Finance 400

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Intertemporal Consumption and Portfolio Choice: The Dynamic Programming Approach

I. Introduction

These notes consider an expected utility maximizing individual's consumption and portfolio choices over many periods. Previously, a similar analysis was presented in a single period or static context. Here, the intertemporal nature of the problem is explicitly modeled. The approach follows the theory developed in the late 1960s and early 1970s by Paul A. Samuelson and Robert C. Merton, among others. The solution technique involves dynamic programming. While this dynamic programming technique is not the only approach to solving problems of this type, it can sometimes be the most convenient and intuitive way of deriving solutions.[1]

When we distinguish between an investor's *decision or trading period* (the time between portfolio rebalancing or consumption choices) versus the investor's *planning horizon* (for example, a remaining lifespan), we explicitly model the individual's multiple consumption and portfolio decisions over a single planning period. This cannot be done in a single-period framework: with only one period an investor's decision period and planning horizon are the same. However, results from a single-period analysis are often be useful because sometimes multi-period problems can be transformed into single-period ones, as will be illustrated below.

Consider an environment in which an individual chooses his level of consumption and the proportions of his wealth invested n different assets, where he takes the stochastic processes followed by the n different assets as given. This implies that security markets are perfectly competitive in the sense that the (small) individual is a price-taker in security markets. An individual's trades do not impact the price of the security. For most investors trading in liquid security markets, this is a reasonably realistic assumption. In addition, we assume that there are no transactions costs or taxes when buying or selling assets, so that security markets are "friction-less."

[1] J. Cox and C.F. Huang (1989) "Optimum Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process," *Journal of Economic Theory* 49, p.33-83, provide an alternative martingale approach to solving consumption and portfolio choice problems.

Solving the individual consumer's consumption and portfolio choice problem is of interest in that it provides a theory of optimal investment strategies. However, the solution to this problem is also an important part of any general equilibrium theory of capital asset pricing. Combining this model of consumer preferences over consumption and securities with a model of firm production technologies can then lead to a general equilibrium model of the economy that determines equilibrium asset price processes, as is done in J. Cox, J. Ingersoll, and S. Ross (1985) "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica* 53, p.363-384.

In these notes, we first solve the individual's consumption-portfolio choice problem assuming that the individual's decision interval is a discrete-time period. Later, we assume that this interval is instantaneous, that is, the individual can make consumption and portfolio choices continuously.[2] This latter assumption often simplifies problems and can lead to sharper results. When we move from discrete-time to continuous-time, continuous-time stochastic processes will be used to model security prices.

II. Assumptions and Notation of the Discrete-Time Model

The following analysis is based on the unpublished class notes on *Portfolio Theory and Capital Markets* by Robert C. Merton.

A.1 *Preferences*:

The individual maximizes his expected utility of consumption over a T period planning horizon. Utility depends on each period's consumption level, $C(0), C(1), ..., C(T-1)$ as well as a terminal bequest, $W(T)$. We assume that utility is given by the additively-separable function

$$\hat{U}\left[C(0), C(1), ..., C(T-1), W(T)\right] \equiv \sum_{t=0}^{T-1} U(C(t), t) + B(W(T), T)$$

The individual's objective of expected utility maximization is then

$$\max E_0 \left[\sum_{t=0}^{T-1} U(C(t), t) + B(W(T), T) \right] \tag{1}$$

[2]The discrete-time presentation is covered in Chapter 11 while the continuous-time material is covered in Chapter 13 of J. Ingersoll (1987) *Theory of Financial Decision Making* Rowman and Littlefield, Totowa, NJ.

where $E_0[\cdot]$ is the expectation operator at date 0. U and B are assumed to be increasing, concave functions.

A.2 *Budget and Accumulation Equation*:

At date t, the individual has tangible wealth equal to $W(t)$ and knows the prices of n different securities, $P_i(t), i = 1, ..., n$. He also receives $y(t)$ in wage income for the period and can use this as well as the proceeds from any security sales made at current prices to consume at level $C(t)$. Hence, the amount he has left over to invest among the n securities is

$$I(t) \equiv W(t) + y(t) - C(t) = \sum_{i=1}^{n} N_i(t) P_i(t) \tag{2}$$

where $N_i(t)$ is the number of shares owned of security i having price $P_i(t)$ per share.

