Finance 400

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Option Pricing in Continuous-Time and the Black-Scholes Equation

Having introduced diffusion processes and Itô's lemma, we can now derive the Black-Scholes-Merton option pricing equation. This partial differential equation is found for the case of an option written on a stock that pays no dividends over the life of the option.

Suppose the price of a share of stock, $S(t)$, follows the process

$$dS = \mu S\, dt + \sigma S\, dz. \tag{1}$$

where it is assumed that μ may be a function of S and t, that is, $\mu(S,t)$, but that σ is a constant. Consider the value of a European call option on this stock, $c(S,\,t)$. The option depends on calendar time since at the expiration date $t = T$, we have

$$c(S(T),T) = max[\,0,\, S(T) - X]. \tag{2}$$

What is the process followed by c prior to maturity? Itô's lemma says

$$dc = \left[\frac{\partial c}{\partial S}\,\mu S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2 \right] dt + \frac{\partial c}{\partial S}\,\sigma S\, dz. \tag{3}$$

Now consider forming a portfolio of -1 units of the derivative security (call option) and $+\frac{\partial c}{\partial S}$ units (shares) of the stock.Let $W(t)$ be the value of this portfolio at time t. Then

$$W(t) = -c(t) + \left(\frac{\partial c}{\partial S}\right) S(t) \tag{4}$$

and the instantaneous change in value of this portfolio is[1]

[1] As will be shown, $c\,(S,t)$ is a non-linear function of S and t, so that $\partial c/\partial S$ varies over time as S and t change. Thus, equation (5) implies that the number of shares invested in the stock is not constant. Rather, continuous rebalancing of the stock is required to create this portfolio process. We defer until our discussion of consumption and portfolio choice problems in continuous-time to show that the continuous-time portfolio process in (5) can be derived as the limit of a discrete-time portfolio process.

$$
\begin{aligned}
dW(t) &= -\,dc(t) + \left(\frac{\partial c}{\partial S}\right) dS(t) \qquad (5)\\
&= -\left[\frac{\partial c}{\partial S}\,\mu S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2\right] dt - \left(\frac{\partial c}{\partial S}\right)\sigma S\,dz + \left(\frac{\partial c}{\partial S}\right)\mu S\,dt + \left(\frac{\partial c}{\partial S}\right)\sigma S\,dz\\
&= -\left[\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2\right] dt.
\end{aligned}
$$

Note that the return on this portfolio is instantaneously riskless. By continually re-adjusting the number of shares held in the stock so that it always equals $\partial c/\partial S$, the risk of the option is perfectly hedged. Dynamic trading in the stock is able to replicate the risk of the option because both the option and stock depend on the same (continuous-time) Brownian motion process, dz. In this sense, when assets following continuous-time stochastic processes, dynamic (continuous) trading can lead to a complete market and price contingent claims.

Since the rate of return on this "hedge" portfolio is riskless, to avoid arbitrage it must equal the competitive risk-free rate of return. Assuming this risk-free rate equals a constant, r, we have:

$$
\begin{aligned}
dW(t) &= -\left[\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2\right] dt. \qquad (6)\\
&= rW(t)dt\\
&= r\left[-c(t) + \left(\frac{\partial c}{\partial S}\right) S(t)\right] dt
\end{aligned}
$$

Thus, equation (6) implies

$$
\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\,\sigma^2 S^2 + r\,S\,\frac{\partial c}{\partial S} - r\,c = 0. \qquad (7)
$$

which is the Black-Scholes partial differential equation. The call option's value must satisfy this partial differential equation subject to the boundary condition

$$
c(S(T),\,T) = max[\,0,\,S(T) - X] \qquad (8)
$$

The solution is

$$c(S(t),\, t) \;=\; S(t)\, N(d_1) \;-\; X\, e^{-r\,(T-t)}\, N(d_2) \tag{9}$$

where

$$\begin{aligned} d_1 \quad &= \quad \frac{\ln\left(S(t)/X\right) \;+\; \left(r + \frac{1}{2}\sigma^2\right)\, (T-t)}{\sigma\, \sqrt{T-t}} \\ d_2 \quad &= \quad d_1 \;-\; \sigma\, \sqrt{T-t}. \end{aligned} \tag{10}$$

and $N(\cdot)$ is the standard normal distribution function. Similar to the Cox-Ross-Rubinstein option pricing formula, the value of the call option does not depend on the stock's expected rate of return, μ, only its current price, $S(t)$, and volatility, σ.