Finance 400

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Modeling Credit Risk

We have considered a number of different bond pricing models, but each of these models assumes that the bonds had no risk of default. Therefore, these models are most applicable to valuing (nominal) bonds issued by a federal government, such as Treasury bills, notes, and bonds. However, many bonds, such as those issued by corporations, have default or "credit" risk and require a different modeling approach. We now consider two somewhat different methods for modeling default risk. The first, outlined by R. Merton (1974) "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance* 29, p.449-70, values a firm's debt as an explicit function of the value of the firm's assets and its capital structure.[1] In contrast to this "structural" approach, a number of recent papers have simply assumed a process for a firm's default probability and, possibly, debtholders' recovery rate in the event that default occurs.[2] This latter approach viewed the exogenously specified default process as the "reduced-form" of a more complicated and complex model of a firm's assets and capital structure. In these notes, we will consider an example of each of the two credit risk modeling techniques.

I. The Structural Approach

Let $A(t)$ denote the date t value of a firm's assets. The firm is assumed to have a very simple capital structure. In addition to shareholders' equity, it has issued a single zero-coupon bond that promises to pay an amount B at date $T > t$. Also let $\tau \equiv T - t$ be the time until this debt matures. The firm is assumed to pay dividends to its shareholders at the continuous rate $\delta A(t)dt$, where δ is the firm's constant proportion of assets paid in dividends per unit time.

[1] F. Longstaff and E. Schwartz (1995) "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt," *Journal of Finance* 50, p.789-820 is similar in spirit. It models firm assets but does not specify the explicit characteristics of a firm's debts. Bankruptcy is assumed to occur when firm assets fall to a critical level. A given formula is then used to specify the defaulted debt's value. A similar modeling of default for sovereign debt is given in S. Claessens and G. Pennacchi (1996) "Estimating the Likelihood of Mexican Default from the Market Prices of Brady Bonds, *Journal of Financial and Quantitative Analysis* 31(1), 109-126.

[2] D. Duffie and K. Singleton (1999) "Modeling Term Structures of Defaultable Bonds," *Review of Financial Studies* 12 (4), 687-720 is an example of this approach.

The value of the firm's assets are assumed to follow the process

$$dA/A = (\alpha - \delta)\, dt + \sigma dz \tag{1}$$

where α denotes the instantaneous expected rate of return on bank assets and σ is the constant standard deviation of return on bank assets. Now let $D(t)$ be the date t market value of the firm's debt which promises to pay the amount B at date T. It is assumed that when the debt matures, the firm pays the promised amount to the debtholders if there is sufficient asset value to do so. If not, the firm defaults (bankruptcy occurs) and the debtholders take ownership of all of the firm's assets. Hence, the payoff to debtholders at date T can be written as

$$\begin{aligned} D(T) &= \min[B, A(T)] \\ &= B - \max[0, B - A(T)] \end{aligned} \tag{2}$$

From the second line in equation (2), we see that the payoff to the debtholders equals the promised payment, B, less the payoff on a European put option written on the firm's assets and having exercise price equal to B. Hence, if we make the usual "frictionless" market assumptions (no transactions costs to trading firm assets or debt), then the current market value of the debt can be derived to equal the present value of the promised payment less the value of a put option on the dividend-paying assets. Assuming that the default-free interest rate is constant and equal to r, we obtain[3]

$$\begin{aligned} D(t) &= e^{-r\tau}B - e^{-r\tau}BN(-d_2) + e^{-\delta\tau}AN(-d_1) \\ &= e^{-r\tau}BN(d_2) + e^{-\delta\tau}AN(-d_1) \end{aligned} \tag{3}$$

where $d_1 = \left[\ln(A/B) + \left(r - \delta + \frac{1}{2}\sigma^2\right)\tau\right] / (\sigma\sqrt{\tau})$ and $d_2 = d_1 - \sigma\sqrt{\tau}$.

