreted as mechanisms for reducing fluctuations in consumption over time. Given that $J_W = U_C$, then $J_{WW} = U_{CC} \partial C / \partial W$, and so

$$A = -\frac{U_C}{U_{CC} (\partial C / \partial W)} > 0 \tag{13}$$

by the concavity of U. Also, since $J_{Wx} = U_{CC} \partial C / \partial x$, we have

$$H = -\frac{\partial C/\partial x}{\partial C/\partial W} \gtreqless 0 \tag{14}$$

Now the first term on the right-hand side of (12) is the usual demand function for a risky asset by a single-period mean-variance utility maximizer. Note that A is proportional to the reciprocal of the individual's absolute risk aversion, so that the more risk averse the individual, the smaller is A and the smaller is the individual's demand for any risky asset.

The second term on the right-hand side of (12) captures the individual's desire to hedge against "unfavorable" shifts in investment opportunities. An unfavorable shift is defined as a change in x such that consumption falls for a given level of current wealth, that is, an increase in x if $\partial C/\partial x < 0$ and a decrease in x if $\partial C/\partial x > 0$. For example, suppose that Ω is a diagonal matrix, so that $\upsilon_{ij} = 0$ for $i \neq j$, but assume $\upsilon_{ii} = 1/\sigma_{ii} > 0$ and $\sigma_{ix} \neq 0$.[1] Then in this special case the hedging demand term in (12) simplifies to

$$H\upsilon_{ii}\sigma_{ix} = -\frac{\partial C/\partial x}{\partial C/\partial W}\upsilon_{ii}\sigma_{ix} > 0 \text{ iff } \frac{\partial C}{\partial x}\sigma_{ix} < 0 \tag{15}$$

Condition (15) says that if an increase in x leads to a decrease in optimal consumption ($\partial C/\partial x < 0$) and if x and asset i are positively correlated ($\sigma_{ix} > 0$), then there is a positive hedging demand for asset i, that is, $H\upsilon_{ii}\sigma_{ix} > 0$ and asset i is held in greater amounts than what would be predicted based on a simple single-period mean-variance analysis. The intuition for this result is that by holding more of asset i, one hedges against a decline in future consumption due to an unfavorable shift in x. If x increases, which would tend to decrease consumption ($\partial C/\partial x < 0$), then asset i would tend to have a high return ($\sigma_{ix} > 0$), which by augmenting wealth, W, helps neutralize the fall in consumption ($\partial C/\partial W > 0$).

III. Equilibrium Asset Returns

Given the asset demands derived in the previous section, we can derive the equilibrium return relations between assets that must occur in order for the market portfolio to be efficient (that is, held by investors), in equilibrium. Consider the following cases:

III.A Constant Investment Opportunity Set

Investors' asset demands for this case (α_i, σ_i, r all constant) were analyzed earlier in our

[1] Alternatively, assume Ω is non-diagonal, but that $\sigma_{jx} = 0$ for $j \neq i$.

coverage of Merton's Consumption and Portfolio Rules paper. Defining δ_k as the demand for risky asset k as a proportion of all other risky assets, it was shown that:

$$\delta_k = \sum_{j=1}^{n} \nu_{kj} \left(\alpha_j - r\right) / \sum_{i=1}^{n} \sum_{j=1}^{n} \nu_{ij} \left(\alpha_j - r\right), \qquad k = 1, ...n. \tag{16}$$

R.C. Merton (1972) "An Analytical Derivation of the Efficient Portfolio Frontier," *Journal of Financial and Quantitative Analysis* 7, p.1851-72, shows that when the market portfolio is efficient (that is, willingly held by investors), equilibrium asset returns must satisfy

$$\alpha_j - r = \beta_i \left(\alpha_M - r\right), \qquad i = 1, ...n. \tag{17}$$

where $\beta_i \equiv \sigma_{iM}/\sigma_M^2$, σ_{iM} where is the covariance between the i^{th} asset's rate of return and the market's rate of return, and α_M and σ_M^2 are the mean and the variance of the rate of return on the market portfolio. Thus, with a constant investment opportunity set, the standard, single-period CAPM holds.

