Finance 400

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Arbitrage, Equivalent Martingale Measures, Risk-Neutral Valuation, and Pricing Kernels

In these notes, we formally show in that when asset prices follow diffusion processes and trading is continuous, then the absence of arbitrage allows us to value assets using the "risk-neutral" or "martingale pricing" technique. Under these conditions, a continuous-time "state price deflator" or "pricing kernel" also exists.

I. No Arbitrage Implies the Existence of an Equivalent Martingale Measure

Let S be the value of a risky asset which follows the process

$$dS = \mu_S dt + \sigma_S dW \tag{1}$$

where both μ_S and σ_S may be functions of S and t. For simplicity, we assume that μ_S, σ_S, and dW are scalar processes. Later we will discuss how the model can be generalized to allow for multiple sources of uncertainty. Now let $f(S,t)$ denote the value of a contingent claim whose payoff depends solely on S and t. From Itô's lemma, we know that this value satisfies

$$df = \mu_f dt + \sigma_f dW \tag{2}$$

where $\mu_f = f_t + \mu_S f_S + \frac{1}{2}\sigma_S^2 f_{SS}$ and $\sigma_f = \sigma_S f_S$, and the subscripts on f denote partial derivatives.

Next, following Black and Scholes (1973), consider forming a portfolio of -1 units of the contingent claim and f_S units of the risky asset. Let V be the value of this portfolio. Then

$$V = -f + f_S S \tag{3}$$

and the change in value of this portfolio over the next instant is

$$dV = -df + f_S dS \tag{4}$$

$$= -\mu_f dt - \sigma_f dW + f_S \mu_S dt + f_S \sigma_S dW$$
$$= [f_S \mu_S - \mu_f]\, dt$$

Since the portfolio is riskless, the absence of arbitrage implies that it must earn the risk-free rate. Denoting the (possibly stochastic) instantaneous risk-free rate as $r(t)$, we have[1]

$$dV = [f_S \mu_S - \mu_f]\, dt = rV dt = r[-f + f_S S] dt \tag{5}$$

which implies

$$f_S \mu_S - \mu_f = r[-f + f_S S] \tag{6}$$

If we substitute $\mu_f = f_t + \mu_S f_S + \frac{1}{2}\sigma_S^2 f_{SS}$ into (6), we obtain a Black-Scholes-Merton equilibrium partial differential equation.

$$\frac{1}{2}\sigma_S^2 f_{SS} + rSf_S - rf + f_t = 0 \tag{7}$$

However, consider a different interpretation of equation (6). Substitute $f_S = \frac{\sigma_f}{\sigma_S}$ into (6) and re-arrange to obtain

$$\frac{\mu_S - rS}{\sigma_S} = \frac{\mu_f - rf}{\sigma_f} \equiv \theta(t) \tag{8}$$

Condition (8) is the usual no-arbitrage condition that requires a unique market price of risk. We denote this price of risk as $\theta(t)$. Then the stochastic process for the contingent claim can be written as

$$df = \mu_f dt + \sigma_f dW = [rf + \theta\sigma_f]\, dt + \sigma_f dW \tag{9}$$

[1] For simplicity, we have assumed that the contingent claim's value depends only on a single risky asset price, $S(t)$. However, when the interest rate is stochastic, the contingent claim's value might be also a function of $r(t)$, that is, $f(S, r, t)$. If, for example, the interest rate followed the process $dr = \mu_r(r)\, dt + \sigma_r(r)\, dW_r$ where dW_r is an additional Brownian motion process affecting interest rate movements, then the contingent claim's process would be given by a bi-variate version of Itô's lemma. Also, to create a portfolio that earns an instantaneous risk-free rate, the portfolio would need to include a bond whose price is driven by dW_r. Later, we discuss how our results generalize to multiple sources of uncertainty. However, the current univariate setting can be fully consistent with stochastic interest rates if the risky asset is, itself, a bond so that $S(r, t)$ and $dW = dW_r$. The contingent claim could then be interpreted as a fixed-income (bond) derivative security.

