Finance 400

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Extending Diffusion Processes to Allow for Jumps

I. Modeling Jumps in Continuous-Time Stochastic Processes

Until now, we have been concerned only with diffusion processes: continuous time stochastic processes whose uncertainty results only from a Brownian motion component. While these processes can provide a realistic modeling of many economic and financial time series, they may be unrealistic for modeling random variables whose values can change very significantly over a short period of time. This is because diffusion processes have continuous sample paths, they do not allow for discontinuities or "jumps" in their values. In situations where it is more realistic to allow for large, sudden changes in value, for example when significant new information results in a sudden change in the market value of an asset, then we need to augment the diffusion process with another type of uncertainty. This is where Poisson "jump" processes can be useful. In particular, we can model an economic or financial time series as the sum of diffusion (Brownian motion-based) processes and Poisson jump processes.

Consider the following continuous-time process

$$dS/S = (\alpha - \lambda k)\, dt + \sigma\, dz + dq \tag{1}$$

where dz is a standard Wiener process and dq is a Poisson jump process having the following characteristics:

$$dq = \begin{cases} (\tilde{Y} - 1) & \text{if a jump occurs} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

During each time interval, dt, the probability that q will "jump" once during this interval is $\lambda\, dt$. For simplicity, the analysis that follows assumes that λ is constant over time, though extensions could allow for λ to change. When a jump occurs, since dq equals $(Y - 1)$, the discontinuous change in S is $dS = (Y - 1)S$, that is, S goes to YS. In general, $\tilde{Y}$ may be a random variable. In other words, given that a jump occurs (with probability $\lambda\, dt$), then the jump size is a drawing from a random distribution, $\tilde{Y} - 1$. Again, for simplicity, we assume that successive jump sizes, $(\tilde{Y}-1)$, are independently and identically distributed.

We define $k \equiv E[\tilde{Y}-1]$, that is, k is the mean jump size. Thus the expected change in S from the jump component dq over the time interval dt is $\lambda\, k\, dt$. Therefore, if we wish to let the parameter α denote the instantaneous total expected rate of return (rate of change) on S, we need to subtract off $\lambda\, k\, dt$ from the drift term of S:

$$
\begin{aligned}
E[dS/S] &= E[(\alpha-\lambda k)\, dt] + E[\sigma\, dz] + E[dq] \\
&= (\alpha-\lambda k)\, dt + 0 + \lambda\, k\, dt = \alpha\, dt
\end{aligned}
\tag{3}
$$

Note that a sample path $S(t)$ for a process described by equation (1) will be continuous most of the time, but have finite jumps of differing signs and amplitudes at discrete points in time. If $S(t)$ is an asset price, these jump events can be thought of as times when important information affecting the valuation of the asset are released.

Assuming α and σ are constants, so that the continuous component of $S(t)$ is lognormally distributed, and conditional upon there being n jumps in the interval $(0,\, t)$

$$
\tilde{S}(t) = S(0)\, e^{(\alpha-\frac{1}{2}\sigma^2-\lambda k)\, t + \tilde{z}(t)}\, \tilde{y}(n) \tag{4}
$$

where $\tilde{z}(t) \sim N(0,\, \sigma^2 t)$. $\tilde{y}(0) = 1$ and $\tilde{y}(n) = \prod_{i=1}^{n} \tilde{Y}_i$ for $n \geq 1$ where $\{\tilde{Y}_i\}_{i=1}^{n}$ is a set of independent identically distributed jumps.

II. Extending Itô's Lemma to Mixed Jump-Diffusion Processes

Let $F(S,\, t)$ be the value of a variable that is a function of $S(t)$, the above process following a mixed jump-diffusion process. For example, $F(S,\, t)$ might be the value of a contingent claim on the asset having value $S(t)$. A generalized version of Itô's lemma for mixed jump-diffusion processes implies that the value of this claim follows the process

$$
dF = F_s\, [\, (\alpha-\lambda k) S\, dt + \sigma S\, dz\,] + \frac{1}{2} F_{ss} \sigma^2 S^2\, dt + F_t\, dt + F\, dq_f \tag{5}
$$

where[1]

$$dq_f = \begin{cases} F(S\tilde{Y},\, t)/F(S,\, t) - 1 & \text{if a jump occurs} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Equation (6) implies that when S jumps, the contingent claim's value has a corresponding jump, where it goes from $F(S,\, t)$ to $F(S\tilde{Y},\, t)$. Now define α_f as the instantaneous expected rate of return on F, that is, $E[dF/F]$. Also, define σ_f as the standard deviation of the instantaneous rate of return on F, conditional on a jump not occurring. Thus, from equation (5) above, we have

