Finance 400

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An Equilibrium Model of the Term Structure of Interest Rates

When bond prices are assumed to be driven by continuous-time stochastic processes, no-arbitrage restrictions and continuous-trading can lead to equilibrium relationships between the prices of different maturity bonds. Many equilibrium bond pricing models assume that a single source of uncertainty affects bonds of all maturities. In these so-called "one-factor" bond pricing models, it is often convenient to think of this uncertainty as being summarized by the yield on the shortest (instantaneous) maturity bond, $r(t)$.[1] Define $P(t,\tau)$ as the date t price of a bond that pays \$1 in τ periods. The instantaneous rate of return on this bond is $\frac{dP(t,\tau)}{P(t,\tau)}$. Also note that $P(t,0) = \$1$. The instantaneous yield, $r(t)$, is defined as

$$\lim_{\tau \to 0} \frac{dP(t,\tau)}{P(t,\tau)} \equiv r(t)\,dt \tag{1}$$

The model of Oldrich Vasicek (1977) "An Equilibrium Characterization of the Term Structure of Interest Rates," *Journal of Financial Economics* 5, p.177-188 assumes $r(t)$ follows an Ornstein-Uhlenbeck process

$$dr(t) = \alpha\left[\gamma - r(t)\right]dt + \sigma dz \tag{2}$$

where α, γ, and σ are > 0. σ measures the instantaneous volatility of $r(t)$ while α measures the strength of the process's mean reversion to γ, the unconditional mean value of the process. In discrete time, (2) is equivalent to a normally distributed, auto-regressive (1) process.

Now assume bond prices of all maturities depend on the current level of $r(t)$, $P(r(t),\tau)$. Itô's lemma implies

$$dP(r,\tau) = \frac{\partial P}{\partial r}dr + \frac{\partial P}{\partial t}dt + \tfrac{1}{2}\frac{\partial^2 P}{\partial r^2}(dr)^2 \tag{3}$$

[1] We note that another approach is to assume that *forward* interest rates of all maturities are affected by one or more sources of risk. David Heath, Robert Jarrow, and Andrew Morton (1992) "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica* 60, p.77-106, pioneered this approach.

$$
\begin{aligned}
&= \left[P_r \alpha(\gamma - r) + P_t + \tfrac{1}{2} P_{rr} \sigma^2\right] dt + P_r \sigma dz \\
&\equiv \mu(r,\tau) P(r,\tau) dt - \sigma_p(\tau) P(r,\tau) dz
\end{aligned}
$$

where $\mu(r,\tau) \equiv \frac{\left[P_r \alpha(\gamma - r) + P_t + \frac{1}{2} P_{rr} \sigma^2\right]}{P(r,\tau)}$ and $\sigma_p(\tau) \equiv -\frac{P_r \sigma}{P(r,\tau)}$ are the mean and standard deviation, respectively, of the bond's instantaneous rate of return.

Consider forming a portfolio containing one bond of maturity τ_1 and $-\frac{\sigma_p(\tau_1)P(r,\tau_1)}{\sigma_p(\tau_2)P(r,\tau_2)}$ units of a bond with maturity τ_2. Then if we continually re-adjust the amount of the τ_2-maturity bonds to equal $-\frac{\sigma_p(\tau_1)P(r,\tau_1)}{\sigma_p(\tau_2)P(r,\tau_2)}$ as $r(t)$ changes, the value of this portfolio is

$$
\begin{aligned}
W(t) &= P(r,\tau_1) - \frac{\sigma_p(\tau_1) P(r,\tau_1)}{\sigma_p(\tau_2) P(r,\tau_2)} P(r,\tau_2) \\
&= P(r,\tau_1)\left[1 - \frac{\sigma_p(\tau_1)}{\sigma_p(\tau_2)}\right]
\end{aligned} \tag{4}
$$

