Finance 400

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Intertemporal Consumption and Portfolio Choice in Continuous-Time

I. Specifying the Consumption and Portfolio Choice Model in Continuous-Time

Having already studied an individual's intertemporal consumption and portfolio choice problem in a discrete-time setting, we now consider a similar problem but where asset prices follow continuous-time stochastic processes and the individual makes consumption and portfolio decisions continuously. Consider an individual's choice regarding the level of his consumption and savings, where his savings can be invested in n different assets. As before, we define

$C(t)$ = individual's level of consumption at date t,

$P(t) = (P_1(t),\, P_2(t),\, \ldots\, \ldots,\, P_n(t))$ the prices of the n assets at date t.

However, now assume the i^{th} asset price follows the process:

$$dP_i(t)\,/\,P_i(t) \;=\; \alpha_i\,(x,t)\;dt \;+\; \sigma_i\,(x,t)\;dz_i \tag{1}$$

where $i = 1, ..., n$, and $(\sigma_i\, dz_i)(\sigma_j\, dz_j) \; = \sigma_{ij}\, dt$. It is assumed that α_i and σ_i need not be constants but could be functions of time and/or functions of a $kx1$ vector of state variables, which we denote by $x(t)$. When the α_i and σ_i are time-varying, the investor is said to face changing investment opportunities. The state variables affecting the moments of the asset prices, $x\,(t)$, can, themselves, follow diffusion processes. Let the i^{th} state variable follow the process

$$dx_i = a_i\,(x,t)\,dt + b_i\,(x,t)\,dq_i \tag{2}$$

where $i = 1, ..., k$, and $(b_i\, dq_i)(b_j\, dq_j) \; = b_{ij}\, dt$ and $(b_i\, dq_i)(\sigma_j\, dz_j) = \phi_{ij}\, dt$.

To derive the proper continuous-time budget constraint for the individual, it is worthwhile to first consider the analogous budget constraint for a discrete-time model where each period is of length h. We will then take the limit as $h \to 0$.

Let $N_i(t)$ be the number of shares held in asset i from date t to $t+h$. The individual begins period t with wealth, $W(t)$, which equals last period's holdings at date t prices:

$$W(t) \;=\; \sum_{i=1}^{n} N_i(t-h)P_i(t). \tag{3}$$

Given these date t prices, the individual decides on his level of consumption over the period, $C(t)\,h$, which must equal the net sales of assets.[1] Note that $C(t)$, the individual's "level" of consumption at time t, now represents the *average rate* of consumption over the interval from t to $t+h$.

$$-C(t)\,h \;=\; \sum_{i=1}^{n}[N_i(t) - N_i(t-h)]\,P_i(t). \tag{4}$$

At the start of the next period, $t+h$, we have:

$$\begin{aligned} -C(t+h)\,h \;&=\; \sum_{i=1}^{n}[N_i(t+h) - N_i(t)]\,P_i(t+h) \\ &=\; \sum_{i=1}^{n}[N_i(t+h) - N_i(t)]\,[P_i(t+h) - P_i(t)] \;+\; \sum_{i=1}^{n}[N_i(t+h) - N_i(t)]\,P_i(t) \end{aligned} \tag{5}$$

and

$$W(t+h) \;=\; \sum_{i=1}^{n} N_i(t)\,P_i(t+h). \tag{6}$$

Taking the limits of (5) and (6) as $h \to 0$, and keeping in mind that the $P_i(t)$ and $N_i(t)$ now represent stochastic processes, we obtain the stochastic differential equation

$$-C(t)\,dt \;=\; \sum_{i=1}^{n} dN_i(t)\,dP_i(t) \;+\; \sum_{i=1}^{n} dN_i(t)\,P_i(t) \tag{7}$$

and

$$W(t) \;=\; \sum_{i=1}^{n} N_i(t)\,P_i(t). \tag{8}$$

[1] It is assumed that there are no other sources of wealth, such as wage income

Applying Itô's Lemma to (8), we can derive the dynamics of the individual's wealth:

$$dW = \sum_{i=1}^{n} N_i(t)\, dP_i(t) + \sum_{i=1}^{n} dN_i(t)\, P_i(t) + \sum_{i=1}^{n} dN_i(t)\, dP_i(t). \tag{9}$$

Substituting (7) into (9), we obtain:

$$dW = \sum_{i=1}^{n} N_i(t)\, dP_i(t) - C(t)\, dt. \tag{10}$$

Equation (10) says that the individual's wealth changes due to capital gains income less the amount consumed.