$$dF/F = [\alpha_f - \lambda k_f\,(t)]\,dt + \sigma_f dz + dq_f \tag{7a}$$

where

$$\begin{aligned} \alpha_f &\equiv \frac{1}{F}\left[F_s\,(\alpha - \lambda k)\,S + \frac{1}{2}F_{ss}\sigma^2 S^2 + F_t\right] + \lambda k_f\,(t) && \text{(7a)} \\ \sigma_f &\equiv \frac{F_s}{F}\sigma S, \text{ and} && \text{(7b)} \\ k_f\,(t) &\equiv \frac{1}{F\,(S,t)}E_t\left[F\left(S\tilde{Y},t\right) - F\,(S,t)\right] && \text{(7c)} \end{aligned}$$

III. Valuing Contingent Claims When Assets Follow Jump-Diffusions

Consider constructing a portfolio that includes a contingent claim (for example, a call option) having price F, an underlying asset whose price follows the process given in equation (1), and a riskless bond paying the constant interest rate r per unit time. Denote the proportions of the portfolio invested in the asset, contingent claim, and bond as w_1, w_2, and $w_3 = 1 - w_1 - w_2$, respectively. The instantaneous rate of return on this portfolio, denoted dP/P, is given by:

$$\begin{aligned} dP/P &= w_1\,dS/S + w_2\,dF/F + (1 - w_1 - w_2)r\,dt \\ &= [\,w_1(\alpha - r) + w_2(\alpha_f - r) + r - \lambda(w_1 k + w_2 k_f)\,]\,dt \\ &\quad + (w_1\sigma + w_2\sigma_f)\,dz + w_1\,dq + w_2\,dq_f \end{aligned} \tag{8}$$

[1]Equivalently, the jump component can be written as $F dq_f = \begin{cases} F(S\tilde{Y},\, t) - F(S,\, t) & \text{if a jump occurs} \\ 0 & \text{otherwise} \end{cases}$.

Consider the possibility of choosing w_1 and w_2 in order to eliminate risk from jumps. Note that while jumps occur simultaneously in the asset and the contingent claim, that is, jump risk is perfectly dependent for these two securities, these risks are not necessarily linearly dependent. This is because the contingent claim price, $F(S,\, t)$, is generally a nonlinear function of the asset price and, unlike Brownian motion generated movements, jumps result in non-local changes in S and $F(S,\, t)$. Since, in general, the jump size $(\tilde{Y}-1)$ is random, the ratio between the size of the jump in S and the size of the jump in F cannot be known beforehand. Hence, the proper hedge ratio, w_1/w_2, cannot be known.[2]

Instead, suppose we pick w_1^* and w_2^* to eliminate only the risk from the continuous Brownian motion movements. This Black-Scholes hedge implies setting $w_1^*/w_2^* = -\sigma_f/\sigma = -F_s S/F$ from our definition of σ_f. This leads to the process for value of the portfolio:

$$dP/P = [w_1^*\,(\alpha - r) \,+\, w_2^*\,(\alpha_f - r) \,+\, r - \lambda\,(w_1^*\,k + w_2^*\,k_f)]\,dt \,+\, w_1^*\,dq \,+\, w_2^*\,dq_f \tag{9}$$

The return on this portfolio is a pure jump process. The return is deterministic, except when jumps occur. Using the definitions of dq, dq_f, and w_1^*, we see that the portfolio jump term equals

$$w_1^*\,dq \,+\, w_2^*\,dq_f \;=\; \begin{cases} w_2^*\left[\frac{F(S\tilde{Y},t)}{F(S,t)} \,-\, 1 \,-\, F_s(S,\,t)\frac{S\tilde{Y}-S}{F(S,t)}\right] & \text{if a jump occurs} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Now consider the case when the contingent claim is an option on a stock with expiration date T and strike price X. An implication of the option price being a strictly convex function of the asset price is $F(SY,\, t) - F(S,\, t) - F_s(S,\, t)[SY - S] \geq 0$ for all Y and t. Thus, the unanticipated return on this hedge portfolio has the same sign as w_2^*. This implies that $w_1^*k + w_2^*k_f$, the expected portfolio jump size, also has the same sign as w_2^*. Therefore, a call option writer who follows this Black-Scholes hedge by being short the call option ($w_2^* < 0$) and long the underlying stock earns, most of the time, more than the expected rate of return. However, on those rare occasions when the stock price jumps, a relatively large loss is incurred. Thus in "quiet" times, option writers appear to make positive excess returns. However, during infrequent "active"

[2] If the size of the jump is deterministic, a hedge that eliminates jump risk can be found.

times, option writers suffer large losses.

