Finance 400

A. Penati - G. Pennacchi

Further Implications of the Basic Model of Intertemporal Consumption and Portfolio Choice

I. Testable Implications of the Model

Let us again assume that an individual's objective of expected utility maximization is of the form

$$\max E_0 \left\{ \sum_{t=0}^{T-1} U[C(t), t] + B[W(T), T] \right\} \tag{1}$$

where $E_0[\,\cdot\,]$ is the expectation operator at date 0. U and B are assumed to be increasing, concave functions. Also, as before, assume that the individual can choose between a single-period riskless asset and m risky assets, so that his wealth accumulation equation is given by

$$W(t+1) = \left[\sum_{i=1}^{m} \omega_i(t)[z_i(t) - R(t)] + R(t) \right] [W(t) + y(t) - C(t)]\,. \tag{2}$$

Defining the derived utility of wealth function, $J[W(t), t]$, as:

$$J[W(t), t] = \max\ E_t \left\{ \sum_{s=t}^{T-1} U[C(s), s] + B[W(T), T] \right\} \tag{3}$$

we showed that the individual's consumption and portfolio choice problem at date t could be written as

$$J[W(t), t] = \max_{\{C(t),\, \omega(t)\}} [\, U[C(t), t] + E_t \{\, J[W(t+1), t+1]\, \}\,]\,. \tag{4}$$

The recursive condition (4) is also known as the Bellman equation.

We then showed that by taking derivatives with respect to $C(t)$ and $\omega_i(t)$, we could obtain the first order conditions

$$\begin{aligned} U_C[C^*(t),\, t] &= R(t)\, E_t\, \{\, J_W[W(t+1),\, t+1]\,\} \,=\, J_W[W(t),\, t] \\ E_t\, \{\, z_i(t)\, J_W[W(t+1),\, t+1]\,\} &= R(t)\, E_t\, \{\, J_W[W(t+1),\, t+1]\,\}\,, \qquad i = 1, \ldots, m. \end{aligned} \tag{5}$$

We also saw that by making specific assumptions regarding the form of the utility function (for example, logarithmic or power), the absence of wage income, and the constancy of the investment opportunity set, one can sometimes derive explicit solutions for $C^*(t)$ and ω_i^* using backward dynamic programing. However, the above model has empirically testable implications even for more general cases where the derivation of explicit consumption levels and portfolio proportions are not possible.

To see this, recall that first condition of (5), $U_C[C^*(t)] = J_W[W(t),\, t]$, also known as the envelope condition, conveys that under an optimal policy, the marginal value of financial wealth equals the marginal utility of consumption. Note that to slightly simplify notation, for $U_C[C^*(t),\, t]$ we write "$U_C[C^*(t)]$". Since (5) holds at any date, we can substitute $U_C[C^*(t{+}1)] = J_W[W(t+1),\, t+1]$ in the first equation of (5) to obtain

$$\begin{aligned} U_C[C^*(t)] &= R(t)\, E_t\, \{\, U_C[C^*(t+1)]\,\}\,, \quad \text{or} \\ 1 &= R(t)\, E_t\, \{\, m_{t,\,t+1}\,\} \end{aligned} \tag{6}$$

where $m_{t,\,t+1} \equiv U_C[C^*(t+1)]\,/\,U_C[C^*(t)]$ is the stochastic discount factor or pricing kernel, which in this context equals the marginal rate of substitution. Combining the first order conditions in (5) and making the same substitution also results in

$$\begin{aligned} U_C[C^*(t)] &= E_t\, \{\, z_i(t)\, U_C[C^*(t+1)]\,\}\,, \quad i = 1, \ldots, m, \qquad \text{or} \\ 1 &= E_t\, \{\, z_i\, m_{t,\,t+1}\,\} \end{aligned} \tag{7}$$

