Finance 400

A. Penati - G. Pennacchi

Consumption- Savings, Portfolio Choice, and Asset Pricing

I. The Consumption - Portfolio Choice Problem

We have studied the portfolio choice problem of an individual who maximizes expected utility of end-of-period wealth. Utility from consuming initial beginning-of-period wealth was not modeled, so that all initial wealth was assumed to be saved (invested in a portfolio of assets). Let us now consider a slightly different problem where the individual obtains utility from consuming at both the initial and terminal dates. Therefore, we model the individual's initial consumption-savings decision as well as his portfolio choice decision. In doing this, we can derive relationships between asset prices and the individual's optimal levels of consumption that generalize many of our previous results.

Let W_0 and C_0 be the individual's initial date 0 wealth and consumption, respectively. At date 1, the end of the period, the individual is assumed to consume all of his wealth which we denote as C_1. The individual's utility function is defined over beginning- and end-of-period consumption and takes the following form

$$U(C_0) + \delta E\left[U\left(\widetilde{C}_1\right)\right] \tag{1}$$

where δ is a subjective discount factor that reflects the individual's rate of time preference and $E[\cdot]$ is the expectations operator conditional on information at date 0.[1] Suppose that the individual can choose to invest in n different assets. Let P_i be the date 0 price per share of asset i, $i = 1, ..., n$, and let R_i be the date 1 random payoff (price plus dividend) of asset i. The individual may also receive labor income of y_0 at date 0 and possibly uncertain labor income of y_1 at date 1. If w_i is the proportion of date 0 savings that the individual chooses to invest in asset i, then his intertemporal budget constraint is

$$C_1 = y_1 + (W_0 + y_0 - C_0) \sum_{i=1}^{n} w_i \frac{R_i}{P_i} \tag{2}$$

[1] δ is sometimes written as $\frac{1}{1+\rho}$ where ρ is the rate of time preference. A value of $\delta < 1$ ($\rho > 0$) reflects impatience on the part of the individual, that is, a preference for consuming early.

The individual's maximization problem can then be stated as

$$\max_{C_0,\{w_i\}} U(C_0) + \delta E[U(C_1)] \tag{3}$$

subject to equation (2) and the constraint $\sum_{i=1}^{n} w_i = 1$. The first order conditions are

$$U'(C_0) - \delta E\left[U'(C_1)\sum_{i=1}^{n} w_i \frac{R_i}{P_i}\right] = 0 \tag{4a}$$

$$\delta E\left[U'(C_1)\frac{R_i}{P_i}\right] - \lambda = 0, \quad i = 1, ..., n \tag{4b}$$

where $\lambda = \lambda'/(W_0 + y_0 - C_0)$ and λ' is the Lagrange multiplier for the constraint $\sum_{i=1}^{n} w_i = 1$. The first order conditions in (4b) describes how the investor chooses between different assets. Substituting out for λ, one obtains:

$$E\left[U'(C_1)\frac{R_i}{P_i}\right] = E\left[U'(C_1)\frac{R_j}{P_j}\right] \tag{5}$$

for any two assets, i and j. Since R_i/P_i is the random return on asset i, equation (5) tells us that the investor trades off investing in asset i for asset j when the expected marginal utilities of their returns are equal. Another result of the first order conditions involves the intertemporal allocation of resources. Substituting (4b) into (4a) gives

$$\begin{aligned} U'(C_0) &= \delta E\left[U'(C_1)\sum_{i=1}^{n} w_i \frac{R_i}{P_i}\right] = \sum_{i=1}^{n} w_i \delta E\left[U'(C_1)\frac{R_i}{P_i}\right] \\ &= \sum_{i=1}^{n} w_i \lambda = \lambda \end{aligned} \tag{6}$$