Having made these investments, the individual's tangible wealth at the start of the next period is then

$$W(t+1) = \sum_{i=1}^{n} N_i(t) P_i(t+1) \tag{3}$$

Hence, using (2) and (3), the change in wealth, $W(t+1) - W(t)$, can be written as

$$\begin{aligned} W(t+1) - W(t) &= \sum_{i=1}^{n} N_i(t) P_i(t+1) + y(t) - C(t) - \sum_{i=1}^{n} N_i(t) P_i(t) \\ &= \sum_{i=1}^{n} N_i(t) [P_i(t+1) - P_i(t)] + y(t) - C(t) \end{aligned} \tag{4}$$

which says that the change in tangible wealth comes from capital gains plus net savings out of wage income.

It is often more convenient to make portfolio proportions, ω_i, rather than numbers of shares, to be the individual's decision variables:

$$\omega_i \equiv \frac{N_i(t) P_i(t)}{I(t)}, \quad i = 1, 2, ..., n \tag{5}$$

where $\sum_{i=1}^{n} \omega_i(t) = 1$. Using (5) to substitute for $N_i(t)$ in equation (4), we have

$$W(t+1) - W(t) = \sum_{i=1}^{n} \frac{\omega_i(t) I(t)}{P_i(t)} [P_i(t+1) - P_i(t)] + y(t) - C(t) \tag{6}$$
$$= I(t) \sum_{i=1}^{n} \omega_i(t) \frac{P_i(t+1)}{P_i(t)} - I(t) + y(t) - C(t)$$

and since from (2) $W(t) = I(t) - y(t) + C(t)$, re-arranging equation (6) gives us

$$W(t+1) = I(t) \sum_{i=1}^{n} \omega_i(t) \frac{P_i(t+1)}{P_i(t)} = [W(t) + y(t) - C(t)] \sum_{i=1}^{n} \omega_i(t) \frac{P_i(t+1)}{P_i(t)} \tag{7}$$

It will also be useful to define $z_i(t) \equiv P_i(t+1)/P_i(t)$ as the return on the i^{th} security over the t^{th} period and $Z(t) \equiv W(t+1)/I(t) = \sum_{i=1}^{n} \omega_i(t) z_i(t)$ as the return on the whole portfolio. Now if we assume that the n^{th} asset is a one-period risk-less security and has a *single-period* riskless return of $R(t)$, then

$$Z(t) = \sum_{i=1}^{m} \omega_i(t) [z_i(t) - R(t)] + R(t) \tag{8}$$

where $m \equiv n - 1$. Using these definitions, equation (7) can be re-written as

$$W(t+1) = Z(t) [W(t) + y(t) - C(t)] \tag{9}$$

III. The Derived Utility of Wealth Function

The derived utility of wealth function, $J(W(t), t)$ is defined as:

$$J[W(t), t] = \max E_t \left\{ \sum_{s=t}^{T-1} U[C(s), s] + B[W(T), T] \right\} \tag{10}$$

where "max" means to choose the decision variables $C(s)$ and $\{\omega_i(s)\}$ for $s = t, t+1, ..., T-1$ and $i = 1, ..., m$ so as to maximize the expected value of the term in brackets. Note that J is a function of current wealth and all information up until and including date t, but not of the current or future decision variables since they are assumed to be set to those values that maximize lifetime expected utility. Hence, J can be described as a "derived" utility of wealth function.