Now, as in the discrete-time case, instead of solving for the individual's optimal $C(t)$ and $N_i(t)$, $i = 1, \ldots, n$, it will be more convenient to solve for $C(t)$ and the proportion of wealth held in each particular asset, defined as $\omega_i(t) = N_i(t)P_i(t)/W(t)$. Substituting this in for N_i and substituting (1) for dP_i, equation (10) becomes:

$$dW = \sum_{i=1}^{n} \omega_i W \alpha_i\, dt - C(t)\, dt + \sum_{i=1}^{n} \omega_i W \sigma_i\, dz_i. \tag{11}$$

It is convenient to assume that one of the assets, let it be the n^{th} asset, is risk-free. This implies $\sigma_n = 0$. Define $r = \alpha_n$ and $m = n - 1$.[2] Then (11) can be rewritten

$$dW = \sum_{i=1}^{m} \omega_i(\alpha_i - r) W\, dt + (rW - C(t))\, dt + \sum_{i=1}^{m} \omega_i W \sigma_i\, dz_i. \tag{12}$$

With these preliminaries out of the way, we can now state the individual's intertemporal consumption and portfolio choice problem:

$$\max_{\{C(t),\, \omega(t)\}} E_0 \left[\int_0^T u(C(t), t)\, dt + B(W(T), T) \right] \tag{13}$$

subject to the constraints

$$W(t) > 0, \text{ for all } t \in [0, T] \tag{14a}$$

[2] In general, r may be a function of $x(t)$. For example, $r(t)$ could follow a diffusion process such as that of the Vasicek (1977) or Cox, Ingersoll, and Ross (1985) term structure models.

$$dW \;=\; \sum_{i=1}^{m} \omega_i(\alpha_i - r)W\,dt \;+\; (rW - C(t))\,dt \;+\; \sum_{i=1}^{m} \omega_i W \sigma_i\,dz_i \tag{14b}$$

where ω denotes the vector of risky-asset portfolio weights $\{\omega_1, \ldots, \omega_m\}$, the utility function, u, is assumed to be strictly concave in C and the bequest function B is assumed to be strictly concave in W. This problem, in which the individual has time-separable utility of consumption, is analogous to the discrete-time problem studied earlier.

Note that some possible constraints are have not been imposed. For example, one might wish to impose the constraint $C(t) \geq 0$ (nonnegative consumption) and/or $\omega_i \geq 0$ (no short sales). However, for some utility functions, negative consumption is never be optimal, so that $C(t) \geq 0$ might be imposed automatically.[3]

Before we solve this problem, let's make a digression on stochastic dynamic programing in a continuous-time setting.

II. A Digression on Stochastic Dynamic Programming in Continuous Time

Consider a simple (single choice variable), generic version of the problem specified in (13).

$$\max_{\{c\}} \; E_t \left[\int_t^T u(c(\mu), x(\mu))\,d\mu \right] \tag{D.1}$$

subject to

$$dx \;=\; a(x,\,c)\,dt \;+\; b(x,\,c)\,dz \tag{D.2}$$

where $c(t)$ is a *control* variable (such as a consumption and/or vector of portfolio proportions) and $x(t)$ is a *state* variable (such as wealth and/or a variable that changes investment opportunities, that is, a variable that affects the α_i's and/or σ_i's). Define the indirect utility function, $J(x(t),\,t,T)$:

$$J(x(t),\,t,T) \;=\; \max_{\{c\}} E_t \left[\int_t^T u(c(\mu), x(\mu))\,d\mu \right] \tag{D.3}$$

[3] For example, if $\underset{C(t)\to 0}{lim} \; \frac{\partial u(C(t),t)}{\partial C} = \infty$, as would be the case if the individual's utility displayed constant relative risk aversion (power utility), then the individual would always avoid non-positive consumption. However, other utility functions, such as constant absoute risk aversion (negative exponential) utility, do not display this property.

which can be written as

$$J(x(t),\, t,\, T) \;=\; \max_{\{c\}} E_t \left[\int_t^{t+\Delta t} u(c(\mu), x(\mu))\, d\mu \;+\; \int_{t+\Delta t}^{T} u(c(\mu), x(\mu))\, d\mu \right]. \tag{D.4}$$