Therefore, the first order conditions in (4b) can be written as:

$$\delta E\left[U'(C_1)\frac{R_i}{P_i}\right] = U'(C_0), \quad i = 1, ..., n \tag{7}$$

or

$$P_i U'(C_0) = \delta E\left[U'(C_1) R_i\right], \quad i = 1, ..., n \tag{8}$$

Equation (8) has an intuitive meaning. It says that when the investor is acting optimally, he invests in asset i until the loss in marginal utility of giving up P_i dollars at date 0 just equals the expected marginal utility of receiving the random payoff of R_i at date 1. To see this more clearly, suppose that one of the assets pays a risk-free return over the period. Call it asset f, and suppose $P_f = 1$ so that R_f is the risk-free return (one plus the risk-free interest rate). For the risk-free asset, equation (8) becomes

$$U'(C_0) = R_f \delta E\left[U'(C_1)\right] \tag{9}$$

which states that the investor trades off date 0 for date 1 consumption until the marginal utility of giving up \$1 of date 0 consumption just equals the expected marginal utility of receiving \$$R_f$ of date 1 consumption. For example, suppose that $U(C) = C^\gamma/\gamma$, for $\gamma < 1$. Then equation (9) can be re-written as

$$\frac{1}{R_f} = \delta E\left[\left(\frac{C_0}{C_1}\right)^{1-\gamma}\right] \tag{10}$$

Hence, when the interest rate is high, so will be the expected growth in consumption. For the special case of there being only one risk-free asset and non-random labor income, so that C_1 is non-stochastic, equation (10) becomes

$$R_f = \frac{1}{\delta}\left(\frac{C_1}{C_0}\right)^{1-\gamma} \tag{11}$$

Taking logs of both sides of the equation, we obtain

$$\ln(R_f) = -\ln\delta + (1-\gamma)\ln\left(\frac{C_1}{C_0}\right) \tag{12}$$

Since $\ln(R_f)$ is the continuously-compounded risk-free interest rate and $\ln(C_1/C_0)$ is the growth rate of consumption, when $0 < \gamma < 1$, a higher interest rate raises second period consumption less than one-for-one. This implies that initial savings decreases as the interest rate rises for someone who is less risk-averse than logarithmic utility (the income effect dominates the substitution effect). Conversely, when $\gamma < 0$, a rise in the interest rate raises second period consumption more than one-for-one, implying that initial savings increases with a higher return on savings (the substitution effect dominates the income effect). For the logarithmic utility investor ($\gamma = 0$), a change in the interest rate has no effect on savings.

II. An Asset Pricing Interpretation

Until now, we have analyzed the consumption-portfolio choice problem of an individual investor. For such an exercise, it makes sense to think of the individual taking the current prices of all assets and the distribution of their payoffs as given when deciding on his optimal consumption - portfolio choice plan. However, the first order conditions we have derived might be re-interpreted as asset pricing relationships. Equation (8) can be re-written as:

$$\begin{aligned} P_i &= E\left[\frac{\delta U'(C_1)}{U'(C_0)} R_i\right] \\ &= E\left[m_{01} R_i\right] \end{aligned} \tag{13}$$

where $m_{01} \equiv \delta U'(C_1)/U'(C_0)$ is the marginal rate of substitution between initial and end-of-period consumption. The term, m_{01}, is also referred to as, alternatively, a *stochastic discount factor*, *state price deflator*, or *pricing kernel*. Equation (13) appears in the form of an asset pricing formula. The current asset price, P_i, is an expected discounted value of its payoffs, where the discount factor is a random quantity because it depends on the random level of future consumption. In states of nature where future consumption turns out to be high (due to high asset portfolio returns or high labor income), marginal utility, $U'(C_1)$, is low and the asset's payoffs in these states are not highly valued. Conversely, in states where future consumption is low, marginal utility is high so that the asset's payoffs in these states are highly desired.

The relationship in (13) holds for any asset that the investor can choose to hold. For example, a dividend-paying stock might have a date 1 random return of $\widetilde{R}_i = \widetilde{P}_{1i} + \widetilde{D}_{1i}$, where $\widetilde{P}_{1i}$ is the date 1 stock price and $\widetilde{D}_{1i}$ is the stock's dividend paid at date 1. Alternatively, for a coupon-paying bond, $\widetilde{P}_{1i}$ would be the date 1 bond price and $\widetilde{D}_{1i}$ would be the bond's coupon paid at date 1.[2] However, in writing down the individual's consumption - portfolio choice

