

## Finance 400

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### The Risk-Neutral Valuation Method

#### I. Basic Concepts and Pricing Forward Contracts

The “risk-neutral” technique is frequently used to value derivative securities. It was developed by John Cox and Stephen Ross in a 1976 article “The Valuation of Options for Alternative Stochastic Processes” *Journal of Financial Economics* 3, p.145-66. This note considers the intuition for this technique and the technique’s important applications for the case of the risk-free interest rate being constant. Later, we will see how the risk-neutral technique can be generalized to the case in which risk-free interest rates are stochastic.

Recall the result from the Cox, Ross, Rubinstein model that we derived simply by ruling out the existence of arbitrage:

$$C = \frac{1}{R_f} [pC_u + (1 - p)C_d]. \quad (1)$$

where  $R_f$  is the risk-free return (one plus the risk-free interest rate). The term  $[pC_u + (1 - p)C_d]$  is the one-period expected payoff on a call option *if* all investors were risk-neutral, since in this case the probability of an “up” move,  $q$ , equals  $p$ . Thus, the equation could be written as

$$C_t = \frac{1}{R_f} \hat{E}[\tilde{C}_{t+1}] \quad (2)$$

where  $C_t$  is the call option at time  $t$ , and  $\hat{E}[\tilde{C}_{t+1}]$  is the *expected value* of the call at time  $t+1$  *in a risk-neutral world*. Therefore,  $C_t$  is the “risk-neutral expected value” of  $\tilde{C}_{t+1}$  discounted at the risk-free rate, that is,

$$\frac{1}{R_f} = e^{-r}. \quad (3)$$

where  $r$  is defined as the continuously-compounded one-period interest rate. As we saw earlier, this “risk-neutral” valuation result is not just coincidental to options but will hold whenever markets are complete. For example, it can hold for valuing other securities, such as forward contracts.

A forward contract is an agreement where two parties agree to exchange an asset at some future date for a pre-agreed price. The long (short) party agrees to receive (deliver) the asset in exchange for paying (receiving) this pre-agreed price at a future date, say date  $T$ . No payment occurs between the parties at the time of the agreement of the contract, only at the maturity of the contract. The payoff to the long party at the maturity of the forward contract can be written as  $\tilde{S}_T - K$ , where  $\tilde{S}_T$  is the underlying asset price at the maturity of the forward contract and  $K$  is the previously agreed upon “delivery” price for the forward contract, which is said to equal the “forward price” at the agreement date of the contract. Assume that the underlying asset pays no dividends and define  $f$  as the current date  $t$  value of the forward contract and  $\tau \equiv T - t$  as its time until maturity. Now consider the following two portfolios:

At date  $t$ :

Portfolio A: A long position in one forward contract written on an asset having current value of  $S$  and having a forward price of  $K$

Portfolio B: One share of the underlying asset plus borrowing an amount  $e^{-r\tau}K$

At date  $T$ :

Portfolio A:  $\tilde{S}_T - K$

Portfolio B:  $\tilde{S}_T - K$

Since the two portfolios produce exactly the same cash flow at date  $T$ , the absence of arbitrage implies that their values at date  $t$  must also be the same, that is,  $f = S - e^{-r\tau}K$ . What can also be shown is that the risk-neutral technique can also be used to derive this no-arbitrage value for  $f$ . Applying the above risk-neutral method, the current value of a long position,  $f$ , is

$$f = e^{-r\tau} \hat{E}[\tilde{S}_T - K] = e^{-r\tau} \hat{E}[\tilde{S}_T] - e^{-r\tau}K. \quad (4)$$

What is  $\hat{E}[\tilde{S}_T]$ , the expected value of  $\tilde{S}_T$  in a risk-neutral world? In a risk-neutral world, the expected rate of return on  $S$  would be  $r$ , and  $\hat{E}[\tilde{S}_T] = Se^{r\tau}$ . Thus,

$$f = e^{-r\tau}Se^{r\tau} - e^{-r\tau}K = S - e^{-r\tau}K \quad (5)$$

which is the same as our earlier no-arbitrage derivation. Thus, the “risk-neutral” technique also works for forward contracts. Note when the contract is agreed to initially,  $f = 0$  and  $K = F$ , where  $F$  is the initial forward price. Therefore,

$$\begin{aligned} 0 &= S - Fe^{-r\tau}, & \text{or} \\ F &= Se^{r\tau}. \end{aligned} \tag{6}$$

To gain more intuition, consider the standard approach to valuing risky cashflows. The present value of a long position,  $f$ , equals its *expected cashflows* discounted at a *risk-adjusted rate of return*:

$$f = e^{-\theta\tau} E[\tilde{S}_T] - Ke^{-r\tau} \tag{7}$$

where  $\theta$  is the *true* expected rate of return on the risky asset and  $E[\tilde{S}_T]$  is the *true* expected value of  $\tilde{S}_T$ .  $K$  is discounted by the risk-free rate,  $r$ , since it is a certain, not risky, cashflow.  $\theta$  depends on risk aversion, and might be the result of the CAPM, for example,  $\theta = r + \beta(E[\tilde{r}_m] - r)$ .

