Finance 400

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Options on an Asset that Yields Continuous Dividends

I. Risk-Neutral Price Appreciation in the Presence of Dividends

Options are often written on what can be interpreted as an asset that pays a continuous dividend that is proportional to the asset's price. Let us consider how to value such options for a general case, then we will look at specific examples. Suppose there is a European call option that is written on an asset that pays a continuous dividend yield equal to q. Using the risk-neutral valuation technique:

$$c = e^{-r\tau} \hat{E}\left[\max\left(0, S_T - X\right)\right] \tag{1}$$

where the risk-neutral expectation, $\hat{E}\left[\cdot\right]$, requires that the expected rate of return on the underlying asset equals the risk-free rate, r. This asset's *total rate of return* is composed of both *dividend payments* and *price appreciation* (capital gains). If no dividends are paid, then $\hat{E}\left[S_T\right] = Se^{r\tau}$. However, with dividends, the expected asset price appreciates at rate $r - q$, rather than r. This is because with dividends paid out at rate q, price appreciation must be at rate $r - q$ to keep the total expected rate of return $= q + r - q = r$. Thus, the risk-neutral expectation of S_T is

$$\begin{aligned} \hat{E}\left[S_T\right] &= Se^{(r-q)\tau} \qquad (2) \\ &= Se^{-q\tau}e^{r\tau} = S^* e^{r\tau} \end{aligned}$$

where we define $S^* \equiv Se^{-q\tau}$.

The above shows that the value of an option on a dividend-paying asset with current price S equals the value of an option on a non-dividend paying asset having current price $S^* = Se^{-q\tau}$. Hence, we can value a European call or put option on a dividend-paying asset using the Black-Scholes formula, but replacing S with $S^* = Se^{-q\tau}$.

$$c = Se^{-q\tau} N\left(d_1\right) - Xe^{-r\tau} N\left(d_2\right) \tag{3}$$

and

$$p = Xe^{-r\tau}N\left(-d_2\right) - Se^{-q\tau}N\left(-d_1\right) \tag{4}$$

where $d_1 = \frac{\ln\left(\frac{Se^{-q\tau}}{X}\right)+\left(r+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$, and $d_2 = d_1 - \sigma\sqrt{\tau}$.

II. Stock Index Options

Several exchanges trade options on U.S. and foreign stock market indices. They are cash settled: upon exercise, the holder receives the difference between the index and the strike price. Index options can be American or European.

A stock index equals the value of a stock portfolio. If this portfolio is assumed to follow a diffusion process with a constant rate of return variance, we can apply the previous formula. q is set equal to the average dividend yield on the stocks during the life of the option.

Example: Suppose a stock index currently equals 485.63. Consider the value of a put option with exercise price of 450 and time until maturity of 15 weeks. Assuming a (historically estimated) $q = 2.7\%$, $\sigma = 17\%$, $r = 6\%$, and $\tau = 15/52$, we have

$$p = Xe^{-r\tau}N\left(-d_2\right) - Se^{-q\tau}N\left(-d_1\right) = \$3.94$$

III. Foreign Currency Options

Foreign currency options are sold over-the-counter and on exchanges. Buying a foreign currency allows us to invest in a foreign bond paying the foreign interest rate r_f. Thus, a unit of foreign currency can be considered an asset having a dividend yield equal to r_f. Define the current U.S. \$ value of a unit of foreign currency as S, which is the spot exchange rate. S can then be viewed as the current U.S. \$ value of an investment paying a dividend of r_f. Thus the risk-neutral expectation of this asset's value at date T is

$$\begin{aligned} \hat{E}\left[S_T\right] &= Se^{(r-q)\tau} \\ &= Se^{\left(r-r_f\right)\tau} \end{aligned} \tag{5}$$

If the exchange rate is assumed to have a constant rate of return volatility equal to σ, the value of European puts and calls on a unit of foreign currency are

$$c = Se^{-r_f \tau} N(d_1) - Xe^{-r\tau} N(d_2) \tag{6}$$

$$p = Xe^{-r\tau} N(-d_2) - Se^{-r_f \tau} N(-d_1) \tag{7}$$

where $d_1 = \frac{\ln\left(\frac{Se^{-r_f \tau}}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$, and $d_2 = d_1 - \sigma\sqrt{\tau}$.