Since the hedge portfolio is not riskless, but is exposed to jump risk, somehow we need to assign a "market price" to this jump risk. One way of doing so is to assume that this jump risk is the result of purely firm specific information and, hence, the jump risk is perfectly diversifiable. This would imply that the market price of jump risk is zero. Under this assumption that all of the risk of the hedge portfolio is diversifiable, its expected rate of return must equal the risk-free rate, r.

$$w_1^*(\alpha - r) \,+\, w_2^*(\alpha_f - r) \,+\, r \,=\, r \tag{11}$$

or

$$w_1^*/w_2^* \,=\, -\sigma_f/\sigma \,=\, -(\alpha_f - r)/(\alpha - r) \tag{12}$$

Now define $\tau \equiv T - t$ as the time until maturity of the contingent claim, and use this as the second argument for $F(S, \cdot)$ rather than calendar time, t. Thus, $F(S, \tau)$ now denotes the price of the contingent claim when the current asset price is S and the time until maturity of the contingent claim is τ. With this redefinition, note that $F_\tau = -F_t$. Using (12) and substituting in for α_f and σ_f from the definitions (7b) and (7c), we obtain the equilibrium partial differential equation

$$\frac{1}{2}\sigma^2 S^2 F_{ss} \,+\, (r - \lambda k) S F_s \,-\, F_\tau \,-\, rF \,+\, \lambda\, E_t \left[F(S\tilde{Y},\, \tau) \,-\, F(S,\, \tau)\right] \,=\, 0 \tag{13}$$

For a call option, this is solved subject to the boundary conditions $F(0,\, \tau) = 0$ and $F(S,\, 0) = [S - X]^+$. Note that when $\lambda = 0$, equation (13) is the standard Black-Scholes equation which has the solution

$$W(S, \tau, X, \sigma^2, r) \,\equiv\, S\, N(d_1) \,-\, X e^{-r\tau}\, N(d_2) \tag{14}$$

where $d_1 = [\,\ln(S/X) + (r + \sigma^2)\tau\,] \,/\, (\sigma\sqrt{\tau})$ and $d_2 = d_1 - \sigma\sqrt{\tau}$. R.C. Merton (1976) "Option Pricing When Underlying Stock Returns are Discontinuous," *Journal of Financial Economics* 3, p.125-44, shows that the general solution to (13) is

$$F(S,\, \tau) \,=\, \sum_{n=0}^{\infty} \frac{e^{-\lambda\,\tau}(\lambda\,\tau)^n}{n!}\, E_t \left[W(S\,\tilde{y}(n)\, e^{-\lambda k \tau},\, \tau,\, X,\, \sigma^2,\, r)\right] \tag{15}$$

where, recall that $\tilde{y}(0) = 1$ and $\tilde{y}(n) = \prod_{i=1}^{n} \tilde{Y}_i$ for $n \geq 1$. The intuition behind the formula in (15) is that the option is a probability weighted average of expected Black-Scholes option prices. Note that if the stock price followed (1), then conditional on no jumps occuring over the life of the option, risk-neutral valuation would imply that the Black-Scholes option price would be $W(Se^{-\lambda k \tau}, \tau, X, \sigma^2, r)$.[3] Similarly, conditional on 1 jump occuring, risk-neutral valuation would imply that the option price would be $W(Sy(1)e^{-\lambda k \tau}, \tau, X, \sigma^2, r)$. Conditional on two jumps, it would be $W(Sy(2)e^{-\lambda k \tau}, \tau, X, \sigma^2, r)$, and thus for n jumps, it would be $W(Sy(n)e^{-\lambda k \tau}, \tau, X, \sigma^2, r)$.

Since $\frac{e^{-\lambda \tau}(\lambda \tau)^n}{n!}$ is the probability of n jumps occuring, we see that (15) is the jump-probability weighted average of expected option values conditioned over all possible numbers of jumps.