If  $\theta$  is the true expected rate of return on the asset, then  $E[\tilde{S}_T] = Se^{\theta\tau}$ . Substituting in, we have

$$f = e^{-\theta\tau} Se^{\theta\tau} - e^{-r\tau} K = S - Ke^{-r\tau} \tag{8}$$

which is the same result as before. Note how the risk-neutral technique “works.” By replacing  $e^{-\theta\tau} E[\tilde{S}_T]$  with  $e^{-r\tau} \hat{E}[\tilde{S}_T]$ , we “pretend” the expected rate of return on the asset is  $r$  (rather than  $\theta$ ) and discount by  $r$  (rather than  $\theta$ ). However, these two “mistakes” cancel out, giving us the correct answer:

$$e^{-r\tau} \hat{E}[\tilde{S}_T] = e^{-\theta\tau} E[\tilde{S}_T] = S. \tag{9}$$

## II. Risk-Neutral Derivation of the Black-Scholes Formula

We now apply the risk-neutral technique to valuing a European call option on a non-dividend

paying stock:

$$c = e^{-r\tau} \hat{E}[\max(0, \tilde{S}_T - X)]. \quad (10)$$

To evaluate the “risk-neutral” expectation, we need to make an assumption regarding the probability distribution of  $\tilde{S}_T$ . If we assume that the stock price follows the binomial “tree” process considered earlier, we obtain the Cox, Ross, Rubinstein binomial option formula.

Let us now assume a different distribution for  $\tilde{S}_T$ , namely, the lognormal distribution. Define

$$\begin{aligned} \mu &= \text{annual expected rate of return on the stock,} \\ \sigma^2 &= \text{annualized variance of the rate of return on the stock.} \end{aligned} \quad (11)$$

If  $\tilde{S}_T$  is a lognormally distributed random variable, then  $\ln(\tilde{S}_T)$  is a normally distributed random variable:

$$\ln(\tilde{S}_T) \sim n(\ln(S) + (\mu - \sigma^2/2)\tau, \sigma^2\tau) \quad (12)$$

where  $n(\cdot)$  is the normal probability density function. The lognormal distribution is attractive because it allows  $\tilde{S}_T$  to take any possible value over the range 0 to  $\infty$ . Continuously compounded returns on the stock over unit time intervals,  $\ln(\tilde{S}_{t+1}) - \ln(\tilde{S}_t) = \ln(\tilde{S}_{t+1}/\tilde{S}_t)$ , are therefore normally distributed.

Let’s now calculate  $\hat{E}[\max(0, \tilde{S}_T - X)]$ , the call’s expected payoff in a risk-neutral world. If investors were risk-neutral, then  $\mu = r$ , so that

$$\ln(\tilde{S}_T) \sim n(\ln(S) + (r - \sigma^2/2)\tau, \sigma^2\tau). \quad (13)$$

Using this distributional assumption, we have<sup>1</sup>

$$c = e^{-r\tau} \hat{E}[\max(0, \tilde{S}_T - X)] = e^{-r\tau} \int_X^\infty (S_T - X) g(S_T) dS_T \quad (14)$$

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<sup>1</sup>As in the Black-Scholes option derivation, we need not assume that the *actual distribution* of the underlying security be lognormal, just that  $\sigma$  be constant.  $\mu$  could be time-varying. However, assuming the risk-free interest rate,  $r$ , is constant implies that the *risk-neutral distribution* of the underlying security is lognormal, that is, equation (13) holds.

where  $g(S_T)$  is the lognormal probability density function assuming  $\mu = r$ . Evaluating the integral in (14), one obtains the Black-Scholes formula

$$c = SN(d_1) - Xe^{-r\tau}N(d_2) \quad (15)$$

where  $N(\cdot)$  is the normal distribution function and

$$\begin{aligned} d_1 &= \frac{\ln(S/X) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\ d_2 &= d_1 - \sigma\sqrt{\tau} \end{aligned}$$

Using put-call parity, the price of a European put option is

$$\begin{aligned} p &= c + Xe^{-r\tau} - S \\ &= Xe^{-r\tau}N(-d_2) - SN(-d_1). \end{aligned} \quad (16)$$