III.B Stochastic Investment Opportunity Set

Consider, as before, the case in which there is a single state variable, x. Recall that equation (10) is the system of n equations that a given individual's portfolio weights satisfy. Let's re-write (10) in matrix form, using the superscript k to denote the k^{th} individual's value of wealth, vector of optimal portfolio weights, and values of A and H:

$$A^k \left(\boldsymbol{\alpha} - r\mathbf{1}\right) = \boldsymbol{\Omega}\mathbf{w^k}W^k - H^k\boldsymbol{\sigma}_\mathbf{x} \tag{18}$$

where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)'$, $\mathbf{1}$ is an n-dimensional vector of one's, $\mathbf{w^k} = (w_1^k, ..., w_n^k)'$ and $\boldsymbol{\sigma}_\mathbf{x} = (\sigma_{1x}, ..., \sigma_{nx})'$. We will use **bold** type to denote vector or matrix variables, while normal type is used for scalar variables. Now if we sum across all individuals and divide both sides by $\sum_k A^k$, we obtain

$$\boldsymbol{\alpha} - r\mathbf{1} = a\boldsymbol{\Omega}\boldsymbol{\mu} - h\boldsymbol{\sigma}_\mathbf{x} \tag{19}$$

where $a \equiv \sum_k W^k / \sum_k A^k$, $h \equiv \sum_k H^k / \sum_k A^k$, and $\boldsymbol{\mu} \equiv \sum_k \mathbf{w^k}W^k / \sum_k W^k$ is the average investment in each asset across investors. These must be the market weights, in equilibrium.

Hence, the i^{th} row (i^{th} risky asset excess return) of equation (19) is:

$$\alpha_i - r = a\sigma_{iM} - h\sigma_{ix} \tag{20}$$

To find the excess return on the market portfolio, we can pre-multiply (19) by $\boldsymbol{\mu}'$ and obtain:

$$\alpha_M - r = a\sigma_M^2 - h\sigma_{Mx} \tag{21}$$

Next, define $\boldsymbol{\eta} \equiv \frac{\boldsymbol{\Omega}^{-1}\boldsymbol{\sigma}_x}{\mathbf{1}'\boldsymbol{\Omega}^{-1}\boldsymbol{\sigma}_x}$. By construction, $\boldsymbol{\eta}$ is the vector of portfolio weights for the risky assets, where this portfolio has the maximum absolute correlation with the state variable, x. In this sense, it provides the best possible hedge against changes in the state variable. To find the excess return on this optimal hedge portfolio, we can premultiply (19) by $\boldsymbol{\eta}'$ and obtain

$$\alpha_\eta - r = a\sigma_{\eta M} - h\sigma_{\eta x} \tag{22}$$

where $\sigma_{\eta M}$ is the covariance between the optimal hedge portfolio and the market portfolio and $\sigma_{\eta x}$ is the covariance between the optimal hedge portfolio and the state variable, x. Equations (21) and (22) are two linear equations in the two unknowns, a and h. Solving for a and h and substituting them back into equation (20), we obtain:

$$\alpha_i - r = \frac{\sigma_{iM}\sigma_{\eta x} - \sigma_{ix}\sigma_{M\eta}}{\sigma_M^2\sigma_{\eta x} - \sigma_{Mx}\sigma_{M\eta}}\left(\alpha_M - r\right) + \frac{\sigma_{ix}\sigma_M^2 - \sigma_{iM}\sigma_{Mx}}{\sigma_M^2\sigma_{\eta x} - \sigma_{Mx}\sigma_{M\eta}}\left(\alpha_\eta - r\right) \tag{23}$$

While the derivation is not given here, (23) can be re-written as[2]

$$\alpha_i - r = \frac{\sigma_{iM}\sigma_\eta^2 - \sigma_{i\eta}\sigma_{M\eta}}{\sigma_M^2\sigma_\eta^2 - \sigma_{M\eta}^2}\left(\alpha_M - r\right) + \frac{\sigma_{i\eta}\sigma_M^2 - \sigma_{iM}\sigma_{M\eta}}{\sigma_\eta^2\sigma_M^2 - \sigma_{M\eta}^2}\left(\alpha_\eta - r\right) \tag{24}$$

$$\equiv \beta_i^M\left(\alpha_M - r\right) + \beta_i^\eta\left(\alpha_\eta - r\right)$$