Under particular assumptions regarding the distribution of $\tilde{Y}$, solutions to (15) can be calculated numerically or, in some cases, in closed-form. Here, we consider a case which leads to a closed-form solution, namely the case in which $\tilde{Y}$ is lognormally distributed. Thus, if $E[\ln \tilde{Y}] \equiv \gamma - \frac{1}{2}\delta^2$ where $var[\ln \tilde{Y}] \equiv \delta^2$, then $E[\tilde{Y}] = e^{\gamma} = 1 + k$. Hence, $\gamma \equiv \ln(1 + k)$. Given this assumption, if α is assumed to be constant, the probability density for $\ln[S(t + \tau)]$ conditional on the value of $S(t)$ is

$$\sum_{n=0}^{\infty} g(\ln[S(t+\tau)/S(t)] \,|\, n)h(n) \tag{16}$$

where $g(\cdot \,|\, n)$ is the conditional density function given that n jumps occur during the interval between t and $t + \tau$, and $h(n)$ is the probability that n jumps occur between t and $t + \tau$. The values of these expressions are

$$g\left(\ln\left[\frac{S\left(t+\tau\right)}{S(t)}\right] | \, n\right) \equiv \frac{1}{\sqrt{2\pi\nu_n^2\tau}} \exp\left[-\frac{\left(\ln\left[\frac{S(t+\tau)}{S(t)}\right] - \left(\alpha - \lambda k + \frac{n\gamma}{\tau} - \frac{\nu_n^2}{2}\right)\tau\right)^2}{2\nu_n^2\tau}\right] \tag{17a}$$

$$h\left(n\right) \equiv \frac{e^{-\lambda\tau}\left(\lambda\tau\right)^n}{n!} \tag{17b}$$

[3]Recall that since the drift is $\alpha - \lambda k$, and risk-neutral valuation sets $\alpha = r$, then λk is like a dividend yield. Hence $W(Se^{-\lambda k \tau}, \tau, X, \sigma^2, r)$ is the Black-Scholes formula for an asset with a dividend yield of λk.

where $\nu_n^2 \equiv \sigma^2 + n\delta^2/\tau$ is the "average" variance per unit time. From (17a), we see that conditional on n jumps occurring, $\ln[S(t+\tau)/S(t)]$ is normally distributed. Using the Cox-Ross "risk-neutral" (equivalent martingale) transformation which allows us to set $\alpha = r$, we can compute the expectation of $[S - X]^+$ at time $t + \tau$, discounted by the risk-free rate, and conditional on n jumps occurring. This is given by

$$\begin{aligned} E_t[\, W(S\tilde{y}(n)e^{-\lambda\, k\, \tau}, \tau, X, \sigma^2, r)\,] &= e^{-\lambda\, k\, \tau}(1+k)^n\, W(S, \tau, X, \nu_n^2, r_n) \\ &= e^{-\lambda\, k\, \tau}(1+k)^n\, f_n(S,\, \tau) \end{aligned} \tag{18}$$

where $f_n(S,\, \tau) \equiv W(S, \tau, X, \nu_n^2, r_n)$ and where $r_n \equiv r - \lambda k + n\gamma/\tau$. The actual value of the option is then the weighted average of these conditional values, where each weight equals the probability that a Poisson random variable with characteristic parameter $\lambda\tau$ will take on the value n. Defining $\lambda' \equiv \lambda(1+k)$, this equals

$$\begin{aligned} F(S,\, \tau) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda\, \tau}(\lambda\, \tau)^n}{n!}\, e^{-\lambda\, k\, \tau}\, (1+k)^n\, f_n(S,\, \tau) \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda'\, \tau}(\lambda'\, \tau)^n}{n!}\, f_n(S,\, \tau) \end{aligned} \tag{19}$$

IV. Valuing Contingent Claims Assuming an Equilibrium Price of "Jump" Risk

In some circumstances, it is unrealistic to assume that "jump" risk is non-priced risk. For example, a paper by David Bates (1991) "The Crash of '87 – Was It Expected? The Evidence from Options Markets," *Journal of Finance* 46, p.1009-1044, investigates the stock market crash of 1987, an event that certainly was not firm specific but affected the entire market for equities. A similar paper that assumes market wide jump risk is Vasanttilak Naik and Moon Lee (1990) "General Equilibrium Pricing of Options on the Market Portfolio with Discontinuous Returns" *Review of Financial Studies* 3, p. 493-521. Both of these papers start with the assumption that aggregate wealth in the economy follows a mixed jump-diffusion process. This could result from a representative agent Cox, Ingersoll, and Ross - type "production" economy in which technologies follow a jump-diffusion process and individuals select investments in these technologies such that their optimally invested aggregate wealth follows a mixed jump-diffusion process (Bates). Or it can simply be assumed that the economy is a Lucas-type "endowment" economy and there is an exogenous firm dividend process that follows a mixed-jump diffusion process, and these dividends cannot be invested but must be consumed (Naik and Moon).

In both of these papers, jumps in aggregate wealth or consumption (endowment) are assumed to be of the lognormal type that we assumed earlier. Further, utility is assumed to be of the constant relative risk aversion type. Given this setup, contingent claims, which are assumed to be in zero net supply, can be priced.