IV. The Solution Technique: Stochastic Dynamic Programming

Note from the definition of J that

$$J\left[W\left(T\right),T\right]=B\left[W\left(T\right),T\right] \tag{11}$$

Now working backwards, consider the individual's optimization problem when he has a single period left in his life at date $T-1$.

$$J\left[W\left(T-1\right),T-1\right]=\max_{C(T-1),\{\omega_i(T-1)\}}\left\{U\left[C\left(T-1\right),T-1\right]+E_{T-1}B\left[W\left(T\right),T\right]\right\} \tag{12}$$

To clarify how $W\left(T\right)$ depends explicitly on $C\left(T-1\right)$ and $\{\omega_i\left(T-1\right)\}$, use equations (8) and (9) to substitute for $W\left(T\right)$ in equation (12)

$$J\left[W\left(T-1\right),T-1\right]=\max_{C(T-1),\{\omega_i(T-1)\}} \tag{13}$$

$$\left\{U\left[C\left(T-1\right),T-1\right]+E_{T-1}B\left[\left(\sum_{i=1}^{m}\omega_i\left[z_i-R\right]+R\right)\left(W\left(T-1\right)+y\left(T-1\right)-C\left(T-1\right)\right),T\right]\right\}$$

Except for the dependence on consumption, $C\left(T-1\right)$, equation (13) represents a standard single-period portfolio selection problem. To solve (13), we differentiate with respect to each decision variable and set the resulting expressions to zero:

$$\frac{\partial}{\partial C}:0=U_C\left[C^*,T-1\right]-E_{T-1}\left\{B_W\left(W\left(T\right),T\right)\left(\sum_{i=1}^{m}\omega_i\left[z_i-R\right]+R\right)\right\} \tag{14a}$$

$$\frac{\partial}{\partial \omega_i}:0=E_{T-1}\left\{B_W\left(W\left(T\right),T\right)\left[z_i-R\right]\right\},\ \text{for } i=1,...,m \tag{14b}$$

where the subscripts on U and B denote partial differentiation. Using (14b), we see that (14a) can be re-written as

$$0 = U_C\left[C^*, T-1\right] - E_{T-1}\left\{B_W\left(W\left(T\right), T\right)R\right\} \tag{15}$$

Thus, we can re-write the first-order optimality conditions as

$$U_C\left[C^*, T-1\right] = RE_{T-1}\left\{B_W\left(W\left(T\right), T\right)\right\} \tag{16a}$$

and

$$\begin{aligned} E_{T-1}\left\{z_i B_W\left(W\left(T\right), T\right)\right\} &= E_{T-1}\left\{z_j B_W\left(W\left(T\right), T\right)\right\} \\ &= RE_{T-1}\left\{B_W\left(W\left(T\right), T\right)\right\}, \text{ for } i, j = 1, ..., m \end{aligned} \tag{16b}$$

Equations (16a) and (16b) represent $m+1$ equations that determine the optimal choices of C^* and $\{\omega_i^*\}$. In particular, equations (16b) should look familiar as they are identical to the first-order "Euler" conditions for a single-period portfolio maximizer (but with "B" replacing "U"). If we substitute these optimal decision variables back into equation (13) we have:

$$J\left[W\left(T-1\right), T-1\right] = U\left[C^*\left(T-1\right), T-1\right] \tag{17}$$

$$+E_{T-1}B\left(\left[\sum_{i=1}^{m}\omega_i^*\left[z_i - R\right] + R\right]\left[W\left(T-1\right) + y\left(T-1\right) - C^*\left(T-1\right)\right], T\right)$$

Now if we differentiate (17) totally with respect to $W\left(T-1\right)$, we have:

$$\begin{aligned} J_W &= U_C\frac{\partial C^*}{\partial W\left(T-1\right)} + E_{T-1}\left\{\left[\frac{\partial W\left(T\right)}{\partial W\left(T-1\right)} + \sum_{i=1}^{m}\frac{\partial W\left(T\right)}{\partial \omega_i^*}\frac{\partial \omega_i^*}{\partial W\left(T-1\right)} + \frac{\partial W\left(T\right)}{\partial C^*}\frac{\partial C^*}{\partial W\left(T-1\right)}\right]B_W\right\} \\ &= U_C\frac{\partial C^*}{\partial W} + \end{aligned} \tag{18}$$

$$+E_{T-1}\left\{\left[\sum_{i=1}^{m}\left[z_i - R\right]\left(W + y - C^*\right)\frac{\partial \omega_i^*}{\partial W} + \left(\sum_{i=1}^{m}\omega_i^*\left[z_i - R\right] + R\right)\left(1 - \frac{\partial C^*}{\partial W}\right)\right]B_W\right\}$$

Using the equalities in (16), we see that (18) simplifies to $J_W = RE_{T-1}\{B_W\}$. Thus using (16a) we finally obtain

$$J_W\left[W\left(T-1\right), T-1\right] = U_C\left[C^*\left(T-1\right), T-1\right] \tag{19}$$

which is referred to as the "envelope condition." It says that the individual's optimal policy equates his marginal utility of current consumption, U_C, to his marginal utility of wealth (future consumption).