Based on this result, one can also solve for the market value of the firm's shareholder's

[3] It is straightforward to extend the model to allow default-free interest rates to be stochastic. R. Merton (1973) "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science* 4, p.141-83, gives the value of call and put options on assets when interest rates follow a stochastic process consistent with the Vasicek (1977) model of the term structure which we previously discussed.

equity, which we denote as $E(t)$. In the absence of taxes and other transactions costs, the value of investors' claims on the firm's assets, $D(t) + E(t)$ must equal the total value of the firm's assets, $A(t)$. This allows us to write

$$\begin{aligned} E(t) &= A(t) - D(t) \\ &= A - e^{-r\tau} BN(d_2) - e^{-\delta\tau} AN(-d_1) \\ &= A\left[1 - e^{-\delta\tau} N(-d_1)\right] - e^{-r\tau} BN(d_2) \end{aligned} \tag{4}$$

See R. Merton (1974), revised and updated as Chapter 12 in R. Merton (1990) *Continuous-Time Finance*, Basil Blackwell, for an in-depth analysis of the comparative statics properties of the debt and equity formulas in equations (3) and (4).

II. The Reduced-Form Approach

This section presents a simple example of the reduced form approach. Assuming frictionless markets, we showed that the absence of arbitrage opportunities led to the martingale pricing equation

$$f(t) = \tilde{E}_t \left[e^{-\int_t^T r(s)ds} f(T) \right] \tag{5}$$

where $f(t)$ is the date t price of a security and $r(t)$ is the date t instantaneous default-free interest rate. If $P(t,T)$ denotes the date t value of a risk-less-in-terms-of-default zero-coupon bond that pays $1 at date T, then equation (5) implies that

$$P(t,T) = \tilde{E}_t \left[e^{-\int_t^T r(s)ds} \right] \tag{6}$$

Given an assumed process followed by $r(t)$ and a specification of the market price of interest rate risk, the risk-neutral expectation in (6) can be evaluated. One popular model, similar to the previously presented Vasicek (1977) model, is that of J. Cox, J. Ingersoll, and S. Ross (1985) "A Theory of the Term Structure of Interest Rates," *Econometrica* 53, p.385-407. It assumes the following process for $r(t)$:[4]

[4]This "square-root" process is attractive since it precludes negative interest rates, that is, $r(t) \geq 0$. If

$$dr = \kappa (\gamma - r)\, dt + \sigma \sqrt{r} dz \tag{7}$$

where the market price of interest rate risk is $\theta(t) = \frac{u_p - r}{\sigma_p} = q\sqrt{r}$ and q is a constant. Equation (7) is the "true" process followed by the short-term interest rate (the probability measure P). However, under the risk-neutral measure, $\widetilde{\mathrm{P}}$, generated by the process $d\widetilde{z} = dz + \theta dt$, the short-term interest rate follows the process

$$\begin{aligned} dr &= \kappa (\gamma - r)\, dt - \theta(t)\, \sigma \sqrt{r} dt + \sigma \sqrt{r} d\widetilde{z} \\ &= \kappa (\gamma - r)\, dt - q \sigma r dt + \sigma \sqrt{r} d\widetilde{z} \\ &= \left[\kappa\gamma - (\kappa + \lambda)\, r\right] dt + \sigma \sqrt{r} d\widetilde{z} \end{aligned} \tag{8}$$

where $\lambda \equiv q\sigma$. Defining $\tau \equiv T - t$ and taking the expectation in (6) where the risk-neutral process for r is given by (8), one obtains

$$P(t,T) = A(\tau)\, e^{-B(\tau) r} \tag{9}$$

where

$$A(\tau) \equiv \left[\frac{2\phi e^{\frac{1}{2}(\kappa+\lambda+\phi)\tau}}{(\kappa+\lambda+\phi)\,(e^{\phi\tau} - 1) + 2\phi} \right]^{\frac{2\kappa\gamma}{\sigma^2}}$$

$$B(\tau) \equiv \frac{2\left(e^{\phi\tau} - 1\right)}{(\kappa+\lambda+\phi)\,(e^{\phi\tau} - 1) + 2\phi}$$

and $\phi \equiv \left[(\kappa+\lambda)^2 + 2\sigma^2\right]^{\frac{1}{2}}$. Having derived the price of a default-free zero-coupon bond, now consider the price of a default-risky zero-coupon bond maturing at date T. Default is modeled as the arrival of an unpredictable Poisson process. Conditional on default having not occurred prior to date t, the probability of default during the interval $(t, t + dt)$ under the equivalent martingale measure $\widetilde{\mathrm{P}}$ is given by $h(t)\, dt$. $h(t)$ is the time-varying default intensity and is