Consider a special case of the Bates (1991) model, where a contingent claim is written on the "market" portfolio.[4] Following the model and notation used in Bates (1991), assume that optimally invested wealth, W_t, follows the process

$$dW/W \;=\; (\mu \,-\, \lambda\bar{k} \,-\, C/W)\,dt \,+\, \sigma\,dz \,+\, k\,dq \tag{20}$$

where μ is constant, $\text{Prob}(dq = 1) = \lambda dt$, and $\tilde{k}\,(\equiv\, \tilde{Y} - 1)$ is the percentage change in wealth, conditional on the Poisson event occurring. $\ln(1 + \tilde{k}) \sim N(\gamma - \frac{1}{2}\delta^2,\, \delta^2)$ and $\bar{k} \equiv E[\tilde{k}] = e^{\gamma} - 1$. Also assume the representative individual maximizes expected lifetime utility of the form

$$E_t\left[\int_t^{\infty} e^{-\rho s}(C(s)^{1-R} - 1)/(1-R)\,ds\right] \equiv J(W,\,t). \tag{21}$$

The Bellman equation for this problem is

$$0 = \max_C\;\left[\,u(C(t),\,t) \,+\, L[J]\,\right], \tag{22}$$

but now $L[J]$ must reflect a potential change due to a jump over the interval dt. Thus,

$$0 = \max_C\;\left[\begin{array}{c} u(C(t),\,t) \,+\, J_t(W) \,+\, (\mu \,-\, \lambda\bar{k} \,-\, C/W)\,W\,J_W(W) \\ +\,\lambda\,E_t[J_W(W(1+\tilde{k})) \,-\, J_W(W)] \,+\, \frac{1}{2}\sigma^2 W^2 J_{WW}(W). \end{array}\right] \tag{23}$$

Following the procedure in John Cox, Jonathan Ingersoll, and Stephen Ross (1985) "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica* 53, p.385-407, if we

[4]For example, Bates assumes that the rate of return on the S&P500 market index equals the rate of return on optimally invested aggregate wealth.

assume that the representative individual can invest in the instantaneous maturity risk-free bond paying rate of return $r(W, t)$, then equating the first order conditions for the demand for this asset to zero (since it is assumed to be in zero net supply), we get that its equilibrium rate of return must be

$$r(W,\, t) \,=\, \mu \,-\, R\sigma^2 \,+\, \frac{\lambda}{J_w} E_t[(J_W(W(1+\tilde{k})) \,-\, J_W(W))\,\tilde{k}]. \tag{24}$$

Now since utility is constant relative risk aversion, we have $J_W = Ae^{-\rho t}W^{-R}$, and thus

$$\frac{1}{J_W(W)}(J_W(W(1+\tilde{k})) \,-\, J_W(W)) \,=\, (1+\tilde{k})^{-R} \,-\, 1 \tag{25}$$

is independent of wealth, which implies that the risk-free rate

$$r \,=\, \mu \,-\, R\sigma^2 \,+\, \lambda E_t[\,((1+\tilde{k})^{-R} \,-\, 1)\,\tilde{k}\,] \tag{26}$$

is constant. Now if $F(W, t)$ is the price of a contingent claim on the "market" portfolio, following a similar exercise to CIR that we have just done for the risk-free bond, one can show that the equilibrium rate of return on this contingent claim, μ_f, satisfies

$$(\mu_f - r)F \,=\, RWF_W\sigma_f\sigma \,+\, \frac{1}{J_W}E_t\left\{\left[J_W(W(1+\tilde{k})) \,-\, J_W(W)\right]\left[F(W(1+\tilde{k})) \,-\, F(W)\right]\right\}. \tag{27}$$

Equating μ_f and σ_f to that defined by Itô's lemma, gives us the equilibrium partial differential equation for this contingent claim.

$$F_t \,+\, \left[r \,-\, \lambda E_t[\,((1+\tilde{k})^{-R} \,-\, 1)\,\tilde{k}\right] WF_W \,+\, \frac{1}{2}\sigma^2W^2F_{WW} \tag{28}$$

$$+\lambda\, E_t\left[\,(1+\tilde{k})^{-R}\left\{F(W(1+\tilde{k})) \,-\, F(W)\right\}\right] \,-\, rF \,=\, 0$$

subject to the boundary condition appropriate to the particular contingent claim. Note that this equation is similar in form to the above equation (13). A series solution of the same form as Merton's can then be found for the case of the call option.