We now apply Girsonov's Theorem, which says that a change in the Brownian motion process, which changes the future probability distribution for asset prices, allows us to adjust the drift of a diffusion process to almost any desired level. We do this by defining a process $\tilde{W}_t = W_t + \int_0^t \theta(\eta)\, d\eta$, so that $d\tilde{W}_t = dW_t + \theta(t)\, dt$. Then (9) can be rewritten:

$$df = rfdt + \sigma_f d\tilde{W} \tag{10}$$

Hence, converting from the Brownian motion process dW to $d\tilde{W}$, which removes the risk-premium $\theta\sigma_f$ from the first term on the right-hand side of (9), results in the expected rate of return of f being equal to the risk-free rate *if* we now viewed $d\tilde{W}$, rather than dW, as a Brownian motion process. The probability distribution of future values of f that are generated by $d\tilde{W}$, which we can denote as $\tilde{P}$, is referred to as the risk-neutral measure.[2] This risk-neutral measure is "equivalent" to the original "physical" probability measure for f, denoted as P, in that any future value of f that has positive probability (density) under the physical measure also has positive probability (density) under the risk-neutral measure. We can next show that the contingent claim's "deflated" price process will be a martingale under the probability measure $\tilde{P}$. Let $B(t)$ be the value of a "money market fund" which invests in the instantaneous maturity risk-free asset. Also assume $B(0) = 1$. Then

$$dB/B = r(t)dt \tag{11}$$

Note that $B(t) = B(0)\, e^{\int_0^t r(s)ds} = e^{\int_0^t r(s)ds}$. Now define $F(t) \equiv f(t)/B(t)$ as the deflated price process for the contingent claim. A trivial application of Itô's lemma gives

$$\begin{aligned} dF &= \frac{1}{B}df - \frac{f}{B^2}dB = rFdt + \frac{\sigma_f}{B}d\tilde{W} - rFdt \\ &= \sigma_F d\tilde{W} \end{aligned} \tag{12}$$

where $\sigma_F \equiv F\sigma_f/f$ is the instantaneous standard deviation of the return on F. Thus, the deflated price process under the equivalent probability measure generated by $d\tilde{W}$ follows a

[2] The risk-neutral probability measure is often referred to as the Q, rather than $\widetilde{P}$, measure.

martingale process:

$$F(t) = \tilde{E}_t \left[F(T) \right] \tag{13}$$

where $\tilde{E}_t$ denotes the expectation operator under the probability measure generated by $d\tilde{W}$. To summarize, the absence of arbitrage implies the existence of an equivalent probability measure such that the deflated price process is a martingale. Note that (13) holds for any deflated contingent claim, including the deflated underlying risky asset, S/B, since we could define the contingent claim as $f = S$.

Now if we re-write (13) in terms of the undeflated contingent claims price, we obtain

$$\begin{aligned} f(t) &= B(t)\tilde{E}_t \left[f(T) \frac{1}{B(T)} \right] \\ &= \tilde{E}_t \left[e^{-\int_t^T r(s)ds} f(T) \right] \end{aligned} \tag{14}$$

Equation (14) can be interpreted as a solution to the partial differential equation (7) and, indeed, is referred to as the Feynman-Kac solution.[3] From a computational point of view, equation (14) says that we can price (value) a derivative security by taking the expected value of its discounted payoff, where we discount at the risk-free rate but also assume that when taking the expectation of $f(T)$ that the rate of return on f (and all other asset prices, such as S) equals the risk-free rate. This an attractive way of calculating the derivative's value because no assumption regarding the market price of risk is required. This procedure is also the idea behind the Cox-Ross risk-neutral valuation technique. Equivalently, using equation (13) requires that to value $f(t)/B(t)$, we take the expectations of the deflated price process, where this deflated process has zero drift.