The portfolio's instantaneous return is

$$
\begin{aligned}
dW(t) &= dP(r,\tau_1) - \frac{\sigma_p(\tau_1) P(r,\tau_1)}{\sigma_p(\tau_2) P(r,\tau_2)} dP(r,\tau_2) \\
&= \mu(r,\tau_1) P(r,\tau_1) dt - \sigma_p(\tau_1) P(r,\tau_1) dz - \frac{\sigma_p(\tau_1)}{\sigma_p(\tau_2)} P(r,\tau_1) \mu(r,\tau_2) dt + \sigma_p(\tau_1) P(r,\tau_1) dz \\
&= \mu(r,\tau_1) P(r,\tau_1) dt - \frac{\sigma_p(\tau_1)}{\sigma_p(\tau_2)} P(r,\tau_1) \mu(r,\tau_2) dt
\end{aligned} \tag{5}
$$

Since the portfolio return is riskless at each instant of time, the absence of arbitrage implies that its rate of return must equal the instantaneous riskless interest rate, $r(t)$:

$$
\begin{aligned}
dW(t) &= \left[\mu(r,\tau_1) - \frac{\sigma_p(\tau_1)}{\sigma_p(\tau_2)} \mu(r,\tau_2)\right] P(r,\tau_1) dt \\
&= r(t) W(t) dt = r(t)\left[1 - \frac{\sigma_p(\tau_1)}{\sigma_p(\tau_2)}\right] P(r,\tau_1) dt
\end{aligned} \tag{6}
$$

The implication of equation (6) is

$$\frac{\mu\left(r,\tau_1\right)-r\left(t\right)}{\sigma_p\left(\tau_1\right)}=\frac{\mu\left(r,\tau_2\right)-r\left(t\right)}{\sigma_p\left(\tau_2\right)} \tag{7}$$

Equation (7) says that the "excess" rates of return on bonds, divided by their standard deviations, must be equal. This must hold for any sets of bonds τ_1, τ_2, τ_3, etc. Assuming this "market price of interest rate risk" is constant over time and equal to q, we have for any bond maturity, τ

$$\frac{\mu\left(r,\tau\right)-r\left(t\right)}{\sigma_p\left(\tau\right)}=q \tag{8}$$

or

$$\mu\left(r,\tau\right)=r\left(t\right)+q\sigma_p\left(\tau\right)$$

Substituting $\mu\left(r,\tau\right)$ and $\sigma_p\left(\tau\right)$ from Itô's lemma into (8) and simplifying, we obtain

$$P_r\alpha\left(\gamma-r\right)+P_t+\tfrac{1}{2}P_{rr}\sigma^2=rP-q\sigma P_r \tag{9}$$

This can be re-written as

$$\tfrac{\sigma^2}{2}P_{rr}+\left(\alpha\gamma+q\sigma-\alpha r\right)P_r-rP+P_t=0 \tag{10}$$

Equation (10) is the equilibrium partial differential equation that all bonds must satisfy. Since $\tau\equiv T-t$, so that $P_t\equiv\frac{\partial P}{\partial t}=-\frac{\partial P}{\partial\tau}\equiv-P_\tau$, equation (10) can be re-written as

$$\tfrac{\sigma^2}{2}P_{rr}+\left(\alpha\gamma+q\sigma-\alpha r\right)P_r-rP-P_\tau=0 \tag{11}$$

and solved subject to the boundary condition that at $\tau=0$, the bond price equals \$1, that is, $P\left(r,0\right)=1$. Doing so, gives the following solution

$$P\left(r\left(t\right),\tau\right)=A\left(\tau\right)e^{-B(\tau)r(t)} \tag{12}$$

where

$$
\begin{aligned}
B(\tau) &\equiv \frac{1-e^{-\alpha\tau}}{\alpha} && (13)\\
A(\tau) &\equiv \exp\left[(B(\tau)-\tau)\left(\gamma+q\frac{\sigma}{\alpha}-\tfrac{1}{2}\frac{\sigma^2}{\alpha^2}\right)-\frac{\sigma^2 B(\tau)^2}{4\alpha}\right] && (14)
\end{aligned}
$$

Using equation (12) we see that

$$\sigma_p(\tau) = -\sigma\frac{P_r}{P} = \sigma B(\tau) = \frac{\sigma}{\alpha}\left(1-e^{-\alpha\tau}\right) \tag{15}$$

which implies that a bond's standard deviation of its rate of return (volatility) increases with τ, but at a decreasing rate. In addition, from equation (8) we see that a bond's expected rate of return increases (decreases) with its time until maturity if q is positive (negative).