Having solved the individual's problem with one period to go in his planning horizon, we next consider his optimal consumption and portfolio choices with two periods to go, at date $T-2$. The individual's objective at this date is

$$\begin{aligned} J\left[W\left(T-2\right), T-2\right] &= \max E_{T-2}\left\{U\left[C\left(T-2\right), T-2\right] + U\left[C\left(T-1\right), T-1\right] + B\left[W\left(T\right), T\right]\right\} \\ &= \max\left\{U\left[C\left(T-2\right), T-2\right] + E_{T-2}\left(U\left[C\left(T-1\right), T-1\right] + B\left[W\left(T\right), T\right]\right)\right\} \end{aligned} \tag{20}$$

because at date $T-2$, $W\left(T-2\right)$ is known and, therefore $C\left(T-2\right)$ can be chosen with certainty. The individual must maximize the above expression by choosing $C\left(T-2\right)$ as well as $\{\omega_i\left(T-2\right)\}$. However, note that he wishes to maximize an expression that is an expectation over quantities $U\left[C\left(T-1\right), T-1\right] + B\left[W\left(T\right), T\right]$ that depend on future decisions, that is, $C\left(T-1\right)$ and $\{\omega_i\left(T-1\right)\}$. What should one assume that these future values of $C\left(T-1\right)$ and $\{\omega_i\left(T-1\right)\}$ will be? The answer comes from the *Principle of Optimality*. It states:

> "An optimal set of decisions has the property that given an initial decision, the remaining decisions must be optimal with respect to the outcome that results from the initial decision."

The "max" in (20) is over all remaining decisions, but the principle of optimallity says that whatever decision is made in period $T-2$, given the outcome, the remaining decisions (for period $T-1$) must be optimal (maximal). In other words

$$\max_{\{(T-2),(T-1)\}}(Y) = \max_{\{T-2\}}\left[\max_{\{T-1,|\text{ outcome from }(T-2)\}}(Y)\right]$$

This implies that (20) can be re-written as

$$J\left[W\left(T-2\right),T-2\right] = \tag{21}$$

$$\max_{\{C(T-2),\omega(T-2)\}}\left\{U\left[C\left(T-2\right),T-2\right]+E_{T-2}\left(\max_{\{C(T-1),\omega(T-1)\}}U\left[C\left(T-1\right),T-1\right]+B\left[W\left(T\right),T\right]\right)\right\}$$

Now note that if X is any stochastic process, from the definition of conditional expectation, we have $E_{T-2}\left[E_{T-1}\left(X\right)\right] = E_{T-2}\left[X\right]$. Thus, (21) can be re-written as

$$J\left[W\left(T-2\right),T-2\right] = \max_{\{C(T-2),\omega(T-2)\}} \tag{22}$$

$$\left\{U\left[C\left(T-2\right),T-2\right]+E_{T-2}\left(\max_{\{C(T-1),\omega(T-1)\}}E_{T-1}\left(U\left[C\left(T-1\right),T-1\right]+B\left[W\left(T\right),T\right]\right)\right)\right\}$$

and then using the definition of J, results in

$$J\left[W\left(T-2\right),T-2\right] = \max_{\{C(T-2),\omega(T-2)\}}\left\{U\left[C\left(T-2\right),T-2\right]+E_{T-2}\left(J\left[W\left(T-1\right),T-1\right]\right)\right\} \tag{23}$$