II. The No Arbitrage Condition Also Determines a Pricing Kernel

This is not the first time that we have taken expectations to value a security. Recall in the standard intertemporal consumption-portfolio choice problem with time-separable utility that

[3] To solve (7), a boundary condition for the derivative is needed. For example, in the case of a European call option it would be $f(T) = \max[0, S(T) - X]$. The solution given by (14) incorporates this boundary condition, $f(T)$.

we obtained the Euler condition

$$f(t) = E_t\left[f(h)\, M_h / M_t\right] \tag{15}$$

or

$$1 = E_t\left[R_h M_h / M_t\right] \tag{15'}$$

where $M_t = U_c(C_t, t)$ was the marginal utility of consumption at date t and R_h is the return on security f between dates t and h. M_t can be referred to as a *state price deflator* or state price density or *pricing kernel*. In general, a pricing kernel is a strictly positive Itô process such that the deflated price process, $f(t)\, M_t$, is a martingale. But note the difference here versus our earlier analysis: the expectation in (15) is taken under the physical probability measure, P, while in (13) and (14) the expectation is taken under the equivalent martingale measure, $\tilde{P}$.

In the standard time-separable utility portfolio choice model, M_t is the marginal utility of consumption. But the concept of a pricing kernel is more general than this. What we will show is that the absence of arbitrage opportunities, which earlier guaranteed the existence of an equivalent martingale measure, also determines a pricing kernel, M_t. In fact, the concepts of an equivalent martingale measure and state pricing kernel are one and the same.

Since we have defined the pricing kernel or state price deflator as an Itô process, we can write it as

$$dM = \mu_m dt + \sigma_m dW \tag{16}$$

Now consider the restrictions that the Black-Scholes no-arbitrage conditions place on μ_m and σ_m if (15) and (16) hold. For any arbitrary security, f, define $f^m = fM$ and apply Itô's lemma:

$$\begin{aligned} df^m &= f dM + M df + (df)(dM) \\ &= \left[f\mu_m + M\mu_f + \sigma_f \sigma_m\right] dt + \left[f\sigma_m + M\sigma_f\right] dW \end{aligned} \tag{17}$$

If $f^m = fM$ satisfies (15), that is, f^m is a martingale, then its drift must be zero, implying

$$\frac{\mu_f}{f} = -\frac{\mu_m}{M} - \frac{\sigma_f \sigma_m}{fM} \tag{18}$$

Now consider the case in which f is the instantaneously riskless asset, that is, $f(t) = B(t)$ is the money market investment following the process in equation (11). This implies that $\sigma_f = 0$ and $\mu_f / f = r(t)$. Using (18), requires

$$r(t) = -\frac{\mu_m}{M} \tag{19}$$

In other words, the expected rate of change of the pricing kernel must equal minus the instantaneous risk-free interest rate. Next, consider the general case where the asset f is risky, so that $\sigma_f \neq 0$. Using (18) and (19) together, we obtain

$$\frac{\mu_f}{f} = r(t) - \frac{\sigma_f \sigma_m}{fM} \tag{20}$$

or

$$\frac{\mu_f - rf}{\sigma_f} = -\frac{\sigma_m}{M} \tag{20'}$$

Comparing (20′) to (8), we see that

$$-\frac{\sigma_m}{M} = \theta(t) \tag{21}$$

Thus, the no-arbitrage condition implies that the form of the pricing kernel must be

$$dM/M = -r(t)\,dt - \theta(t)\,dW \tag{22}$$

Note that if we define $m \equiv \ln M$, then $dm = -\left[r + \frac{1}{2}\theta^2\right] dt - \theta dW$. Hence, in using the pricing kernel to value any contingent claim, we can re-write (15) as

$$f(t) = E_t\left[f(h)\, M_h/M_t\right] = E_t\left[f(h)\, e^{m_h - m_t}\right] \tag{23}$$

$$= \quad E_t \left[f(h) \, e^{-\int_t^h \left[r(s) + \frac{1}{2}\theta^2(s) \right] ds - \int_t^h \theta(s) dW(s)} \right]$$

Given processes for $r(s)$, $\theta(s)$ and the contingent claim's payoff, $f(h)$, it may be easier to compute (23) rather than, say (13) or (14). Of course, in computing (23), we need to use the actual drift for f, that is, we compute expectations under P, not $\tilde{P}$.