The *duration* (or interest rate sensitivity) of a coupon bond, or a portfolio of zero-coupon bonds, is defined to be the *maturity of a zero coupon bond* that has the same rate of return standard deviation, $\sigma_p(\tau)$, as the coupon bond.[2] Thus, if σ_c is the standard deviation of the rate of return of a given coupon bond, its duration, D, is that which satisfies

$$\sigma_c = \sigma_p(D) \tag{16}$$

or for the above Vasicek model

$$\sigma_c = \frac{\sigma}{\alpha}\left(1-e^{-\alpha D}\right) \tag{17}$$

which implies

$$D = -\frac{1}{\alpha}\ln\left(1-\frac{\alpha\sigma_c}{\sigma}\right) \tag{18}$$

<u>Valuing options on bonds</u>

The *Black-Scholes* option-pricing model assumed that interest rates are *constant.* Often, this may be a reasonable approximation when valuing options on assets, such as stocks, where

[2]This definition is that used by J. Cox, J. Ingersoll, and S. Ross (1979) "Duration and the Measurement of Basis Risk," *Journal of Business* 52, 51-61.

changes in the value of the asset do not result primarily from changes in interest rates. However, for options written on bond-like assets (fixed-income securities), it is clearly unreasonable to assume that they are unaffected by interest rates.

Another assumption of the Black-Scholes formula is at odds with valuing options on bonds. The Black-Scholes derivation assumes that the volatility of the underlying asset, σ, is constant over the life of the option. This is counter to the observation that a bond's volatility, σ_p, tends to shrink as it approaches maturity: unlike stocks, a bond's value must equal its known face (par) value when it matures. For short-term options on long maturity bonds (for example, Treasury bonds), assuming σ_p is constant over the life of the option is not unreasonable. Making this assumption of a constant σ_p, one can use the Black-Scholes as an approximation. Defining r as the interest rate on an investment that matures at the expiration of the option, then the values of call and put options on a bond with current value P are:

$$c \approx PN(d_1) - e^{-r\tau} XN(d_2) \tag{19}$$

and

$$p \approx e^{-r\tau} XN(-d_2) - PN(-d_1) \tag{20}$$

where $d_1 = \frac{\ln\left(\frac{P}{X}\right) + \left(r + \frac{\sigma_p^2}{2}\right)\tau}{\sigma_p\sqrt{\tau}}$ and $d_2 = d_1 - \sigma_p\sqrt{\tau}$.

To more precisely model the notion that a discount bond's volatility shrinks as it approaches maturity, we use the Vasicek model bond volatility

$$\sigma_p(T) = \frac{\sigma}{\alpha}\left(1 - e^{-\alpha T}\right) \tag{21}$$

where T is the bond's time until maturity and σ and α are positive constants. Note that as $T \to 0$, then $\sigma_p \to 0$ as well.

With this new modeling of bond volatility, $\sigma_p(T)$, one can solve for the value of European call and put options on a discount bond (zero-coupon bond) with maturity T. Again defining r as the interest rate on an investment that matures at the expiration of the option, the value of options on a discount bond with current value P and time to maturity T are

$$c = PN(d_1) - e^{-r\tau} XN(d_2) \tag{22}$$

and

$$p = e^{-r\tau} XN(-d_2) - PN(-d_1) \tag{23}$$

where $d_1 = \frac{\ln\left(\frac{P}{X}\right) + r\tau + \frac{\sigma_b^2}{2}}{\sigma_b}$, $d_2 = d_1 - \sigma_b$, and $\sigma_b^2 = \frac{\sigma^2}{2\alpha^4}\left(1 - e^{-2\alpha\tau}\right)^2 \left(1 - e^{-\alpha T}\right)^2$. See Robert C. Merton (1973) "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science* 4, p.141-183 and F. Jamshidian (1989) "An Exact Bond Option Formula," *Journal of Finance* 44, p.205-209 for a proof of this.

There are numerous other models that are used to value options on fixed-income securities. Many of these models use more complex ways of modeling bond volatility. In addition, numerical methods for estimating options on fixed-income securities using binomial tree techniques (c.f. the Cox-Ross-Rubinstein approach) are frequently used. See T. Ho and S. Lee (1986) "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance* 41, p.1011-1029 for an example of this approach.