Finance 400

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Arbitrage-Free Binomial Models of the Term Structure

Earlier, in our discussion of martingale pricing theory, we showed that the absence of arbitrage implied that the date t price of any asset, $f(t)$, satisfies:

$$f(t) \,=\, B(t)\,\tilde{E}_t\left[f(T)\,\frac{1}{B(T)}\right] \,=\, \tilde{E}_t\left[e^{-\int_t^T r_s\,ds} f(T)\right] \tag{1}$$

where $B(t)$ was the value of a "money market fund" which invests in the instantaneous maturity risk-free asset, so that $dB/B = r_t\,dt$, where r_t is the date t value of the instantaneous-maturity interest rate. $\tilde{E}_t[\,\cdot\,]$ is the expectations operator with respect to the "risk-neutral" probability measure. Since condition (1) must hold for any asset, it must also hold for bonds of any maturity. Let $P_t(\tau)$ be the date t price of a bond that pays 1 in τ periods. Letting $t = 0$, $T = 1$, and assuming $\tau \geq 1$, we can apply condition (1) to obtain

$$P_0(\tau) \,=\, \tilde{E}_0\left[e^{-\int_0^1 r_s\,ds}\,P_1(\tau-1)\right]. \tag{2}$$

If our goal is to price interest rate contingent claims that have possibly complex features, such as American-type (early exercise) options, it will be convenient to construct a discrete-time, discrete-state model (also called a "lattice" model) of bond prices, similar to the Cox, Ross, Rubinstein binomial model of stock prices that was used to price stock options.

Binomial models of the term structure are inherently more difficult than binomial models of stock prices because at each point in time one needs to keep track of the whole term structure of interest rates (bond prices for every different maturity) rather than just a single stock price. This is complicated, because assuming an arbitrary evolution of the term structure (set of bond prices) may be inconsistent with the absence of arbitrage opportunities. In other words, a zero net investment trading strategy involving long and short positions in different maturity bonds could result in a riskless profit (arbitrage) if one assumed arbitrary binomial models for different maturity bonds or interest rates.

Fortunately, however, if we use condition (2) to guide our modeling of the term structure, which we know rules out the existence of arbitrage, then we can avoid this problem. Hence, let us consider a discrete-time model where the shortest time period is one unit of time, that is, $\Delta t = 1$. Thus, the shortest maturity bond is one period and its date t price-yield relationship is given by

$$P_t(1) \,=\, e^{-r_t(1)} \tag{3}$$

where $r_t(1)$ is the date t one-period interest rate, which we will refer to as the short-term interest rate, or "short rate." Similarly, the date t price of a τ period bond is

$$P_t(\tau) \,=\, e^{-r_t(\tau)\tau}. \tag{4}$$

Using equation (3), noting that $\int_t^{t+1} r_s\, ds = r_t(1)$ for a discrete-time model with the length of each period equaling one, we can rewrite the risk-neutral valuation equation (2) as

$$P_0(\tau) \,=\, \tilde{E}_0\left[e^{-r_0(1)}\,P_1(\tau-1)\right] \,=\, \tilde{E}_0\left[\,P_1(\tau-1)\,\right]P_0(1). \tag{5}$$

Now we are ready to specify the form of the binomial tree. In doing so, we wish to make the tree *consistent with the currently observed set of traded bond prices and their volatilities*, that is, we wish to constrain the binomial model to "fit" the initially observed term structure of interest rates and "term structure of interest rate volatilities." This is analogous to the desire that the Cox, Ross, Rubinstein model's binomial tree for stock prices have an initial stock price equal to the currently observed stock price and that the up and down moves for the stock price be calibrated to the actual volatility of the stock. Similarly, we would also like our binomial model of bond prices to reflect actual observed bond prices and volatilities.

The following method for constructing a bond price-interest rate binomial model is based on ideas presented in Black, F., E. Derman, and W. Toy, "A One Factor Model of Interest Rates and Its Application to Treasury Bond Options," *Financial Analysts Journal* 46, (1990) p.33-39. However, in presenting this method, we will use the somewhat more simple distributional assumptions for interest rate movements that are based on Ho, T., and S. Lee, "Term Structure

Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance* 41, (1986), p.1011-30.