[2] The coupon payment would be uncertain if default on the payment is possible and/or the coupon is not fixed

problem, we implicitly assumed that returns are expressed in real or "purchasing power" terms. The reason is that an individual's utility should depend on the real, not nominal (currency denominated), value of consumption. Therefore, in the budget constraint (2), if C_1denotes real consumption, then asset returns and prices (as well as labor income) need to be in real terms. Thus if P_i and R_i are assumed to be in nominal terms, we need to deflate them by a price index. Letting CPI_t denote the consumer price index at date t, the pricing relationship in (13) becomes

$$\frac{P_i}{CPI_0} = E\left[\frac{\delta U'(C_1)}{U'(C_0)} \frac{R_i}{CPI_1}\right] \tag{14}$$

or if we define $I_{ts} = CPI_s/CPI_t$ as one plus the inflation rate between dates t and s, equation (14) can be re-written as

$$\begin{aligned} P_i &= E\left[\frac{1}{I_{01}} \frac{\delta U'(C_1)}{U'(C_0)} R_i\right] \\ &= E\left[M_{01} R_i\right] \end{aligned} \tag{15}$$

where $M_{01} \equiv (\delta/I_{01}) U'(C_1)/U'(C_0)$ is the stochastic discount factor (pricing kernel) for discounting nominal returns. Hence, this nominal pricing kernel is simply the real pricing kernel, m_{01}, discounted by the (random) rate of inflation between dates 0 and 1.

The relation in (13) can be re-written to shed light on an asset's risk premium. Let us define $r_i = R_i/P_i - 1$ as the random (real) rate of return on asset i. Then dividing each side of equation (13) by P_i results in

$$\begin{aligned} 1 &= E\left[m_{01}(1+r_i)\right] \\ &= E\left[m_{01}\right] E\left[1+r_i\right] + Cov\left[m_{01}, r_i\right] \\ &= E\left[m_{01}\right] \left(E\left[1+r_i\right] + \frac{Cov\left[m_{01}, r_i\right]}{E\left[m_{01}\right]}\right) \end{aligned} \tag{16}$$

Define $r_f \equiv R_f - 1$ as the risk-free real interest rate, and recall from (9) that $E\left[\delta U'(C_1)/U'(C_0)\right] =$

but floating (tied to a market interest rate).

$E\left[m_{01}\right] = 1/R_f$. Then (16) can be re-written as

$$1 + r_f = E\left[(1 + r_i)\right] + \frac{Cov\left[m_{01}, r_i\right]}{E\left[m_{01}\right]} \tag{17}$$

or

$$\begin{aligned} E\left[r_i\right] &= r_f - \frac{Cov\left[m_{01}, r_i\right]}{E\left[m_{01}\right]} \\ &= r_f - \frac{Cov\left[U'\left(C_1\right), r_i\right]}{E\left[U'\left(C_1\right)\right]} \end{aligned} \tag{18}$$

Equation (18) states that the risk premium for asset i equals minus the covariance between the marginal utility of end-of-period consumption and the asset return divided by the expected end-of-period marginal utility of consumption. If an asset pays a higher return when consumption is high, its return has a negative covariance with the marginal utility of consumption, and therefore the investor demands a positive risk premium over the risk free rate.

Conversely, if an asset pays a higher return when consumption is low, so that its return positively covaries with the marginal utility of consumption, then it has an expected return less than the risk-free rate. Investors will be satisfied with this lower return because the asset is providing insurance against low consumption states of the world, that is, it is helping to smooth consumption across states.

Now suppose there exists a portfolio with return, $\tilde{r}_m$, that is perfectly negatively correlated with the marginal utility of date 1 consumption, $U'\left(\widetilde{C}_1\right)$, implying that it is also perfectly negatively correlated with the pricing kernel, m_{01}:

$$U'(\tilde{C}_1) \;=\; -\gamma\, \tilde{r}_m, \qquad \gamma > 0. \tag{19}$$

Then this implies

$$Cov[U'(C_1),\, r_m] \;=\; -\gamma\, Cov[r_m,\, r_m] \;=\; -\,\gamma\, Var[r_m] \tag{20}$$

and

$$Cov[U'(C_1),\, r_i] \;=\; -\gamma\, Cov[r_m,\, r_i]. \tag{21}$$

For the portfolio having return $\tilde{r}_m$, the risk premium relation (18) is

$$E[r_m] \;=\; r_f - \frac{Cov[U'(C_1),\, r_m]}{E[U'(C_1)]} \;=\; r_f + \frac{\gamma\, Var[r_m]}{E[U'(C_1)]} \tag{22}$$