Note that $\sigma_{i\eta} = 0$ iff $\sigma_{ix} = 0$. For the case in which the state variable, x, is uncorrelated with the market, equation (24) simplifies to:

[2]See J. Ingersoll (1987) *Theory of Financial Decision Making*, Rowman and Littlefield, Totowa, NJ, p.286.

$$\alpha_i - r = \frac{\sigma_{iM}}{\sigma_M^2}\left(\alpha_M - r\right) + \frac{\sigma_{i\eta}}{\sigma_\eta^2}\left(\alpha_\eta - r\right) \tag{25}$$

In this case, the first term on the right-hand side of (25) is that found in the standard CAPM. The assumption that x is uncorrelated with the market is not as restrictive as one might first believe, since one could re-define the state variable x as a factor that cannot be explained by current market returns, that is, a factor that is uncorrelated with the market.

An equation such as (24) can be derived when more than one state variable exists. In this case, there will be an additional "beta" for each state variable. Relations such as (24) and (25) bear more than a coincidental relationship to Ross's Linear Factor model (APT).

IV. Extending the Model to State-Dependent Utility

Suppose individual utility is directly dependent on the state of nature, x, that is, $u\left(C\left(t\right), x\left(t\right), t\right)$. It is straightforward to verify that the optimality condition (8) and the first order conditions for C and the w_i's remain unchanged. Hence our results on the equilibrium returns on assets, equation (24), continue to hold. The only change is in the interpretation of H, the individual's hedging demand coefficient. With state-dependent utility,

$$J_{Wx} = U_{CC}\frac{\partial C}{\partial x} + U_{Cx} \tag{26}$$

so that

$$H = -\frac{\partial C/\partial x}{\partial C/\partial W} - \frac{U_{Cx}}{U_{CC}\frac{\partial C}{\partial W}} \tag{27}$$

It can be shown that in this case individuals do not hold consumption variance minimizing portfolios, but marginal utility variance-minimizing portfolios.

V. Breeden's Consumption-Based CAPM

The article by Douglas T. Breeden (1979) "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," *Journal of Financial Economics* 7, p.265-96 provides a way of simplifying the asset return relationship given in Merton's ICAPM. Breeden's model shows that the single-period consumption-portfolio choice result that an asset's expected rate of return depends upon its covariance with the marginal utility of consumption can be generalized to a multi-period, continuous-time context.

Breeden considers the same model as Merton, and hence in the case of multiple state variables, derives equation (12). Substituting in for A and H, equation (12) can be written in matrix form, and for the case of s (multiple) state variables the optimal portfolio weights for the k^{th} investor are given by:

$$\mathbf{w^k}W^k = -\frac{U_C^k}{U_{CC}^k C_W^k}\boldsymbol{\Omega}^{-1}(\boldsymbol{\alpha} - r\mathbf{1}) - \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}_{\mathbf{ax}}\mathbf{C_x^k}/C_W^k \tag{28}$$

where $C_W^k = \partial C^k/\partial W^k$, $\mathbf{C_x^k} = \left(\frac{\partial C^k}{\partial x_1}...\frac{\partial C^k}{\partial x_s}\right)'$ and $\boldsymbol{\Omega}_{\mathbf{ax}}$ is an $n \times s$ matrix of covariances of asset returns with changes in the state variables. Pre-multiplying (28) by $C_W^k\boldsymbol{\Omega}$ and rearranging terms, we have

$$-\frac{U_C^k}{U_{CC}^k}(\boldsymbol{\alpha} - r\mathbf{1}) = \boldsymbol{\Omega}_{\mathbf{aW}^k}C_W^k + \boldsymbol{\Omega}_{\mathbf{ax}}\mathbf{C_x^k} \tag{29}$$

where $\boldsymbol{\Omega}_{\mathbf{aW}^k}$ is the $n \times 1$vector of covariances between asset returns with the change in wealth of individual k. Now individual k's optimal consumption, $C^k\left(W^k, \mathbf{x}, t\right)$ is a function of wealth, W^k, the vector of state variables, $\mathbf{x}$, and time, t. Thus, from Itô's lemma, we know that the stochastic terms for dC^k will be

$$C_W^k\left(w_1^k W^k\sigma_1 dz_1 + ... + w_n^k W^k\sigma_n dz_n\right) + (\sigma_{x1}dz_{x1} + ... + \sigma_{xs}dz_{xs})\,\mathbf{C_x^k} \tag{30}$$