Now let us apply Bellman's *Principle of Optimality.* This says that an optimal policy must be such that for a given future realization of the state variable, $x(t+\Delta t)$, (whose value may be affected by the optimal control policy at date t and earlier), any remaining decisions at date $t+\Delta t$ and later must be optimal with respect to $x(t+\Delta t)$. In other words, an optimal policy must be time consistent. This allows us to write

$$\begin{aligned} J(x(t),\, t,\, T) \;&=\; \max_{\{c\}} E_t \left[\int_t^{t+\Delta t} u(c(\mu), x(\mu))\, d\mu \;+\; \max_{\{c\}} E_{t+\Delta t} \left[\int_{t+\Delta t}^{T} u(c(\mu), x(\mu))\, d\mu \right] \right] \\ &=\; \max_{\{c\}} E_t \left[\int_t^{t+\Delta t} u(c(\mu), x(\mu))\, d\mu \;+\; J(x(t+\Delta t),\, t+\Delta t,\, T) \right]. \end{aligned} \tag{D.5}$$

Now approximate the first integral and also expand $J(x(t+\Delta t),\, t+\Delta t,\, T)$ around $x(t)$ and t in a Taylor series to get

$$\begin{aligned} J(x(t),\, t,\, T) \;&=\; \max_{\{c\}} E_t \big[u(c(t), x(t))\, \Delta t \;+\; J(x(t),\, t,\, T) \;+\; J_x \Delta x \;+\; J_t \Delta t \\ &\quad + \frac{1}{2} J_{xx} (\Delta x)^2 \;+\; J_{xt} (\Delta x)(\Delta t) \;+\; \frac{1}{2} J_{tt} (\Delta t)^2 \;+\; o(\Delta t) \Big] \end{aligned} \tag{D.6}$$

where

$$\Delta x \;\approx\; a(x,\, c)\Delta t \;+\; b(x,\, c)\Delta z \;+\; o(\Delta t). \tag{D.7}$$

Substituting (D.7) into (D.6), and subtracting $J(x(t),\, t,\, T)$ from both sides, one obtains:

$$0 \;=\; \max_{\{c\}} E_t \left[u(c(t), x(t))\Delta t \;+\; \Delta J \;+\; o(\Delta t) \right] \tag{D.8}$$

where

$$\Delta J \;=\; \left[J_t \;+\; J_x a \;+\; \frac{1}{2} J_{xx} b^2 \right] \Delta t \;+\; J_x b \Delta z. \tag{D.9}$$

Note that (D.9) is just the discrete-time version of Itô's Lemma. Hence, if we note that in (D.8) the term $E_t\left[J_x b \Delta z\right] = 0$, then divide both sides of (D.8) by Δt, and, finally, take the limit as $\Delta t \to 0$, we have

$$0 \;=\; \max_{\{c\}} \left[u(c(t),\, x(t)) \;+ J_t \;+\; J_x a \;+\; \frac{1}{2} J_{xx} b^2 \right] \tag{D.10}$$

which is the stochastic, continuous-time Bellman equation. It is sometimes written as

$$0 \;=\; \max_{\{c\}} \left[u(c(t),\, x(t)) \;+\; L[J] \;\right] \tag{D.11}$$

where $L[\cdot]$ is the *Dynkin operator*, being the *"drift" term (expected change) in* $dJ(x,\,t)$ *that one obtains by applying Itô's Lemma to* $J(x,\,t)$. (D.11) then gives us a condition that the optimal stochastic control policy, c, must satisfy. Let us now return to the consumption and portfolio choice problem and apply this tool.

III. Solving the Continuous-Time Consumption and Portfolio Choice Problem

Define the indirect utility of wealth function, $J(W,\,x,\,t)$, as

$$J(W, x,\, t) \;=\; \max_{\{C(t),\, \omega(t)\}} E_t \left[\int_t^T u(C(s), s)\, ds \;+\; B(W(T), T) \right] \tag{15}$$

and define L as the Dynkin operator with respect to the state variables W and x_i, $i = 1,\, \ldots,\, k$. In other words

$$L \;=\; \frac{\partial}{\partial t} \;+\; \left[\sum_{i=1}^{m} \omega_i(\alpha_i - r)W + (rW - C) \right] \frac{\partial}{\partial W} \;+\; \sum_{i=1}^{k} a_i \frac{\partial}{\partial x_i} \;+$$
$$\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \sigma_{ij} \omega_i \omega_j W^2 \frac{\partial^2}{\partial W^2} \;+\; \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} b_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} \;+\; \sum_{i=1}^{k} \sum_{j=1}^{m} W \omega_j \phi_{ij} \frac{\partial^2}{\partial x_i\, \partial W}. \tag{16}$$

Thus, using (D.11), we have

$$0 \;=\; \max_{\{C(t),\, \omega(t)\}} \left[u(C(t), t) \;+ L[J] \right]. \tag{17}$$