Recall that the forward exchange rate, F, is given by[1]

$$F = Se^{(r - r_f)\tau} \tag{8}$$

Substituting $Fe^{-r\tau} = Se^{-r_f \tau}$ into the above currency option formulas, they can be re-written in terms of the forward rate.

$$c = e^{-r\tau} \left[FN(d_1) - XN(d_2) \right] \tag{9}$$

$$p = e^{-r\tau} \left[XN(-d_2) - FN(-d_1) \right] \tag{10}$$

where $d_1 = \frac{\ln\left(\frac{F}{X}\right) + \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}$, and $d_2 = d_1 - \sigma\sqrt{\tau}$.

IV. Futures Options

Upon exercising a call option on a futures contract, the writer must deliver a *long position* in an underlying futures contract *plus cash* equal to the *difference* between the current *futures price* and the option *strike price*, $F_T - X$. Upon exercising a put option on a futures contract, the writer must deliver a *short position* in an underlying futures contract *plus cash* equal to the *difference* between the *strike price* and the current *futures price*, $X - F_T$. Since the holder can immediately close out the futures position at no cost, a futures option is essentially one that is written on a futures price. Options are written on futures of commodities, equities, bonds, and currencies. Futures options can be attractive since it may be easier to deliver an underlying futures contract rather than a physical commodity or asset.

[1] This is the same formula derived earlier, but with interest rates quoted on a continuously-compounded basis.

The relationship between spot and futures prices is often written in terms of the "cost of carry" formula:

$$F = Se^{(c-y)\tau} \equiv Se^{\alpha\tau} \tag{11}$$

where $\alpha \equiv c - y$ is the cost of carry less any convenience yield. For example, the cost of carry for a dividend-yielding security is $c = r - q$. Commodities are often considered to have a convenience yield, in which case y is non-zero. Assuming α is non-random, if S has a volatility of σ, so does F. Since futures options are written on F_T, not S_T, we need to consider the risk-neutral expectation of F_T.

The undiscounted profit from a long position in a futures contract which was entered into at the initial futures price of F and held until the maturity of the contract is $S_T - F = F_T - F$.[2] Since it costs zero to enter into a futures contract, what would be the expected profit in a risk-neutral world? It must be zero. Therefore,

$$\hat{E}\left[F_T - F\right] = 0 \tag{12}$$

or

$$F = \hat{E}\left[F_T\right] \tag{13}$$

Thus, while a non-dividend paying asset would be expected to grow at rate r, futures prices would be expected to grow at rate 0. Hence, futures are like assets with a dividend yield $q = r$. From this we can derive the value of futures options as:

$$c = e^{-r\tau}\left[FN\left(d_1\right) - XN\left(d_2\right)\right] \tag{14}$$

$$p = e^{-r\tau}\left[XN\left(-d_2\right) - FN\left(-d_1\right)\right] \tag{15}$$

[2]Technically, this is the profit on a forward contract. Futures contracts are "marked-to-market" daily, so that this would be the profit on a futures contract over a single day. However, in a risk-neutral world, the relationship in (12) holds for either forward or futures contracts.

where $d_1 = \frac{\ln\left(\frac{F}{X}\right)+\frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}$, and $d_2 = d_1 - \sigma\sqrt{\tau}$.

Example: Let the current futures price written on a stock index and having a time until maturity of 15 weeks be 490.10. Consider the value of a futures put option on this stock index having a strike price of 475 and also having a maturity of 15 weeks. Assuming $\sigma = 15\%$, $r = 6\%$, $\tau = 15/52$, and setting $F = 490.10$ and $X = 475$, we have

$$p = e^{-r\tau}\left[XN(-d_2) - FN(-d_1)\right] = \$8.95 \tag{16}$$

One can derive a put-call parity relationship for European futures options:

At date t:

Portfolio A: one futures call plus a bond having an initial value equal to $Xe^{-r\tau}$.

Portfolio B: one futures put plus a bond having an initial value equal to $Fe^{-r\tau}$ plus a long position in a futures contract.

At date T:

Portfolio A: $\max(0, F_T - X) + X = \max(F_T, X)$

Portfolio B: $\max(0, X - F_T) + F + F_T - F = \max(F_T, X)$

Since Portfolios A and B have the same date T value, their values at date t must be the same, implying

$$c + Xe^{-r\tau} = p + Fe^{-r\tau} \tag{17}$$