Recall that we had previously derived equations (6) and (7) in the context of a one-period consumption-portfolio choice problem. Here we see that the same relationships holds for the multiperiod case. These conditions provide us with the further economic intuition that under an optimal policy, the marginal rate of substitution between consumption at any two dates, such as t and $t+1$, equals the marginal rate of transformation. Consumption at date t can be "transformed" into consumption at date $t+1$ by investing in the riskless asset at rate $R(t)$ or investing in the risky asset having stochastic return $z_i(t)$. Note that equations of the form of (6) and (7) hold between any two dates since using (7)

$$\begin{aligned} U_C[C^*(t)] &= E_t\{\,z_i(t)\,E_{t+1}\{z_j(t+1)\,U_C[C^*(t+2)]\,\}\,\} \\ &= E_t\{\,E_{t+1}\{\,z_i(t)\,z_j(t+1)\,U_C[C^*(t+2)]\,\}\,\} \\ &= E_t\{\,z_i(t)\,z_j(t+1)\,U_C[C^*(t+2)]\,\} \end{aligned} \tag{8}$$

$$1 = E_t\{\,z_i\,(t)\,z_j\,(t+1)\;m_{t,\,t+2}\,\}$$

where we note that $z_i(t)z_j(t+1)$ is the return on an investment that holds risky asset i over the period from t to $t+1$ then risky asset j over the period $t+1$ to $t+2$. Of course, i could equal j but need not, in general.

Equations like (6), (7), or (8) have formed the basis of numerous empirical tests, using both individual consumption data (Stephen Zeldes (1989) "Consumption and Liquidity Constraints: An Empirical Investigation" *Journal of Political Economy* 97, p.305-346) as well as aggregate data (Robert Hall (1978) "Stochastic Implications of the Life Cycle Permanent Income Hypothesis: Theory and Evidence," *Journal of Political Economy* 86, p.971-87). Note that the use of aggregate consumption data requires, at a minimum, the additional assumption that all individuals have the same utility function.

To illustrate how equations (6)-(8) can be used in empirical work, suppose that utility is assumed to be of the constant relative risk aversion (power) type so that $U[C(t),\,t] = \frac{\delta^t C(t)^{1+\gamma}}{1+\gamma}$, $-\infty < \gamma < 0$. Then $U_C[C(t),\,t] = \delta^t C(t)^\gamma$ and plugging this in to equations (6) and (7) gives us

$$1 = \delta\,R(t)\,E_t\left\{\left[\frac{C(t+1)}{C(t)}\right]^\gamma\right\}, \tag{9}$$

and

$$1 = \delta\, E_t \left\{ z_i(t) \left[\frac{C(t+1)}{C(t)} \right]^{\gamma} \right\}, \quad i = 1, \ldots, m. \tag{10}$$

Equations (9) and (10) represent $n = m + 1$ moment conditions. A number of empirical papers have estimated the parameters γ and δ and tested the cross-equation restrictions using the Generalized Method of Moments procedure (Lars P. Hansen (1982) "Large Sample Properties of Generalized Method of Moments Estimators," *Econometrica* 50, p.1029-54). Assuming power utility, researchers have generally found rather large magnitude values of γ, such as $\gamma = -20$. (Aggregate consumption does not vary much relative to asset returns, resulting in high estimates of risk aversion.) Also, tests of cross-equation restrictions are often rejected. This has led to researchers making alternative assumptions regarding the form of utility, including a general time-separable form (Lars Peter Hansen and Ravi Jagannathan (1991) "Implications of Security Market Data for Models of Dynamic Economies," *Journal of Political Economy* 99, p.225-262) and non-time separable forms.

II. The Consumption CAPM

Notice that we can combine the first-order conditions (6) and (7) to write

$$\begin{gathered} 0 = E_t\left\{ U_C[C^*(t+1)] \cdot (z_i(t) - R(t)) \right\}, \qquad i = 1, \ldots, m \quad \text{or} \\ 0 = E_t\left\{ U_C[C^*(t+1)] \right\} \cdot E_t\left\{ z_i(t) - R(t) \right\} + \mathrm{Cov}_t\left\{ U_C[C^*(t+1)],\, z_i(t) - R(t) \right\}. \end{gathered} \tag{11}$$

Rearranging results in

$$E_t\left\{ z_i(t) \right\} = R(t) - \frac{\mathrm{Cov}_t\left\{ U_C[C^*(t+1)],\, z_i(t) \right\}}{E_t\left\{ U_C[C^*(t+1)] \right\}}. \tag{12}$$

Now, remember that the first order conditions, represented by (12) hold on an individual-by-individual basis. In other words, individuals who may differ by the form of their utility function, wealth, and wage generating process choose their individual portfolios so that each person sets his marginal utility of consumption to satisfy (12).