Suppose that the following is the initially observed term structure information:[1]

Initial Term Structure Information

Maturity (Years)	Bond Price (\$)	Yield to Maturity (%)	Volatility of Bond Yields σ_0
1	0.92312	8.0	–
2	0.83527	9.0	0.010
3	0.75201	9.5	0.009

What we mean by the volatility of bond yields will be clarified below. Now let us assume that the short rate is the state variable determining the evolution of all bond prices. According to the above table, its starting value (at date 0) equals to 8 percent. Let us assume that it takes one of two possible values next period:

$$r_0(1) = 8\% \begin{array}{l} \nearrow \; r_{11}(1) \\ \\ \searrow \; r_{10}(1) \end{array} \tag{6}$$

$r_{ij}(1)$ refers to the value of the short rate at date i for tree level (branch) j. Let us also assume that under the *risk-neutral probability measure* (not necessarily the actual measure), that the probability of an up or down move is the same, equal to 0.5. What we have not yet specified is the size of the up and down moves. We will now do this so that the model fits the term structure information in the above table. Note that the two possible one-period bond prices at date 1 are

$$\begin{aligned} P_{10}(1) &= e^{-r_{10}(1)} \\ P_{11}(1) &= e^{-r_{11}(1)}. \end{aligned} \tag{7}$$

[1] This example is from Peter Ritchken (1995) *Derivative Markets*, Harper Collins.

Therefore, applying the martingale pricing condition (5) for $\tau = 2$, we have that the initial two period bond price must equal

$$\begin{aligned} P_0(2) &= \tilde{E}_0\left[P_1(1)\right] P_0(1) \\ &= \left[\tfrac{1}{2}P_{11}(1) + \tfrac{1}{2}P_{10}(1)\right] P_0(1) \\ &= \tfrac{1}{2}\left[e^{-r_{11}(1)} + e^{-r_{10}(1)}\right] P_0(1). \end{aligned} \tag{8}$$

Substituting in for $P_0(1)$ and $P_0(2)$ from the above Table and rearranging, we obtain

$$\frac{2P_0(2)}{P_0(1)} = 1.8097 = e^{-r_{11}(1)} + e^{-r_{10}(1)}. \tag{9}$$

Note that condition (9) is not sufficient to determine unique values for $r_{11}(1)$ and $r_{10}(1)$. However, if we also use information on the volatility of interest rates, these values can be uniquely determined. Let $\sigma_0(1)$ be the date 0 standard deviation of the short rate at date 1. If we assume that in the continuous time limit (as $\Delta t \to 0$) that the short rate is normally distributed, which is the assumption made by Ho and Lee (1986), then the corresponding variance of short rates for the binomial tree is[2]

$$\mathrm{Var}_0[\,r_1(1)\,] = \sigma_0^2(1) = p\,(1-p)\,[\,r_{11}(1) - r_{10}(1)\,]^2 \tag{10}$$

where p is the probability of an up move, which we have assumed equals 0.5.

Substituting $p = 0.5$ into (10), taking the square root, and substituting $\sigma_0(1) = 0.01$ from the above table, gives

$$\sigma_0(1) = 0.0100 = \frac{1}{2}[\,r_{11}(1) - r_{10}(1)\,]. \tag{11}$$

[2] Black, Derman, and Toy (1990) assume that in the continuous-time limit (as $\Delta t \to 0$) the short rate is lognormally distributed. Under this assumption, the natural log of the short rate is normal, so that we obtain $\mathrm{Var}_0[\,r_1(1)\,] = \sigma_0^2(1) = p\,(1-p)\,[\ln r_{11}(1) - \ln r_{10}(1)\,]^2 = p\,(1-p)\,[\ln(r_{11}(1)/r_{10}(1))\,]^2$. This alternative lognormal assumption is attractive in that it rules out the possibility of negative interest rates, an event that could occur under the Ho and Lee normality assumption. However, for ease of presentation, we will make the Ho and Lee (1986) assumption of normality.

Solving equations (9) and (11), we obtain

$$\begin{aligned} r_{11}(1) &= 0.11 \\ r_{10}(1) &= 0.09. \end{aligned} \tag{12}$$

Thus, using equation (7), the date 1 possible prices of one-period zero coupon bonds are

$$\begin{aligned} P_{11}(1) &= 0.8958 \\ P_{10}(1) &= 0.9139. \end{aligned} \tag{13}$$

Let us now continue to construct the tree for date 2. We have

$$\begin{array}{ccccc} & & & & r_{22}(1) \\ & & & \nearrow & \\ & & r_{11}(1) = 11\% & & \\ & \nearrow & & \searrow & \\ r_0(1) = 8\% & & & & r_{21}(1) \\ & \searrow & & \nearrow & \\ & & r_{10}(1) = 9\% & & \\ & & & \searrow & \\ & & & & r_{20}(1). \end{array} \tag{14}$$

