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Rational Speculative Asset Price Bubbles

Consider the following multi-period discrete-time model, which can be considered to be a special case of the Lucas (1978) model. It is assumed that investors are risk neutral and that there is a constant, positive one-period risk-free interest rate equal to r. There is also a risky asset with price p_t at date t. The risky asset is assumed to pay a dividend of d_t, which is declared at date t and paid at the end of the period.

Given these assumptions, the expected rate of return on the risky asset, which equals the dividend price ratio plus the expected capital gain, will equal the risk free rate.

$$\frac{E_t\left[p_{t+1}\right]-p_t}{p_t}+\frac{d_t}{p_t}=r \tag{1}$$

where $E_t\left[\cdot\right]=E\left[\cdot \mid I_t\right]$ is the expectation operator given information at date t, I_t. Re-arranging the equilibrium condition (1), we have:

$$p_t=aE_t\left[p_{t+1}\right]+ad_t \tag{2}$$

where $a\equiv\frac{1}{1+r}<1$. Thus, a is simply the one-period discount factor. The difference equation (2) can be solved by repeated substitution. Update (2) by one period and insert it in for p_{t+1} in the original equation (2).

$$p_t=aE_t\left[aE_{t+1}\left[p_{t+2}\right]+ad_{t+1}\right]+ad_t \tag{3}$$

$$=a^2E_t\left[p_{t+2}+d_{t+1}\right]+ad_t$$

since $E\left[E\left[x\mid I_{t+1}\right]\mid I_t\right]=E\left[x\mid I_t\right]$. Repeating this procedure results in

$$p_t=\sum_{i=0}^{T}a^{i+1}E_t\left[d_{t+i}\right]+a^{T+1}E_t\left[p_{t+T+1}\right] \tag{4}$$

For the first term to converge (that is, be finite), the expected path of dividends must grow at a rate less than r. Now if

$$\lim_{T \to \infty} a^{T+1} E_t \left[p_{t+T+1} \right] = 0 \tag{5}$$

Then we will obtain the solution:

$$p_t = \sum_{i=0}^{\infty} a^{i+1} E_t \left[d_{t+i} \right] \tag{6}$$

This is known as the "fundamental" solution. If we then specify a process for dividends, we can then solve explicitly for p_t. For example, suppose dividends followed the auto-regressive process

$$d_t - \bar{d} = \rho \left(d_{t-1} - \bar{d} \right) + e_t \tag{7}$$

where e_t is known at date t but $E_t \left[e_{t+1} \right] = 0$. In this case

$$E_t \left[d_{t+i} \right] = \bar{d} + \rho^i \left(d_t - \bar{d} \right) \tag{8}$$

Substituting (8) into (6) and assuming $\rho < 1 + r$, one obtains

$$p_t = \frac{a}{1-a} \bar{d} + \frac{a}{1 - a\rho} \left(d_t - \bar{d} \right) \tag{9}$$

Note that for the special case of $\rho = 0$ and $e_t = 0$, so that $d_t = \bar{d}$, we have

$$p_t = \frac{a}{1-a} \bar{d} \tag{10}$$

While the "fundamental" solution, equation (6), is one solution to the stochastic difference equation (4), it is not the only solution. We can have a "rational speculative bubble" solution as well. To see this, define the fundamental solution in equation (6) as p_t^*, that is, $p_t^* \equiv \sum_{i=0}^{\infty} a^{i+1} E_t \left[d_{t+i} \right]$. Next, consider another candidate solution of the form

$$p_t = p_t^* + b_t \tag{11}$$

What restrictions must we impose on b_t for equation (11) to be a solution to equation (4)? Note that $E_t \left[p_{t+1} \right] = E_t \left[p_{t+1}^* \right] + E_t \left[b_{t+1} \right]$. Thus, using equation (2) we have

$$p_t = p_t^* + b_t = aE_t\left[p_{t+1}^*\right] + aE_t\left[b_{t+1}\right] + ad_t \tag{12}$$

By the definition of p_t^* in (6), this implies

$$b_t = aE_t\left[b_{t+1}\right] \tag{13}$$

or

$$E_t\left[b_{t+1}\right] = a^{-1}b_t \tag{14}$$

Hence, for any stochastic process b_t that satisfies (14), p_t will be another solution to equation (4). Since $a < 1$, b_t explodes in expected value:

$$\lim_{i\to\infty} E_t\left[b_{t+i}\right] = \lim_{i\to\infty} a^{-i}b_t = \begin{cases} +\infty \text{ if } b_t > 0 \\ -\infty \text{ if } b_t < 0 \end{cases} \tag{15}$$

The exploding nature of b_t provides a rationale for viewing the solution (11) as a "bubble" solution. Only when $b_t = 0$, do we get the fundamental solution.

Suppose that b_t follows a time trend, that is,

$$b_t = b_0 a^{-t} \tag{16}$$

and suppose $d_t = \bar{d}$, so that the fundamental solution is a constant: $p_t^* = \frac{a}{1-a}\bar{d}$. In this case,

$$p_t = \frac{a}{1-a}\bar{d} + b_0 a^{-t} \tag{17}$$

implies that the risky asset price grows exponentially forever. In other words, we have an ever-expanding speculative bubble.

Next, consider a possibly more realistic modeling of b_t:

$$b_{t+1} = \begin{cases} (aq)^{-1} b_t + e_{t+1} & \text{with probability } q \\ e_{t+1} & \text{with probability } 1-q \end{cases} \tag{18}$$

with $E_t[e_{t+1}] = 0$. Note that this process satisfies the condition in (14), so that $p_t = p_t^* + b_t$ is again a valid bubble solution. In this case, the bubble continues with probability q each period but "bursts" with probability $1 - q$. If it bursts, it returns in expected value to zero. To compensate for the probability of a "crash", the expected return, conditional on not crashing, is higher that in the previous example of a never ending bubble. The disturbance e_t allows bubbles to have additional noise and allows new bubbles to begin after the previous bubble has crashed. This bursting bubble model can be generalized to allow q to be stochastic.