Finance 400

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Option Pricing

Earlier we derived general pricing relationships for contingent claims in terms of an equilibrium stochastic discount factor or in terms of elementary securities. Let us now consider a specific type of contingent claim, namely an option written on another "underlying" asset. We will show that the absence of arbitrage opportunities places restrictions on the price of an option in terms of its underlying asset's price.[1] In general, these restrictions do not determine an exact price for the option: an exact pricing formula requires additional assumptions regarding the probability distribution of the underlying asset's returns. However, the bounds on option prices presented in these notes must hold for any possible distribution of underlying asset returns.

Basic Characteristics of Option Prices

The owner of a call option has the right, but not the obligation, to buy a given asset or commodity in the future at a pre-agreed price, know as the exercise price or "strike" price. Similarly, the owner of a put option has the right, but not the obligation, to sell a given asset or commodity in the future at a pre-agreed price. For each owner (buyer) of an option, there is an option seller, also referred to as the option "writer." If the owner of a call (put) option chooses to exercise, the seller must deliver (receive) the underlying asset or commodity in return for receiving (paying) the pre-agreed exercise price.

Options can have different features regarding which future date(s) that exercise can occur. A European option can be exercised only at the maturity of the option contract, while an American option can be exercised at any time prior to the maturity of the contract.

Since an option always has a non-negative value to the owner, this buyer of the option must make an initial payment to the seller of the option.[2] Let us define the following notation:

$$S = \text{current price of the underlying asset (stock)}$$

[1] Thus, our approach is in the spirit of considering the underlying asset as an "elementary" security and using no-arbitrage restrictions to derive implications for the option's price.

[2] The owner of an option will choose to exercise it only if it is profitable to do so. The owner can always let the option expire unexercised, in which case its resulting value would be zero.

$$
\begin{aligned}
X &= \text{excercise (strike) price of the option} \\
t &= \text{current date. } T = \text{maturity date of option. } \tau \equiv T - t \\
S_T &= \text{asset price at date } T \\
r &= \text{risk-free interest rate between } t \text{ and } T \\
c &= \text{value of a European call option on the asset} \\
p &= \text{value of a European put option on the asset} \\
C &= \text{value of an American call option on the asset} \\
P &= \text{value of an American put option on the asset}
\end{aligned}
$$

The maturity values of European call and put options can be written as:

$$
c_T = max(S_T - X, 0) \tag{1}
$$

$$
p_T = max(X - S_T, 0) \tag{2}
$$

<u>Results on Option Prices</u>

- <u>Result 1: Upper Bounds for Option Prices</u>

$$
c(X, S, \tau) \leq C(X, S, \tau) \leq S \tag{3}
$$

Suppose not. If $C > S$ then an arbitrage is to buy the stock and sell the call. This would result in an initial profit equal to $C - S > 0$. At expiration of the option (expiry), another profit equal to $S_T - max(0, S_T - X) > 0$ would be made.

$$
p(X, S, \tau) \leq X e^{-r\tau} \tag{4}
$$

$$
P(X, S, \tau) \leq X \tag{5}
$$

Suppose not. If $p > X e^{-r\tau}$, then an arbitrage is to sell the put and invest the proceeds at the riskless interest rate. At expiry,
the profit equals $pe^{r\tau} - max(X - S_T, 0) \geq X - max(X - S_T, 0) \geq 0$.

- Result 2: Lower Bounds for Option Prices

$$C(X,\, S,\, \tau) \;\; \geq \;\; c(X,\, S,\, \tau) \, \geq \, 0 \tag{6}$$

$$P(X,\, S,\, \tau) \;\; \geq \;\; p(X,\, S,\, \tau) \, \geq \, 0 \tag{7}$$

- Result 3: Lower Bounds for Options on Non-Dividend-Paying Assets

  – Call options

$$c(X,\, S,\, \tau) \, \geq \, max(0,\, S - X\, e^{-r\tau}) \tag{8}$$

    * Proof: Consider the following two portfolios:
    * At date t:
      · Portfolio A: A bond having an initial value equal to $X\, e^{-r\tau}$ and one call option
      · Portfolio B: One share of the underlying asset (stock)
    * At date T:
      · Portfolio A = $X + \max(0,\, S_T - X)$ =max$(S_T,\, X)$
      · Portfolio B = S_T
    * Hence, Portfolio $A \geq B$, which implies

$$c \, + \, X\, e^{-r\tau} \, \geq \, S \tag{9}$$

  – Put options

$$p(X,\, S,\, \tau) \, \geq \, max(0,\, X\, e^{-r\tau} - S) \tag{10}$$

    * Proof: Consider the following two portfolios:
    * At date t:
      · Portfolio C: One put option and one share of the underlying asset (stock)
      · Portfolio D: A bond having an initial value equal to $X\, e^{-r\tau}$
    * At date T:
      · Portfolio C: $max(0,\, X - S_T) + S_T = max(X,\, S_T)$

      - Portfolio D: X
    - Hence, Portfolio $C \geq D$, which implies

$$p + S \geq X\, e^{-r\tau} \tag{11}$$

  - Further, note that $P \geq max(0,\, X - S)$.