Notice that we have assumed that the binomial model "reconnects," that is, if the short rate first moves down and then up, we end up at the same short rate, r_{21}, that would be attained if the short rate first moved up and then down. This feature is attractive because it reduces the possible number of states in future periods, making computation easier. The requirement that the model reconnect is equivalent to assuming that the volatility of the short rate at date 1 be independent of its level, that is

$$\sigma_{11}(1) = \sigma_{10}(1). \tag{15}$$

Under the Ho and Lee (1986) assumption of normality, this implies[3]

$$\frac{1}{2}[\, r_{22}(1) - r_{21}(1)\,] \;=\; \frac{1}{2}[\, r_{21}(1) - r_{20}(1)\,] \tag{16}$$

or

$$r_{21}(1) \;=\; \frac{1}{2}[\, r_{22}(1) + r_{20}(1)\,]. \tag{17}$$

Equation (17) is one condition, we need two more to solve for all three short rates at date 2. If we again employ the no-arbitrage condition (5), we obtain

$$\begin{aligned} P_{11}(2) &= \tfrac{1}{2}[\, P_{22}(1) + P_{21}(1)\,]\, P_{11}(1) \\ P_{10}(2) &= \tfrac{1}{2}[\, P_{21}(1) + P_{20}(1)\,]\, P_{10}(1). \end{aligned} \tag{18}$$

or

$$\begin{aligned} e^{-2\, r_{11}(2)} &= \tfrac{1}{2}[\, e^{-r_{22}(1)} + e^{-r_{21}(1)}\,]\, P_{11}(1) \\ e^{-2\, r_{10}(2)} &= \tfrac{1}{2}[\, e^{-r_{21}(1)} + e^{-r_{20}(1)}\,]\, P_{10}(1). \end{aligned} \tag{19}$$

Using condition (5) once again also implies

$$\begin{aligned} P_0(3) &= \tfrac{1}{2}\,[\, P_{11}(2) \;+\; P_{10}(2)\,]\, P_0(1) \\ &= \tfrac{1}{2}\left[\, e^{-2\, r_{11}(2)} \;+\; e^{-2\, r_{10}(2)} \right] P_0(1). \end{aligned} \tag{20}$$

Substituting (19) into (20) gives

$$\begin{aligned} P_0(3) &= \tfrac{1}{2}\left[\, \tfrac{1}{2}[\, P_{22}(1) + P_{21}(1)\,]\, P_{11}(1) \;+\; \tfrac{1}{2}[\, P_{21}(1) + P_{20}(1)\,]\, P_{10}(1) \right] P_0(1) \\ &= \tfrac{1}{4}\left[\, P_{11}(1)e^{-r_{22}(1)} \;+\; [P_{11}(1) + P_{10}(1)]e^{-r_{21}(1)} \;+\; P_{10}(1)e^{-r_{20}(1)} \right] P_0(1). \end{aligned} \tag{21}$$

Since from the table, we have $P_0(3) = 0.75201$, and we have already solved in equation (13) for $P_{11}(1)$ and $P_{10}(1)$, equation (21) is a second condition that $r_{22}(1)$, $r_{21}(1)$, and $r_{20}(1)$ must satisfy. The remaining condition comes from the volatility of the two period bond yield which,

[3] If, instead, we made the Black, Derman, and Toy (1990) assumption of interest rate lognormality, then the reconnecting condition (15) would imply $\ln(r_{22}/r_{21}) = \ln(r_{21}/r_{20})$ or $r_{21} = \sqrt{r_{22}\, r_{20}}$.

given the normality assumption, is

$$\sigma_0(2) \;=\; \frac{1}{2}[\, r_{11}(2) - r_{10}(2)\,]. \tag{22}$$

Note from (19) that

$$e^{2\,[\,r_{11}(2)-r_{10}(2)\,]} \;=\; \frac{P_{10}(1)\;\left[\,e^{-r_{21}(1)} + e^{-r_{20}(1)}\,\right]}{P_{11}(1)\;\left[\,e^{-r_{22}(1)} + e^{-r_{21}(1)}\,\right]}. \tag{23}$$

Substituting (22) into (23), we have

$$e^{4\,\sigma_0(2)} \;=\; \frac{P_{10}(1)\;\left[\,e^{-r_{21}(1)} + e^{-r_{20}(1)}\,\right]}{P_{11}(1)\;\left[\,e^{-r_{22}(1)} + e^{-r_{21}(1)}\,\right]}. \tag{24}$$