Given the concavity of u and B, equation (17) implies that the optimal choices of $C(t)$ and $\omega(t)$ satisfy the conditions we obtain from differentiating $u(C(t),\, t) + L[J]$ and setting the result equal to zero. Hence, the first order conditions are:

$$0 \;=\; \frac{\partial u}{\partial C}(C^*) \;-\; \frac{\partial J}{\partial W} \tag{18}$$

$$0 \;=\; \frac{\partial J}{\partial W}(\alpha_i - r)W \;+\; \frac{\partial^2 J}{\partial W^2}\sum_{j=1}^{m}\sigma_{ij}\omega_j^* W^2 \;+\; \frac{\partial^2 J}{\partial x_i\,\partial W}\sum_{j=1}^{k} W\phi_{ij}, \quad i = 1,\ldots,m. \tag{19}$$

Note that equations (18) and (19) are n equations in the n unknowns $C^*(t), \omega_1^*(t), \ldots \omega_{n-1}^*(t)$. For certain functional forms for u and the α_i's, σ_{ij}'s, and ϕ_{ij}'s we can solve for C^* and the ω_i^*'s as functions of the state variables W, x, and derivatives of J, that is, J_W, J_{Wx_i}, $J_{x_i x_j}$. Then, plugging these values for C^* and ω_i^* back into equation (17), we will have a partial differential equation (PDE) for J. If this equilibrium PDE can be solved for J, then explicit solutions for C^* and the ω_i^* in terms of the model's parameters and state variables can be obtained.

III.A. Lognormally-Distributed Asset Prices

Let's do this for the special case in which asset prices are lognormally distributed, that is, all of the α_i's (including r) and σ_i's are constants. This means that each asset's expected rate of return and variance of its rate of return do not change; there is a *constant investment opportunity set.* Hence, investment and portfolio choice decisions are independent of the state variables, x, since they do not affect u, B, the α_i's, or the σ_i's. The only state variable affecting consumption and portfolio choice decisions is wealth, W. This simplifies the above analysis, since now J is only a function of W, not x.

Since the utility function is strictly concave, we can define the inverse function $G \equiv \left[\frac{\partial u}{\partial C}\right]^{-1}$. From equation (18), this implies

$$C^* \;=\; G(J_W,\, t). \tag{20}$$

Also define $\Omega \equiv [\sigma_{ij}]$ to be the $m \times m$ covariance matrix, and $[\nu_{ij}] \equiv \Omega^{-1}$ to be the inverse of Ω. For this constant investment opportunity set case, the first order conditions (19) becomes:

$$0 \;=\; J_W(\alpha_i - r)W \;+\; J_{WW}\sum_{j=1}^{m}\sigma_{ij}\omega_j W^2, \quad i = 1,\ldots,m. \tag{21}$$

Note that (21) gives m linear equations in m unknowns. The solutions are

$$\omega_i^* = -\frac{J_W}{J_{WW}W}\sum_{j=1}^{m}\nu_{ij}(\alpha_j - r), \quad i = 1, \ldots, m. \tag{22}$$

Plugging (20) and (22) back into the optimality equation (17), and using the fact that $[\nu_{ij}] \equiv \Omega^{-1}$, we have

$$0 = u(G, t) + J_t + J_W(rW - G) - \frac{J_W^2}{2J_{WW}}\sum_{i=1}^{m}\sum_{j=1}^{m}\nu_{ij}(\alpha_i - r)(\alpha_j - r). \tag{23}$$

The equilibrium PDE given by equation (23) may not have an analytic solution for any general utility function, u.[4] However, we can still draw some implications of the individual's investment behavior by looking at equation (22). Note that (22) says something rather interesting but which should be intuitive given a constant investment opportunity set. Since ν_{ij}, α_j, and r are constants, the proportion of each risky asset that is optimally held will be proportional to $-J_W/(J_{WW}W)$, which only depends on the total wealth state variable, W. Thus, the proportion of wealth in risky asset i to risky asset k is a constant, that is,

$$\frac{\omega_i^*}{\omega_k^*} = \frac{\sum_{j=1}^{m}\nu_{ij}(\alpha_j - r)}{\sum_{j=1}^{m}\nu_{kj}(\alpha_j - r)} \tag{24}$$

and the proportion of risky asset k to all risky assets is

$$\delta_k = \frac{\sum_{j=1}^{m}\nu_{kj}(\alpha_j - r)}{\sum_{i=1}^{m}\sum_{j=1}^{m}\nu_{ij}(\alpha_j - r)}. \tag{25}$$

This means that the individual splits his portfolio between the risk-free asset, paying return r, and a portfolio of the risky assets that holds the m risky assets in constant proportions, given by (25). In general, only the current level of wealth, $W(t)$, and the investor's time horizon

[4] The solution could be obtained numercially, however.

determine how much is put in the first fund and how much is allocated to the second.