### III. Some Properties of the Black-Scholes Formula

Consider the value of the European calls and puts when  $S$  becomes large. As  $S \rightarrow \infty$ ,  $d_1 \rightarrow \infty$  and  $d_2 \rightarrow \infty$ . Therefore  $N(d_1) \rightarrow 1$  and  $N(d_2) \rightarrow 1$ , but  $N(-d_1) \rightarrow 0$  and  $N(-d_2) \rightarrow 0$ . Hence, as  $S \rightarrow \infty$ , for European options:

$$\begin{aligned} \text{Call:} \quad & \lim_{S \rightarrow \infty} [SN(d_1) - Xe^{-r\tau}N(d_2)] = \infty, \\ \text{Put:} \quad & \lim_{S \rightarrow \infty} [Xe^{-r\tau}N(-d_2) - SN(-d_1)] = 0 \end{aligned}$$

as we would expect.

Conversely, as  $S \rightarrow 0$ ,  $d_1 \rightarrow -\infty$  and  $d_2 \rightarrow -\infty$ . Therefore  $N(d_1) \rightarrow 0$  and  $N(d_2) \rightarrow 0$ , but

$N(-d_1) \rightarrow 1$  and  $N(-d_2) \rightarrow 1$ . Hence, as  $S \rightarrow 0$ , for European options:

$$\begin{aligned} \text{Call:} \quad & \lim_{S \rightarrow 0} [S N(d_1) - X e^{-r\tau} N(d_2)] = 0, \\ \text{Put:} \quad & \lim_{S \rightarrow 0} [X e^{-r\tau} N(-d_2) - S N(-d_1)] = X e^{-r\tau} \end{aligned}$$

Consider what happens when the stock's volatility,  $\sigma$ , becomes small. If  $\ln(S/X) + r\tau > 0$  (equivalent to  $S > X e^{-r\tau}$ ), then as  $\sigma \rightarrow 0$ ,  $d_1 \rightarrow \infty$  and  $d_2 \rightarrow \infty$  so that

$$\begin{aligned} \text{Call:} \quad & \lim_{\sigma \rightarrow 0} c \Big|_{S > X e^{-r\tau}} = S - X e^{-r\tau}, \\ \text{Put:} \quad & \lim_{\sigma \rightarrow 0} p \Big|_{S > X e^{-r\tau}} = 0. \end{aligned}$$

Instead, if  $S < X e^{-r\tau}$ , as  $\sigma \rightarrow 0$ ,  $d_1 \rightarrow -\infty$  and  $d_2 \rightarrow -\infty$  so that

$$\begin{aligned} \text{Call:} \quad & \lim_{\sigma \rightarrow 0} c \Big|_{S < X e^{-r\tau}} = 0, \\ \text{Put:} \quad & \lim_{\sigma \rightarrow 0} p \Big|_{S < X e^{-r\tau}} = X e^{-r\tau} - S. \end{aligned}$$

This makes intuitive sense as well.

#### IV. Estimating the Volatility Parameter

The Black-Scholes formula depends on  $S$ ,  $X$ ,  $r$ ,  $\tau$ , and  $\sigma$ .  $r$  is usually taken as the risk-free interest rate on an investment maturing in  $\tau$  periods, at the expiration of the option contract. The only parameter that is not directly observable is the stock's volatility,  $\sigma$ . This can be estimated using historical data on stock prices or stock returns.

Recall that if  $\mu$  is assumed to be constant, then  $\ln(\tilde{S}_T) \sim n (\ln(S) + (\mu - \sigma^2/2)\tau, \sigma^2\tau)$ , so that  $\ln(\tilde{S}_T/S) \sim n ((\mu - \sigma^2/2)\tau, \sigma^2\tau)$ . Now suppose we observe  $n+1$  stock prices (or  $n$  stock returns) at intervals of length  $\tau$ . Define  $u_i \equiv \ln(S_i/S_{i-1})$  as the continuously compounded stock return over the  $i^{\text{th}}$  interval. Then  $u_i \sim n ((\mu - \sigma^2/2)\tau, \sigma^2\tau)$  and we can use the usual estimates for a sample mean and sample variance to calculate  $\sigma$

$$\text{Sample mean: } \bar{u} = \frac{1}{n} \sum_{i=1}^n u_i \tag{17}$$

$$\text{Sample variance: } \hat{s}^2 = \frac{1}{n-1} \sum_{i=1}^n (u_i - \bar{u})^2 \quad (18)$$

Since,  $\hat{s}^2$  is the estimate of  $\sigma^2\tau$ , the estimate for  $\sigma$  to be used in the Black-Scholes formula is

$$\sigma = \sqrt{\hat{s}^2/\tau} = \frac{\hat{s}}{\sqrt{\tau}} \quad (19)$$

where  $\hat{s}$  = sample standard deviation of the  $u_i$ 's.