- Result 4: An American Call on a Non-Dividend-Paying Stock Should Not Be Exercised Early
  - Note that an implication of this result is $C(X,\, S,\, \tau) = c(X,\, S,\, \tau)$ for non-dividend paying assets.
  - Proof: Consider the following two portfolios:
    - At date t:
      - Portfolio A: A bond having an initial value equal to $X\, e^{-r\tau}$ and one American call option
      - Portfolio B: One share of the underlying asset (stock)
    - Now consider exercising the call at date t^* where $t < t^* < T$.
    - At date t^*:
      - Portfolio A $= X\, e^{-r(T-t^*)} + S_{t^*} - X = S_{t^*} - X\left(1 - e^{-r(T-t^*)}\right) \leq S_{t^*}$
      - Portfolio B $= S_{t^*}$
    - Thus, if exercised early, Portfolio A is worth less than Portfolio B. However, we showed earlier that if not exercised early, then
    - At date T:
      - Portfolio A $= X + Max(0,\, S_T - X) = max(X,\, S_T)$
      - Portfolio B $= S_T$
    - Therefore, if held to expiry, Portfolio A $\geq$ B. Thus, early exercise destroys value.
  - Alternative Proof:
    - Since we showed

$$C(X,\, S,\, \tau) \,\geq\, c(X,\, S,\, \tau) \,\geq\, S - X\, e^{-r\tau}, \tag{12}$$

if we exercise early, then

$$C = S - X < S - X e^{-r\tau} \le c, \tag{13}$$

so early exercise cannot be optimal.

- Result 5: Early Exercise of an American Put on a Non-Dividend-Paying Stock May Be Optimal

  For a put that is sufficiently "in the money," it may be optimal to exercise the option early, selling the asset immediately and receiving $\$X$ now, rather than waiting and receiving $\$X$ at date T (which would have a present value of $X e^{-r\tau}$).

  To see this, note that from Result 3 that we have the boundary condition for a European put:

$$p(X, S, \tau) \ge X e^{-r\tau} - S \tag{14}$$

  but the stronger condition for an American put:

$$P(X, S, \tau) \ge X - S \tag{15}$$

  since it can be exercised early.

- Result 6: Put-Call Parity for Options on a Non-Dividend-Paying Stock

$$c(X, S, \tau) + X e^{-r\tau} = p(X, S, \tau) + S \tag{16}$$

  - Proof: Consider the following two portfolios:
    - At date t:
      - Portfolio A: A bond having an initial value equal to $X e^{-r\tau}$ and a European call option
      - Portfolio B: A European put option and one share of the underlying asset
    - At date T:
      - Portfolio A $= X + \max(0, S_T - X) = max(S_T, X)$
      - Portfolio B $= max(0, X - S_T) + S_T = max(S_T, X)$

* Since the two portfolios have exactly the same value at date T, their values at date t must be the same. If not, an arbitrage is to sell the over-valued portfolio, buy the under-valued.

- Result 7: Lower Bounds for Options on Dividend-Paying Assets

  Let D equal the present value of dividends paid by the asset during the life of the option contract.

$$c(X, S, \tau) \geq max(0, S - X e^{-r\tau} - D) \tag{17}$$

$$p(X, S, \tau) \geq max(0, X e^{-r\tau} + D - S) \tag{18}$$

  - Proof:
    * To prove the lower bound on a call, consider the following two portfolios:
    * At date t:
      · Portfolio A: A bond having an initial value equal to $D + X e^{-r\tau}$ and one call option
      · Portfolio B: One share of the underlying asset (stock)
    * At date T:
      · Portfolio A $= X + D e^{r\tau} + max(0, S_T - X) = max(S_T, X) + D e^{r\tau}$
      · Portfolio B $= S_T + D e^{r\tau}$
    * Hence, Portfolio A $\geq$ B, which implies

$$c + X e^{-r\tau} + D \geq S. \tag{19}$$

    * The proof for the lower bound on a put is similar.

- Result 8: Early Exercise of an American Call on a Dividend-Paying Stock May Be Optimal

  It may be optimal to exercise an American call just before the underlying stock's ex-dividend date, so as to obtain the dividend.

- Result 9: Put-Call Parity With Dividends

$$c(X,\, S,\, \tau) + X\, e^{-r\tau} + D \;=\; p(X,\, S,\, \tau) + S \tag{20}$$

Proof: Consider the following two portfolios:

  - At date t:

    - Portfolio A: A bond having an initial value equal to $D+\; X\, e^{-r\tau}$ and a European call option

    - Portfolio B: A European put option and one share of the underlying asset (stock)

  - At date T:

    - Portfolio A $= X + D\, e^{r\tau} + max(0,\, S_T - X) = max(S_T,\, X) + D\, e^{r\tau}$

    - Portfolio B $= max(0, X -\; S_T) + S_T + D\, e^{r\tau} = max(S_T,\, X) + D\, e^{r\tau}$

  - With equal date T values, no-arbitrage implies that date t values must also be equal.