Now what is interesting about the form of equation (23), the individual's problem at time $T-2$, is that if we compare it to equation (12), the individual's problem at time $T-1$, the two problems are quite similar. The only difference is that in (23) we replace the known function of wealth next period, B, with another (known in principle) function of wealth next period, J. But the solution to (23) will be of the same form as that for equation (12). Thus, the optimality conditions for (23) are:

$$U_C\left[C^*\left(T-2\right),T-2\right]=RE_{T-2}\left\{J_W\left[W\left(T-1\right),T-1\right]\right\}=J_W\left[W\left(T-2\right),T-2\right] \tag{24a}$$

$$E_{T-2}\left\{J_W\left[W\left(T-1\right),T-1\right]z_i\right\}=RE_{T-2}\left\{J_W\left[W\left(T-1\right),T-1\right]\right\},\text{ for } i=1,...,m \tag{24b}$$

Based on the above pattern, inductive reasoning implies that for any $t=0,1,...,T-1$, we have

$$J\left[W\left(t\right),t\right]=\max_{\{C(t),\omega(t)\}}U\left[C\left(t\right),t\right]+E_t\left\{J\left[W\left(t+1\right),t+1\right]\right\} \tag{25}$$

and, therefore, the date t optimality conditions are

$$U_C\left[C^*\left(t\right),t\right]=R\left(t\right)E_t\left\{J_W\left[W\left(t+1\right),t+1\right]\right\}=J_W\left[W\left(t\right),t\right] \tag{26a}$$

$$E_t\left\{z_i\left(t\right)J_W\left[W\left(t+1\right),t+1\right]\right\}=R\left(t\right)E_t\left\{J_W\left[W\left(t+1\right),t+1\right]\right\} \tag{26b}$$

Now (26a) says that the optimal policy is to choose today's consumption such that the marginal utility of current consumption equals the derived marginal utility of wealth (future consumption). In addition, (26b) says that portfolio weights are chosen in a similar manner to the single-period utility of wealth maximization problem, except that the derived utility of wealth function, $J\left(W\right)$, replaces the single-period problem's end of period utility of wealth function $U\left(W\right)$. For example, see the notes "State Preference Theory" for the solution to the single period portfolio problem.

While (26a) and (26b) are fairly simple expressions, we must remember that their dependence on $J\left(W\right)$, the *derived* utility of wealth (future consumption) function, implies that, in general, they depend on future contingent investment opportunities (the distributions of future asset returns $\left(z_i\left(t+j\right),R\left(t+j\right)\right)$, future income flows, $y\left(t\right)$, and states of the world that may affect future utilities $\left(U\left(\cdot,t+j\right)\right)$.

Solving the above system involves starting from the end of the planning horizon and dynamically programing backwards toward the present. Thus, for the last period, T, we know that $J[W(T), T] = B[W(T), T]$. As we did previously, we substitute $B[W(T), T]$ for $J[W(T), T]$ in equations (25) and (26) for date $T-1$ and solve for $J[W(T-1), T-1]$. This is then substituted into equations (25) and (26) for date $T-2$ and one then solves for $J[W(T-2), T-2]$. If we proceed using this "bootstrap" technique, we eventually obtain $J[W(0), 0]$ and the solution is complete. The optimal policy will be of the form:

$$C^*(t) = g[W(t), y(t), t] \tag{27a}$$

$$\omega_i^*(t) = h[W(t), y(t), t] \tag{27b}$$

V. An Example Using Bernoulli Log Utility

Assume that $U[C(t), t] \equiv \delta^t \ln[C(t)]$, $B[W(T), T] \equiv \delta^T \ln[W(T)]$, and $y(t) \equiv 0$, where $\delta = \frac{1}{1+\rho}$ and $\rho \geqslant 0$ is the individual's subjective rate of time preference. Now at date $T-1$, using condition (14a), we have

$$\begin{aligned} U_C &= \delta^{T-1}\frac{1}{C^*} = E_{T-1}\left\{B_W\left(\sum_{i=1}^{m}\omega_i^*[z_i - R] + R\right)\right\} \\ &= \delta^T E_{T-1}\left\{\frac{\sum_{i=1}^{m}\omega_i^*[z_i - R] + R}{\left(\sum_{i=1}^{m}\omega_i^*[z_i - R] + R\right)(W(T-1) - C^*)}\right\} \\ &= \delta^T E_{T-1}\left\{\frac{1}{(W(T-1) - C^*)}\right\} = \delta^T\frac{1}{W - C^*} \end{aligned} \tag{28}$$