If daily data is used to calculate  $\sigma$ , rather than using  $\tau = \frac{1}{365}$ , the practice is to use  $\tau = \frac{1}{250}$ , where 250 is approximately the number of trading days (days stock exchanges are open) per year. Reason: empirical evidence finds stock volatility is much higher on trading days. We can ignore days when exchanges are closed.

Example: You are given 180 daily percentage holding period returns,  $r_i \equiv \frac{S_i - S_{i-1}}{S_{i-1}}$ , for a stock. How would you calculate  $\sigma$ ? Note that  $r_i = \frac{S_i - S_{i-1}}{S_{i-1}} = \frac{S_i}{S_{i-1}} - 1$ . Thus, we form  $u_i = \ln(1 + r_i)$  and find the sample standard deviation of the  $u_i$ 's with  $n = 180$ :

$$\hat{s} = \sqrt{\frac{1}{179} \sum_{i=1}^{180} (u_i - \bar{u})^2}. \quad (20)$$

Since there are approximately 250 trading days per year,  $\tau = \frac{1}{250}$  and so  $\sigma = \frac{\hat{s}}{\sqrt{\tau}} = \sqrt{250} \hat{s}$ .

*Implied volatility* is the value of  $\sigma$  such that the *actual option price* equals the *theoretical* (Black-Scholes formula) *option price*. Thus, if the actual market price of a call option is  $c_a$  and  $c(S, X, r, \tau, \sigma)$  is the theoretical formula, implied volatility,  $\hat{\sigma}$ , satisfies

$$c_a = c(S, X, r, \tau, \hat{\sigma}). \quad (21)$$

Example: An October 70 call on Amgen sells for \$6 and the non-dividend paying stock price is

\$66. Implied volatility satisfies

$$6 = c(S = 66, X = 70, r = .06, \tau = \frac{8}{12}, \hat{\sigma}). \quad (22)$$

Numerically solving, one obtains  $\hat{\sigma} \cong 0.3053$ .

An implied volatility,  $\hat{\sigma}$ , has several uses:

1. The  $\hat{\sigma}$  from one option can be used to price another option on the same stock. This will give an indication of which options are relatively overvalued or undervalued.
2. The  $\hat{\sigma}$  from an option (or a weighted average of  $\hat{\sigma}$ 's from many options on the same stock) can be viewed as a measure of investors' forecast of the future risk of a stock. For example, an increase in the  $\hat{\sigma}$  of a firm's stock may indicate that the firm is now believed to be a takeover candidate.

## V. Modifying the Formula for Dividends

Consider a European option on a dividend paying stock. Assume that the stock's dividends are known over the life of the option. The current value of the stock,  $S$ , equals the present value of (assumed) riskless dividends paid during the life of the option,  $D$ , and a risky component,  $\hat{S}$ , equal to the expected present value of both capital gains and other risky dividends.

$$S = D + \hat{S} \quad (23)$$

Since the price of the stock will decline when dividends are paid,  $D$  will not be received by the holder of a call, nor paid by the holder of a put. Therefore, the option is essentially written on  $\hat{S}$ , not on  $S = D + \hat{S}$ . Thus, we can value a European call or put by using the Black-Scholes formula replacing  $S$  with  $\hat{S} = S - D$ .

Example: A stock is currently priced at \$75 and has a  $\sigma$  of 0.35. It will pay two \$1 dividends in 1 month and 4 months. If  $r = 6\%$ , what is the value of a European put option with an exercise



price of \$65 maturing in 6 months?

$$D = \$1 e^{-0.06(\frac{1}{12})} + \$1 e^{-0.06(\frac{4}{12})} \cong \$1.9752.$$

Thus,  $\hat{S} = S - D \cong \$75 - \$1.9752 \cong \$73.02$ .

$$\begin{aligned} d_1 &= \frac{\ln(\hat{S}/X) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\ &\cong \frac{\ln(73.02/65.00) + (0.06 + \frac{1}{2}0.35^2)\frac{6}{12}}{0.35\sqrt{\frac{6}{12}}} \\ &\cong 0.715 \\ d_2 &= d_1 - \sigma\sqrt{\tau} \\ &\cong 0.715 - 0.35\sqrt{\frac{6}{12}} \\ &\cong 0.468. \end{aligned}$$

$N(-0.715) \cong 0.2373$  and  $N(-0.468) \cong 0.3199$ , so

$$\begin{aligned} p &= X e^{-r\tau} N(-d_2) - \hat{S} N(-d_1) \\ &\cong \$65 e^{-0.06(\frac{6}{12})} (0.3199) - \$73.02 (0.2373) \\ &\cong \$20.179 - \$17.328 \\ &\cong \$2.85. \end{aligned}$$