Technical Aside:

M_t can be related to the Radon-Nikodym derivative of $\tilde{P}$ with respect to P. Let P be the probability density at some future date, say $h > t$, that is generated by the physical Brownian motion process, $W(h)$, and let $\tilde{P}$ be the equivalent probability density generated by the risk-neutral process $\tilde{W}(h) = W(h) + \int_0^h \theta(\eta) \, d\eta$. Also let $X(h)$ be any random variable such that $\tilde{E}_t [\mid X(h) \mid] < \infty$. Then since P and $\tilde{P}$ are equivalent probability distributions, there exists a sequence of strictly positive random variables, ξ_t, such that

$$\tilde{E}_t[\xi_t X(h)] = E_t[\xi_h X(h)] \tag{24}$$

or

$$\tilde{E}_t[X(h)] = E_t[\xi_h X(h)]/\xi_t \tag{24'}$$

ξ is referred to as the Radon-Nikodym derivative of $\tilde{P}$ with respect to P, that is, $\xi_t \equiv E_t[d\tilde{P}/dP]$. For an economy in which prices are measured in terms of the money market investment, $F(t) = f(t)/B(t)$, so that the instantaneous risk-free rate is zero, then M_t is identical to ξ_t.

The above analysis has been done under the assumption of a single source of uncertainty, dW. In a straightforward manner, it can be extended to allow for n independent sources of risk. If we had asset returns depending, in general, on an $n \times 1$ vector of independent Brownian motion processes, $dW = (dW_1 ... dW_n)'$ with corresponding $n \times 1$ vector of market prices, $\theta = (\theta_1 ... \theta_n)'$, and if σ_f was a $1 \times n$ vector $\sigma_f = (\sigma_{f1} ... \sigma_{fn})$, then we would have the no-arbitrage condition

$$\mu_f - rf = \sigma_f \theta \tag{25}$$

Equations (13) and (14) would still hold, and now the pricing kernel's process would be given by

$$dM/M = -r(t)\,dt - \theta(t)'\,dW \tag{26}$$

III. Different Price Deflators

In section I above, we found it convenient to deflate a contingent claim price by the money market fund's price, $B(t)$. Sometimes, however, it may be convenient to deflate or "normalize" a contingent claims price by the price of a different type of security. Such a situation can occur when a contingent claim's payoff depends on multiple risky assets. Let's now consider an example of this, in particular, where the contingent claim is an option written on the difference between two securities' (stocks') prices. The date t price of stock 1, $S_1(t)$, follows the process

$$dS_1/S_1 = \alpha_1 dt + \sigma_1 dW_1 \tag{27}$$

and the date t price of stock 2, $S_2(t)$, follows the process

$$dS_2/S_2 = \alpha_2 dt + \sigma_2 dW_2 \tag{28}$$

where σ_1 and σ_2 are assumed to be constants and dW_1 and dW_2 are Brownian motion processes for which $dW_1 dW_2 = \rho dt$. Let $C(t)$ be the date t price of a European option written on the difference between these two stocks' prices. Specifically, at this option's maturity date, T, the value of the option equals

$$C(T) = max\left[0, S_1(T) - S_2(T)\right] \tag{29}$$

Now define $c(t) = C(t)/S_1(t)$, $s(t) \equiv S_1(t)/S_2(t)$, and $B(t) = S_2(t)/S_2(t) = 1$ as the deflated price processes, where the prices of the option, stock 1, and stock 2 all are normalized by the price of stock 2. With this normalized price system, the terminal payoff corresponding to (29) is now