This can be rewritten as

$$e^{-r_{22}(1)}\,P_{11}(1)\,e^{4\,\sigma_0(2)} \;+\; e^{-r_{21}(1)}\,\left[P_{11}(1)\,e^{4\,\sigma_0(2)} \;-\; P_{10}(1)\right] \;=\; e^{-r_{20}(1)}\,P_{10}(1). \tag{25}$$

Since $\sigma_0(2) = 0.009$ from the above table, and $P_{11}(1)$ and $P_{10}(1)$ are given in equation (13), equation (25) is a third condition for the date 2 states of the short rate. Solving equations (17), (21), and (25) numerically for $r_{22}(1)$, $r_{21}(1)$, and $r_{20}(1)$, one obtains the final short rate tree

$$\begin{array}{ccccc}
 & & & & r_{22}(1) = 12.103\% \\
 & & & \nearrow & \\
 & & r_{11}(1) = 11\% & & \\
 & \nearrow & & \searrow & \\
r_0(1) = 8\% & & & & r_{21}(1) = 10.503\% \\
 & \searrow & & \nearrow & \\
 & & r_{10}(1) = 9\% & & \\
 & & & \searrow & \\
 & & & & r_{20}(1) = 8.903\%.
\end{array} \tag{26}$$

Using our accumulated results, we can also construct a complete bond price tree:

$$
\begin{array}{ccccc}
 & & & & P_{22}(1) = 0.8860 \\
 & & & \nearrow & \\
 & & \begin{array}{c} P_{11}(1) = 0.8958 \\ P_{11}(2) = 0.8001 \end{array} & & \\
 & \nearrow & & \searrow & \\
\begin{array}{c} P_0(1) = 0.9231 \\ P_0(2) = 0.8353 \\ P_0(3) = 0.7520 \end{array} & & & & P_{21}(1) = 0.9003 \\
 & \searrow & & \nearrow & \\
 & & \begin{array}{c} P_{10}(1) = 0.9139 \\ P_{10}(2) = 0.8294 \end{array} & & \\
 & & & \searrow & \\
 & & & & P_{20}(1) = 0.9148.
\end{array}
\tag{27}
$$

What we have shown is that by assuming the absence of arbitrage opportunities (equation 1), specifying an initial term structure of interest rates and term structure of interest rate volatilities (as we did in the table), and making an assumption regarding the probability distribution of interest rates (for example, normality as in Ho and Lee, or lognormality as in Black, Derman, and Toy), one can uniquely determine a binomial lattice model of interest rates and bond prices that exactly fits initially observed bond prices and volatilities. While the construction of this model requires numerically solving a system of nonlinear equations at each stage of the tree, this is straightforward given current computer technology.

Example: Consider the value of a European call option that matures at date 2 and is written on a one period bond. The exercise price is assumed to be $X = 0.8900$. Using the above bond price tree, since $c_{2j} = max[\, P_{2j}(1) - X,\, 0\,]$, we have

$$
\begin{tikzcd}
 & & c_{22} = 0 \\
 & c_{11} \arrow[ur] \arrow[dr] & \\
c_0 \arrow[ur] \arrow[dr] & & c_{21} = 0.0103 \\
 & c_{10} \arrow[ur] \arrow[dr] & \\
 & & c_{20} = 0.0248.
\end{tikzcd}
\tag{28}
$$

To find the value of the call option at date 1, we apply the martingale pricing relation (1) to obtain

$$
c_{1j} \;= \tilde{E}_1[\, c_2\,]\, P_{1j}(1) \;= \begin{cases} \frac{1}{2}[c_{22} \,+\, c_{21}]\, P_{11}(1) \;= 0.0046 & \text{for } j = 1 \\ \frac{1}{2}[c_{21} \,+\, c_{20}]\, P_{10}(1) \;= 0.0160 & \text{for } j = 0, \end{cases} \tag{29}
$$

so that the option price tree is now

$$
\begin{tikzcd}
 & & c_{22} = 0 \\
 & c_{11} = 0.0046 \arrow[ur] \arrow[dr] & \\
c_0 \arrow[ur] \arrow[dr] & & c_{21} = 0.0103 \\
 & c_{10} = 0.0160 \arrow[ur] \arrow[dr] & \\
 & & c_{20} = 0.0248.
\end{tikzcd}
\tag{30}
$$

To compute the price of the option at the initial date, we once again apply equation (1):

$$c_0 \;=\; \tilde{E}_0[\,c_1\,]\,P_0(1) \;=\; \frac{1}{2}[\,c_{11} \;+\; c_{10}\,]\,P_0(1) \;=\; 0.0095. \tag{31}$$