However, if we assume a representative individual (or a form of utility, wages, etc. that allows for aggregation), then the utility of consumption on the right-hand side of (12) can be viewed as the utility of aggregate consumption. In this case, equation (12) can be interpreted as a model of the process for asset returns, that is, a capital asset pricing model. In this light, equation (12) says that assets whose rates of return have a higher covariance with the marginal utility of consumption have, in equilibrium, a lower expected return. Since marginal utility is high in states when consumption is low, equation (12) implies that consumers accept a lower expected return on assets that have a relatively higher return in states when consumption is low. The intuition is that these assets provide "insurance" in low consumption states. For most risky assets, one would expect that $\mathrm{Cov}_t\,\{U_C[C^*],\, z_i\} < 0$, that is, asset returns are usually high when aggregate consumption is high, so that most risky assets will have an expected rate of return that exceeds the risk-free rate. See Lars P. Hansen and Kenneth J. Singleton (1983) "Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns," *Journal of Political Economy* 91, p.249-265 for an empirical test of this type.

We can see how the above "consumption" capital asset pricing model is related to the standard Sharpe-Lintner-Mossin CAPM. Let us define the return from date t to $t+1$ on the (value-weighted) portfolio of all risky assets, that is, the "market" portfolio, as $z_M(t)$. Under the assumption that the return on this market portfolio is *perfectly negatively correlated* with the marginal utility of consumption we have

$$\mathrm{Cov}_t\,\{\,U_C[C^*(t+1)],\, z_i(t)\,\} \;=\; \kappa(t)\,\mathrm{Cov}_t\,\{z_M(t),\, z_i(t)\}\,, \quad i = 1, \dots, m, \tag{13}$$

where $\kappa(t) \;\equiv\; -\left[\,\mathrm{Var}_t\,\{\,U_C\,[C^*(t+1)]\,\}\;/\;\mathrm{Var}_t\,\{z_M(t)\}\,\right]^{\frac{1}{2}}$. Applying equation (12) to the market portfolio gives

$$\begin{aligned} E_t\,\{z_M(t)\} \quad &= \quad R(t) \;-\; \frac{\kappa(t)\,\mathrm{Var}_t\,\{z_M(t)\}}{E_t\,\{\,U_C[C^*(t+1)]\,\}}, \qquad \text{or} \\ \frac{E_t\,\{z_M(t)\} \,-\, R(t)}{\kappa(t)\,\mathrm{Var}_t\,\{z_M(t)\}} \quad &= \quad -\,\frac{1}{E_t\,\{\,U_C[C^*(t+1)]\,\}}. \end{aligned} \tag{14}$$

Substituting (14) into (12) gives

$$E_t\{z_i(t)\} - R(t) = \frac{\text{Cov}_t\{U_C[C^*(t+1)],\, z_i(t)\}}{\kappa(t)\,\text{Var}_t\{z_M(t)\}}\,[\,E_t\{z_M(t)\} - R(t)\,]\,. \tag{15}$$

Finally, substituting (13) into (15) gives

$$\begin{aligned} E_t\{z_i(t)\} - R(t) &= \frac{\text{Cov}_t\{z_i(t),\, z_M(t)\}}{\text{Var}_t\{z_M(t)\}}\,[\,E_t\{z_M(t)\} - R(t)\,] \\ &= \beta(t)\,[\,E_t\{z_M(t)\} - R(t)\,] \end{aligned} \tag{16}$$

which is the standard CAPM. Note that to derive (16) we did not need to assume any particular probability distribution for asset returns, but we did need to assume perfectly negative correlation between the marginal utility of consumption and the market portfolio. If utility was quadratic, so that marginal utility was linear in consumption, then this assumption implies that consumption would be perfectly correlated with the return on the market portfolio.