or

$$C^*(T-1) = \frac{1}{1+\delta}W(T-1) \tag{29}$$

Using (14b), we also have

$$E_{T-1}\{B_W z_i\} = E_{T-1}\{B_W R(T-1)\} \tag{30}$$

$$\delta^T E_{T-1}\left\{\frac{z_i}{\left(\sum_{i=1}^m \omega_i\left[z_i-R\right]+R\right)\left(W-C^*\right)}\right\}=\delta^T E_{T-1}\left\{\frac{R\left(T-1\right)}{\left(\sum_{i=1}^m \omega_i\left[z_i-R\right]+R\right)\left(W-C^*\right)}\right\}$$

$$E_{T-1}\left\{\frac{z_i}{\sum_{i=1}^m \omega_i^*\left[z_i-R\right]+R}\right\}=R\left(T-1\right)E_{T-1}\left\{\frac{1}{\sum_{i=1}^m \omega_i^*\left[z_i-R\right]+R}\right\},\ \text{for } i=1,...,m$$

It is interesting to note that the optimal consumption policy for a log utility investor, given by equation (29), is to consume a fixed proportion of wealth. The consumption rule is independent of any investment opportunities (for example, the current level of $R\left(t\right)$, expected returns on z_i's, et cetera) and, therefore, is independent of the portfolio decision.

From equation (30), we see that the portfolio decision rules that determine $\{\omega_i^*\}$ do not depend on $W\left(T-1\right)$, $C\left(T-1\right)$, or δ. All log utility investors choose the same portfolio, and it is independent of their consumption decisions.

Also, note the following. From equation (16a) we see

$$\begin{aligned} U_C &= \delta^{T-1}\frac{1}{C^*}=\delta^T R\left(T-1\right)E_{T-1}\left\{\frac{1}{\left(\sum_{i=1}^m \omega_i^*\left[z_i-R\right]+R\right)\left(W-C^*\right)}\right\} \qquad (16a)\\ &= \frac{\delta^T R\left(T-1\right)}{\left(W-C^*\right)}E_{T-1}\left\{\frac{1}{\sum_{i=1}^m \omega_i^*\left[z_i-R\right]+R}\right\}\end{aligned}$$

plugging in for C^*, we see that

$$1=RE_{T-1}\left\{\frac{1}{\sum_{i=1}^m \omega_i^*\left[z_i-R\right]+R}\right\} \qquad (31)$$

Thus, the optimal portfolio policy, which does not depend on W, is such that

$$E_{T-1}\left\{\frac{z_i}{\sum_{i=1}^m \omega_i^*\left[z_i-R\right]+R}\right\}=E_{T-1}\left\{\frac{R}{\sum_{i=1}^m \omega_i^*\left[z_i-R\right]+R}\right\}=1 \qquad (32)$$

The next step is to solve for $J\left[W\left(T-1\right),T\right]$ by plugging in the optimal consumption and portfolio rules. From equation (12) we have

$$
\begin{aligned}
J\left[W\left(T-1\right),T-1\right] &= \delta^{T-1}\ln\left[C^{*}\right]+\delta^{T}E_{T-1}\left\{\ln\left[\left(\sum_{i=1}^{m}\omega_{i}^{*}\left[z_{i}-R\right]+R\right)\left(W\left(T-1\right)-C^{*}\right)\right]\right\} \\
&= \delta^{T-1}\left(-\ln\left[1+\delta\right]+\ln\left[W\left(T-1\right)\right]\right)+ \\
&\quad \delta^{T}\left[E_{T-1}\left\{\ln\left[\sum_{i=1}^{m}\omega_{i}^{*}\left[z_{i}-R\right]+R\right]\right\}+\ln\left[\frac{\delta}{1+\delta}\right]+\ln\left[W\left(T-1\right)\right]\right] \\
&= \delta^{T-1}\left[\left(1+\delta\right)\ln\left[W\left(T-1\right)\right]+H\left(T-1\right)\right] \qquad (33)
\end{aligned}
$$

where $H\left(T-1\right)\equiv-\ln\left[1+\delta\right]+\delta\ln\left[\frac{\delta}{1+\delta}\right]+\delta E_{T-1}\left\{\ln\left[\sum_{i=1}^{m}\omega_{i}^{*}\left[z_{i}-R\right]+R\right]\right\}$. Notably, from equation (30) we saw that ω_{i}^{*} did not depend on $W\left(T-1\right)$, and, therefore, $H\left(T-1\right)$ does *not* depend on $W\left(T-1\right)$.