$$c(T) = \max\left[0, s(t) - 1\right] \tag{30}$$

Applying Itô's lemma, the risk-neutral process for $s(t)$ is given by

$$ds/s = \alpha_s dt + \sigma_s dW_3 \tag{31}$$

where $\alpha_s \equiv \alpha_1 - \alpha_2 + \sigma_1^2 - \rho\sigma_1\sigma_2$, $\sigma_s dW_3 \equiv \sigma_1 dW_2 - \sigma_2 dW_2$, and $\sigma_s^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$. Further, when prices are measured in terms of stock 2, the deflated price of stock 2 becomes the riskless asset, with the riskless rate of return given by $dB/B = 0dt$, that is, the riskless rate of return equals zero. Using Itô's lemma once again, the deflated option price, $c\left(s\left(t\right), t\right)$, follows the process

$$dc = \left[c_s\, \alpha_s s + c_t + \frac{1}{2} c_{ss}\, \sigma_s^2 s^2\right] dt + c_s\, \sigma_s s\, dW_3. \tag{32}$$

With this deflated price system, the usual Black-Scholes (1973) hedge portfolio can be created from the option and stock 1. The hedge portfolio's value is given by

$$V = -c + c_s s \tag{33}$$

and the instantaneous change in value of the portfolio is

$$\begin{aligned} dV &= -\,dc + c_s ds \\ &= -\left[c_s \alpha_s s + c_t + \frac{1}{2} c_{ss}\, \sigma_s^2 s^2\right] dt - c_s\, \sigma_s s\, dW_3 + c_s \alpha_s s\, dt + c_s \sigma_s s\, dW_3 \\ &= -\left[c_t + \frac{1}{2} c_{ss}\, \sigma_s^2 s^2\right] dt \end{aligned} \tag{34}$$

When measured in terms of stock 2's price, the return on this portfolio is instantaneously riskless. In the absence of arbitrage, it must earn the riskless return, which as noted above, equals zero under this deflated price system. Thus we can write

$$dV = -\left[c_t + \frac{1}{2} c_{ss}\, \sigma_s^2 s^2\right] dt = 0 \tag{35}$$

which implies

$$c_t + \frac{1}{2} c_{ss}\, \sigma_s^2 s^2 = 0 \tag{36}$$

which is the usual Black-Scholes partial differential equation, but with the risk-free rate, r, set to zero. Solving it subject to the boundary condition (30), which implies a unit exercise price, gives the usual Black-Scholes formula

$$c(S,\, t) \;=\; s\,N(d_1) \;-\; N(d_2) \tag{37}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(s(t)\right) \;+\; \frac{1}{2}\sigma_s^2\left(T-t\right)}{\sigma_s\sqrt{T-t}} \\ d_2 &= d_1 \;-\; \sigma_{\,s}\sqrt{T-t}. \end{aligned} \tag{38}$$

To convert back to the undeflated price system, we simply multiply (37) by $S_2\left(t\right)$ and obtain

$$C(\,t) \;=\; S_1\,N(d_1) \;-S_2\,N(d_2) \tag{39}$$

Note that the option price does not depend on the risk-free rate for the non-deflated price system, $r\left(t\right)$. Hence, the formula holds even for stochastic interest rates. The same formula could be derived using the Black-Scholes hedging argument with the non-deflated price system. In this case, the hedge portfolio would involve a unit short position in the option, $C_{S_1} \equiv \partial C/\partial S_1$ shares of stock 1, and $c_{S_2} \;\equiv\; \partial C/\partial S_2$ shares of stock 2. The resulting equilibrium partial differential equation would then be

$$C_t \,+\, \frac{1}{2}C_{S_1S_1}\,\sigma_1^2{S_1}^2 + \frac{1}{2}C_{S_2S_2}\,\sigma_2^2{S_2}^2 + C_{S_1S_2}\rho\sigma_1\sigma_2S_1S_2 + C_{S_1}S_1r + C_{S_2}S_2r - rC = 0 \tag{40}$$

which when solved subject to the boundary condition $C\left(T\right) \,=\, \max\left[0, S_1\left(T\right) - S_2\left(T\right)\right]$ gives exactly the solution (39). Hence, hedging arguments based on a non-deflated price system are valid also for a deflated price system. Similarly, risk-neutral pricing and the existence of a pricing kernel also holds for the deflated price system.