This implies that with a constant investment opportunity set, one can think of the investment decision as being just a two-asset decision, where the choice is between the risk-free asset paying rate of return r and a risky asset having expected rate of return α and variance σ^2 where

$$\begin{aligned}\alpha &\equiv \sum_{i=1}^{m} \delta_i \alpha_i \\ \sigma^2 &\equiv \sum_{i=1}^{m} \sum_{j=1}^{m} \delta_i \delta_j \sigma_{ij}.\end{aligned} \tag{26}$$

Let's now look at a special case of the above, that is, when utility is of the hyperbolic absolute risk aversion (HARA) class.

III.B. Lognormally-Distributed Asset Prices and HARA Utility

HARA utility functions are defined by

$$\begin{aligned} u(C, t) &= e^{-\rho t} v(C) \\ \text{where} \quad v(C) &= \tfrac{1-\gamma}{\gamma} \left(\tfrac{\beta C}{1-\gamma} + \eta \right)^{\gamma}. \end{aligned} \tag{27}$$

Special cases of HARA include power (constant relative risk-aversion) utility, exponential (constant absolute risk aversion), and quadratic utility. Robert C. Merton (1971) "Optimum Consumption and Portfolio Rules in a Continuous-Time Model" *Journal of Economic Theory* 3, p.373-413 finds explicit solutions for this class of utility functions.

For this HARA case, and assuming a zero bequest function, $B \equiv 0$, we have from equation (18) that optimal consumption is of the form

$$C^* = \frac{1-\gamma}{\beta} \left[\frac{e^{\rho t} J_W}{\beta} \right]^{\frac{1}{\gamma - 1}} - \frac{(1-\gamma)\eta}{\beta} \tag{28}$$

and using (22) and (24), the proportion put in the risky asset is

$$\omega^* = -\frac{J_W}{J_{WW} W} \frac{\alpha - r}{\sigma^2}. \tag{29}$$

This solution is incomplete since C^* and ω^* are in terms of J_W and J_{WW}. But we can solve for J in the following manner. Plug (28) and (29) into the optimality equation (17), or, alternatively, directly simplify equation (23) to obtain

$$0 \;=\; \frac{(1-\gamma)^2}{\gamma} e^{-\rho t} \left[\frac{e^{\rho t} J_W}{\beta}\right]^{\frac{\gamma}{\gamma-1}} \;+\; J_t \;+\; \left(\frac{(1-\gamma)\eta}{\beta} + rW\right) J_W \;-\; \frac{J_W^2}{J_{WW}} \frac{(\alpha - r)^2}{2\sigma^2}. \qquad (30)$$

This is the equilibrium PDE for J that can be solved subject to the boundary condition $J(W, T) = 0$, which is implied by the zero bequest. Note that this is a nonlinear PDE. They usually do not have an analytic solution but must be solved numerically. However, the above equation is of the Bernoulli-type and can be simplified by a change in variable $Y = J^{\frac{\gamma}{\gamma-1}}$.

Merton (1971) finds a solution for this equation, $J(W, t)^*$. It is given by equation (5.47) on page 139 of the reprint of his 1971 article in his 1990 book *Continuous-Time Finance*, Basil Blackwell, Oxford. Given this solution for J, we can then calculate J_W to solve for C^* (given by Merton's equation (5.48)) and then calculate J_{WW} to solve for ω^*. It is interesting to note that for this class of HARA utility, C^* is of the form

$$C(t)^* \;=\; aW(t) \;+\; b \qquad (31)$$

and

$$\omega^* \;=\; g \;+\; \frac{h}{W} \qquad (32)$$

where a, b, g, and h are, at most, functions of time. For the special case of constant relative risk aversion where $v(C) = \frac{C^\gamma}{\gamma}$, the explicit solution is[5]

$$C(t)^* \;=\; aW(t) \qquad (33)$$

and

$$\omega^* \;=\; \frac{\alpha - r}{(1-\gamma)\sigma^2} \qquad (34)$$

where $a \equiv \frac{\gamma}{\gamma-1}\left[\frac{\rho}{\gamma} - r - \frac{1}{2}\left(\frac{\alpha-r}{\sigma}\right)^2 \frac{1}{1-\gamma}\right]$. Consumption is a constant proportion of wealth. One also sees that as risk aversion increases (γ becomes more negative), the individual holds less

[5] See Ingersoll (1987) page 275.

wealth in the risky asset portfolio.