Hence, the instantaneous covariances of asset returns with changes in individual k's consumption are given by calculating the instantaneous covariance between each asset (having stochastic term $\sigma_i dz_i$) with the terms given in (30). The result, in matrix form, is that the $n \times 1$ vector of covariances between asset returns and changes in the individual's consumption is

$$\boldsymbol{\Omega}_{\mathbf{aC}^k} = \boldsymbol{\Omega}_{\mathbf{aW}^k}C_W^k + \boldsymbol{\Omega}_{ax}\mathbf{C_x^k} \tag{31}$$

Note that the right-hand side of (31) equals the right-hand side of (29), and therefore

$$\boldsymbol{\Omega}_{\mathbf{aC}^k} = -\frac{U_C^k}{U_{CC}^k}(\boldsymbol{\alpha} - r\mathbf{1}) \tag{32}$$

Equation (32) holds for each individual, k. Next, define C as the aggregate rate of consumption and define T as an aggregate rate of risk tolerance, where

$$T \equiv \sum_k -\frac{U_C^k}{U_{CC}^k} \tag{33}$$

Then (32) can be aggregated over all individuals to obtain.

$$\boldsymbol{\alpha} - r\mathbf{1} = T^{-1}\boldsymbol{\Omega}_{\mathbf{aC}} \tag{34}$$

where $\boldsymbol{\Omega}_{\mathbf{aC}}$ is the $n \times 1$ vector of covariances between asset returns and changes in aggregate consumption. If we multiply and divide the right hand side of (34) by current aggregate consumption, one obtains

$$\boldsymbol{\alpha} - r\mathbf{1} = (T/C)^{-1}\,\boldsymbol{\Omega}_{\mathbf{a},\ln\mathbf{C}} \tag{35}$$

where $\boldsymbol{\Omega}_{\mathbf{a},\ln\mathbf{C}}$ is the $n \times 1$ vector of covariances between asset returns and changes in the logarithm of consumption (percentage rates of change of consumption).

Consider a portfolio, M, with vector of weights $\mathbf{w^M}$. Pre-multiplying (35) by $\mathbf{w^{M\prime}}$, we have:

$$\alpha_M - r = (T/C)^{-1}\,\sigma_{M,\ln C} \tag{36}$$

where α_M is the expected return on portfolio M and $\sigma_{M,\ln C}$ is the (scalar) covariance between returns on portfolio M and changes in the log of consumption. Substituting for $(T/C)^{-1}$ in (35), we have

$$\boldsymbol{\alpha} - r1 = (\boldsymbol{\Omega}_{\mathbf{a},\ln\mathbf{C}}/\sigma_{M,\ln C})\,(\alpha_M - r) \tag{37}$$

$$= (\boldsymbol{\beta}_{aC}/\beta_{MC})\,(\alpha_M - r)$$

where $\boldsymbol{\beta}_{aC}$ and β_{MC} are the "consumption betas" of asset returns and of portfolio M's return. The consumption beta for any asset is defined as

$$\beta_{iC} = cov\,(dP_i/P_i, d\ln C)\,/var\,(d\ln C) \tag{38}$$

Portfolio M may be any portfolio of assets, not necessarily the market portfolio. Equation (37) says that the ratio of expected excess returns on any two assets or portfolios of assets is equal to the ratio of their betas measured relative to aggregate consumption. Hence, the risk of a security's return can be summarized by a single consumption beta. Aggregate optimal consumption, $C(W, \mathbf{x}, t)$, encompasses the effects of levels of wealth and the state variables, and in this way is a sufficient statistic for the value of asset returns in different states of the world.

Breeden's consumption CAPM is a considerable simplification relative to Merton's multi-beta ICAPM. Furthermore, while the multiple state variables in Merton's model may not be directly identified or observed, and hence the multiple state variable "betas" may not be computed, Breeden's consumption beta can be computed given that we have data on aggregate consumption. However, the results of empirical tests of the consumption beta model have been "mixed" at best.