$2\kappa\gamma \geq \sigma^2$, then $r(t) > 0$.

assumed to follow the following continuous-time process

$$dh = (\alpha - \beta h)\, dt + \sigma_h \sqrt{h} d\tilde{z}_h \tag{10}$$

where $d\tilde{z}d\tilde{z}_h = 0$, that is, the changes in the default-free interest rate and the default intensity are assumed to be uncorrelated.[5] Now consider the value of a hypothetical bond that promises \$1 at date T but pays nothing if the firm defaults at or prior to date T. In other words, the bond pays \$1 at date T unless default occurs prior to this date, in which case the defaulted bond is worthless. The date t value of this "all or nothing" bond, denoted as $V(t,T)$, is given by

$$V\left(t,T\right) = \tilde{E}_t\left[e^{-\int_t^T [r(s)+h(s)]ds}\right] \tag{11}$$

Thus, we see that the value of this bond is similar to that of a default-free bond but with the discount rate reflecting $r\left(s\right) + h\left(s\right)$ rather than just $r\left(s\right)$. Under the assumption that $r\left(s\right)$ and $h\left(s\right)$ are uncorrelated processes, equation (11) becomes

$$\begin{aligned} V\left(t,T\right) &= \tilde{E}_t\left[e^{-\int_t^T [r(s)+h(s)]ds}\right] \\ &= \tilde{E}_t\left[e^{-\int_t^T r(s)ds}\right]\tilde{E}_t\left[e^{-\int_t^T h(s)ds}\right] \\ &= P\left(t,T\right)A_h\left(\tau\right)e^{-B_h(\tau)h} \\ &= A\left(\tau\right)A_h\left(\tau\right)e^{-B(\tau)r-B_h(\tau)h} \end{aligned} \tag{12}$$

where

$$A_h\left(\tau\right) \equiv \left[\frac{2\phi_h e^{\frac{1}{2}(\beta+\phi_h)\tau}}{\left(\beta+\phi_h\right)\left(e^{\phi_h\tau}-1\right)+2\phi_h}\right]^{\frac{2\alpha}{\sigma_h^2}}$$

$$B_h\left(\tau\right) \equiv \frac{2\left(e^{\phi_h\tau}-1\right)}{\left(\beta+\phi_h\right)\left(e^{\phi_h\tau}-1\right)+2\phi_h}$$

[5] See G. Duffee (1999) "Estimating the Price of Default Risk," *Review of Financial Studies* 12, p.197-226 for a slight generalization of this modeling that allows for non-zero correlation between the default-free term structure and default probabilities.

and $\phi_h \equiv \left[\alpha^2 + 2\sigma_h^2\right]^{\frac{1}{2}}$. Next, we can generalize the consequences of default to allow for a non-zero recovery value.[6] In the event of default, it is assumed that the bond pays, at the time of default, a fixed fraction, δ, of an otherwise equivalent default-free bond, that is, $\delta P(t,T)$.[7] The equivalent assumption is that when default occurs the debtholders receive the amount δ at date T. Denote the price of this positive recovery bond as $D(t,T)$. Then the absence of arbitrage implies that after default the price of this bond equals $\delta P(t,T)$ and that prior to default its value is

$$\begin{aligned} D(t,T) &= \delta P(t,T) + (1-\delta) V(t,T) \\ &= \delta P(t,T) + (1-\delta) P(t,T) A_h(\tau) e^{-B_h(\tau)h} \\ &= P(t,T)\left[\delta + (1-\delta) A_h(\tau) e^{-B_h(\tau)h}\right] \\ &= A(\tau) e^{-B(\tau)r}\left[\delta + (1-\delta) A_h(\tau) e^{-B_h(\tau)h}\right] \end{aligned} \tag{13}$$

While the default-risky bond price, $D(t,T)$, depends on the current unobserved default intensity, $h(t)$, this default intensity could be inferred from the market prices of one or more of the issuer's bonds. This could help determine whether a given bond issued by a particular issuer is relatively over- or under-priced relative to another bond of the same issuer.

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[6] D. Dufffie and K. Singleton (1999) describe alternative specifications of recovery value in the context of reduced-form models.

[7] For example, G. Duffee (1999) reports that the recovery rate, δ, estimated by Moody's for senior unsecured bondholders is approxmately 44 percent.