Next, let's move back one more period and consider the individual's optimal consumption and portfolio decisions at time $T-2$. From equation (23) we have

$$
\begin{aligned}
J\left[W\left(T-2\right),T-2\right] &= \max_{\{C(T-2),\omega(T-2)\}}\left\{U\left[C\left(T-2\right),T-2\right]+E_{T-2}\left(J\left[W\left(T-1\right),T-1\right]\right)\right\} \\
&= \max_{\{C(T-2),\omega(T-2)\}}\left[\delta^{T-2}\ln\left[C\left(T-2\right)\right]\right] \\
&\quad +\delta^{T-1}E_{T-2}\left\{\left(1+\delta\right)\ln\left[W\left(T-1\right)\right]+H\left(T-1\right)\right\} \qquad (34)
\end{aligned}
$$

Substituting $W\left(T-1\right)=\left[\sum_{i=1}^{m}\omega_{i}^{*}\left(T-2\right)\left[z_{i}\left(T-2\right)-R\left(T-2\right)\right]+R\left(T-2\right)\right]\left[W\left(T-2\right)-C\left(T-2\right)\right]$ and using equation (24a), the optimality conditions for C is

$$
\begin{aligned}
U_{C} &= \frac{\delta^{T-2}}{C^{*}}=RE_{T-2}\left\{J_{W}\left[W\left(T-1\right),T-1\right]\right\} \qquad (35) \\
&= \delta^{T-1}E_{T-2}\left\{\left(1+\delta\right)\frac{R}{\left(\sum_{i=1}^{m}\omega_{i}^{*}\left[z_{i}-R\right]+R\right)\left(W\left(T-2\right)-C^{*}\right)}\right\}
\end{aligned}
$$

Using (24b), we then see that the optimality conditions for the $\{\omega_{i}^{*}\}$ turn out to be of the same form as at $T-1$:

$$E_{T-2}\left\{\frac{z_i\,(T-2)}{\sum_{i=1}^{m}\omega_i^*\,[z_i-R]+R}\right\}=E_{T-2}\left\{\frac{R\,(T-2)}{\sum_{i=1}^{m}\omega_i^*\,[z_i-R]+R}\right\}=1,\ \text{ for } i=1,...,m \tag{36}$$

Using this result, equation (35) becomes

$$U_C=\frac{\delta^{T-2}}{C^*}=\delta^{T-1}\,(1+\delta)\,\frac{1}{(W\,(T-2)-C^*)} \tag{37}$$

Re-arranging, we then have

$$C^*\,(T-2)=\frac{1}{1+\delta+\delta^2}W\,(T-2) \tag{38}$$

Recognizing the above pattern, we see that the optimal consumption and portfolio rules for any prior date, t, are

$$C^*=\frac{1}{1+\delta+...+\delta^{T-t}}W\,(t)=\frac{1-\delta}{1-\delta^{T-t+1}}W\,(t) \tag{39}$$

$$E_t\left\{\frac{z_i\,(t)}{\sum_{i=1}^{m}\omega_i^*\,[z_i-R]+R}\right\}=E_t\left\{\frac{R\,(t)}{\sum_{i=1}^{m}\omega_i^*\,[z_i-R]+R}\right\}=1,\ \text{ for } i=1,...,m \tag{40}$$

Thus, we discover that consumption and portfolio rules are mutually independent. Importantly, (36) shows that the portfolio proportions depend only on the distribution of one-period returns and are independent of future (investment) return possibilities. This is described as *myopic behavior* because decisions made by the multi-period log investor are identical to a one-period log investor, no matter what are the serial dependencies (intertemporal correlations) of asset returns. It should be emphasized that these independence results are highly specific to the log utility assumption and cannot be extended, in general, to other utility functions.