III. The Lucas Model of Asset Pricing

The Lucas model (Robert E. Lucas (1978) "Asset Prices in an Exchange Economy," *Econometrica* 46, p.1429-1445) derives the equilibrium prices of risky assets for the case of an endowment economy. In describing the economy as an *endowment* economy, we mean that real output for the economy is exogenous, and output cannot be reinvested to produce more output in the future. Thus, all output is immediately consumed, implying that equilibrium aggregate consumption equals the exogenous level of output each period. In contrast, a *production* economy allows for an aggregate consumption - savings (investment) decision. Not all of current output need be consumed, but some can be physically invested (saved) to produce more output in the future. John C. Cox, Jonathan E. Ingersoll, and Stephen A. Ross (1985) "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica* 53, p.363-384 derived a continuous-time general equilibrium asset pricing model of this type.

The Lucas model is attractive in that much of its derivation does not depend on the assumption of exogenous production. To see this, we start with equation (7) again but now let us assume that the return on the i^{th} risky asset, $z_i(t)$, includes a dividend payment made at date

$t+1$, $d_i(t+1)$, along with its capital gain, $P_i(t+1)-P_i(t)$. Hence $P_i(t)$ denotes the ex-dividend price of the risky asset at date t.

$$z_i(t) \;=\; \frac{d_i(t+1) \;+\; P_i(t+1)}{P_i(t)}. \tag{17}$$

Substituting (17) into (7) and rearranging gives

$$P_i(t) \;=\; E_t\left\{ \frac{U_C[C^*(t+1)]}{U_C[C^*(t)]}\,\left[d_i(t+1) \;+\; P_i(t+1)\right]\right\}. \tag{18}$$

Similar to what was done in equation (8), if we substitute for $P_i(t+1)$ using equation (18) updated one period, and use the properties of conditional expectation, we have

$$\begin{aligned} P_i(t) \;&=\; E_t\left\{ \tfrac{U_C[C^*(t+1)]}{U_C[C^*(t)]}\,\left[d_i(t+1) \;+\; \tfrac{U_C[C^*(t+2)]}{U_C[C^*(t+1)]}\,(d_i(t+2) \;+\; P_i(t+2))\right]\right\} \\ &=\; E_t\left\{ \tfrac{U_C[C^*(t+1)]}{U_C[C^*(t)]}\,d_i(t+1) \;+\; \tfrac{U_C[C^*(t+2)]}{U_C[C^*(t)]}\,(d_i(t+2) \;+\; P_i(t+2))\right\}. \end{aligned} \tag{19}$$

Repeatedly making this kind of substitution, that is, solving forward the difference equation (18) gives us

$$P_i(t) \;=\; E_t\left\{ \sum_{j=1}^{N} \frac{U_C[C^*(t+j)]}{U_C[C^*(t)]}\,d_i(t+j) \;+\; \frac{U_C[C^*(t+N)]}{U_C[C^*(t)]}\,P_i(t+N)\right\}. \tag{20}$$

Now suppose utility reflects some rate of time preference, so that $U[C(t)] \;\equiv\; \delta^t u[C(t)]$, where $\delta < 1$. Then (20) looks like

$$P_i(t) \;=\; E_t\left\{ \sum_{j=1}^{N} \delta^j\,\frac{u_C[C^*(t+j)]}{u_C[C^*(t)]}\,d_i(t+j) \;+\; \delta^N\,\frac{u_C[C^*(t+N)]}{u_C[C^*(t)]}\,P_i(t+N)\right\}. \tag{21}$$

Assuming that we have an infinitely lived individual and an infinitely lived asset, then as long as $\lim_{N\to\infty} \delta^N E_t \left\{ \frac{u_C[C^*(t+N)]}{u_C[C^*(t)]} P_i(t+N) \right\} \to 0$, we have no speculative price "bubbles" and

$$\begin{aligned} P_i(t) &= E_t \left\{ \sum_{j=1}^{\infty} \delta^j \frac{u_C[C^*(t+j)]}{u_C[C^*(t)]} d_i(t+j) \right\} \\ &= E_t \left\{ \sum_{j=1}^{\infty} m_{t,\,t+j}\, d_i(t+j) \right\}. \end{aligned} \tag{22}$$

Equation (22) looks like (and is) a present value formula, where the stochastic discount factors are the marginal rates of substitution between the present and when the dividends are paid. Note that (22) holds for any individual following an optimal consumption - portfolio choice policy. We have not yet made any strong assumption about consumer homogeneity or the structure of the economy. For example, (22) should hold for a production economy with heterogeneous individuals.

To make (22) into a (quasi-) general equilibrium model of asset pricing, Lucas assumes that there is a *representative* individual (all individuals are identical) and that each risky asset represents a claim on a real output process, where risky asset i pays a real dividend of $d_i(t)$ at date t. One can think of a share of risky asset i as an ownership stake in a particular tree that produces a stochastic stream of perishable, seedless fruit. Thus, equation (22) holds for aggregate consumption which is given by

$$C^*(t) = \sum_{i=1}^{n} d_i(t). \tag{23}$$

With these assumptions about individual preferences and the structure of the economy, whatever one assumes about the production (equivalently, dividend) process for a firm, industry, or the entire economy determines the value (price) of the security representing an ownership claim on this process. For example, if the representative individual is risk-neutral, so that u_C is a constant, then (22) becomes

$$P_i(t) = E_t \left\{ \sum_{j=1}^{\infty} \delta^j\, d_i(t+j) \right\}. \tag{24}$$

In words, the price of risky asset i is the expected value of dividends discounted at a constant rate, the rate of time preference.

Next, consider the value of the market portfolio (ownership claim on all assets in the economy). Let $P(t)$ represent the value of this portfolio which pays the dividend $d(t) \equiv \sum_{i=1}^{n} d_i(t)$. If utility is assumed to be logarithmic, $u[C(t)] = \ln[C(t)]$, then recalling that $C^*(t) = d(t)$ we obtain

$$\begin{aligned} P(t) &= E_t \left\{ \sum_{j=1}^{\infty} \delta^j \, \frac{d(t)}{d(t+j)} d(t+j) \right\} \\ &= E_t \left\{ \sum_{j=1}^{\infty} \delta^j \, d(t) \right\} \\ &= \frac{\delta}{1-\delta} \, d(t) \end{aligned} \tag{25}$$

implying that the value of the market portfolio moves in step with the current level of dividends. It does not depend on expected future dividends. Why? Higher expected future dividends increase expected $d(t+j)$ but, since dividends equal consumption, the marginal utility of future consumption, and hence $m_{t,t+j}$, falls. For logarithmic utility, the effects offset each other leaving the value of a claim on this output process unchanged. Note that this will not be the case for the more general specification of power utility (constant relative risk aversion). If $u[C(t)] = C(t)^{1+\gamma}$, then

$$\begin{aligned} P(t) &= E_t \left\{ \sum_{j=1}^{\infty} \delta^j \, \left[\frac{d(t)}{d(t+j)} \right]^{\gamma} d(t+j) \right\} \\ &= E_t \left\{ \sum_{j=1}^{\infty} \delta^j \, \frac{C(t+j)^{\gamma+1}}{C(t)^{\gamma}} \right\}. \end{aligned} \tag{26}$$

Note that a hypothetical riskless asset that paid a one-period dividend of \$1 would have a value given by

$$P_R(t) = \delta \, E_t \left\{ \left[\frac{C(t+1)}{C(t)} \right]^{\gamma} \right\}. \tag{27}$$

By estimating the process followed by actual U.S. aggregate consumption, Rajnish Mehra and

Edward C. Prescott (1985) "The Equity Premium: A Puzzle," *Journal of Monetary Economics* 15, p.145-161 used equations such as (26) and (27) to estimate the value of γ that would produce a risk-premium (excess average return over a risk-free asset return) on the market portfolio of U.S. stocks. They found that for reasonable values of γ, they could not come close to the historical risk-premium of 6 percent. Again, the problem is that aggregate consumption appears to vary little in relation to the variance of the market portfolio, implying that one needs a high value of risk aversion to match equations like (26) and (27).