Using (18) and (22) to substitute for $E[U'(C_1)]$, and using (21), we obtain

$$\frac{E[r_m] - r_f}{E[r_i] - r_f} \;=\; \frac{\gamma\, Var[r_m]}{\gamma\, Cov[r_m,\, r_i]} \tag{23}$$

and re-arranging

$$E[r_i] - r_f \;=\; \frac{Cov[r_m,\, r_i]}{Var[r_m]}\; (E[r_m] - r_f) \tag{24}$$

or

$$E[r_i] \;=\; r_f + \beta_i\; (E[r_m] - r_f)\,. \tag{25}$$

So we obtain the CAPM if the market portfolio is perfectly negatively correlated with the marginal utility of end-of-period consumption, that is, perfectly negatively correlated with the pricing kernel. Note that for an arbitrary distribution of asset returns and non-random labor income, this will always be the case if utility is quadratic because marginal utility is linear in consumption and the market return is linear in consumption.

Another implication of the stochastic discount factor is that it places bounds on the means and standard deviations of individual securities and, therefore, determines an efficient frontier. To see this, re-write the first line in (18) as

$$E\left[r_i\right] = r_f - \rho_{m_{01},r_i} \frac{\sigma_{m_{01}}\sigma_{r_i}}{E\left[m_{01}\right]} \tag{26}$$

where $\sigma_{m_{01}}$, σ_{r_i}, and ρ_{m_{01},r_i} are the standard deviation of the discount factor, the standard deviation of the return on asset i, and the correlation between the discount factor and the return on asset i, respectively. Re-arranging (26) leads to

$$\frac{E\left[r_i\right] - r_f}{\sigma_{r_i}} = -\rho_{m_{01},r_i} \frac{\sigma_{m_{01}}}{E\left[m_{01}\right]} \tag{27}$$

The left hand side of (27) is known as the "Sharpe ratio," after William Sharpe a developer of the CAPM. Since $-1 \leq \rho_{m01,r_i} \leq 1$, we know that

$$\left|\frac{E\left[r_i\right]-r_f}{\sigma_{r_i}}\right| \leq \frac{\sigma_{m01}}{E\left[m_{01}\right]}=\sigma_{m01}\left(1+r_f\right) \tag{28}$$

If there exists an asset portfolio whose return is perfectly negatively correlated with the discount factor, m_{01}, then the bound in (28) holds with equality. As we just showed in equations (19) to (25), such a situation implies the CAPM, so that the slope of the capital market line, $S_e \equiv \frac{E[r_m]-r_f}{\sigma_{r_m}}$, equals $\sigma_{m01}\left(1+r_f\right)$. Thus, the slope of the capital market line, which represents (efficient) portfolios that have a maximum Sharpe ratio, can be related to the standard deviation of the discount factor.

The inequality in (28) has empirical implications. σ_{m01} can be estimated if we could observe an individual's consumption stream and we know his or her utility function. Then, according to (28), the Sharpe ratio of any portfolio of traded assets should be less than or equal to $\sigma_{m01}/E\left[m_{01}\right]$. For power utility, $U\left(C\right)=C^{\gamma}/\gamma$, $\gamma<1$, so that $m_{01}\equiv\delta\left(C_1/C_0\right)^{\gamma-1}=\delta e^{(\gamma-1)\ln(C_1/C_0)}$. If $C_1$is assumed to be lognormally distributed, with parameters μ_c and σ_c then

$$\begin{aligned}
\frac{\sigma_{m01}}{E\left[m_{01}\right]} &= \frac{\sqrt{Var\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]}}{E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]}=\frac{\sqrt{E\left[e^{2(\gamma-1)\ln(C_1/C_0)}\right]-E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]^2}}{E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]} \\
&= \sqrt{E\left[e^{2(\gamma-1)\ln(C_1/C_0)}\right]/E\left[e^{(\gamma-1)\ln(C_1/C_0)}\right]^2-1} \\
&= \sqrt{e^{2(\gamma-1)\mu_c+2(\gamma-1)^2\sigma_c^2}/e^{2(\gamma-1)\mu_c+(\gamma-1)^2\sigma_c^2}-1}=\sqrt{e^{(\gamma-1)^2\sigma_c^2}-1} \\
&\approx \left(1-\gamma\right)\sigma_c
\end{aligned} \tag{29}$$

where in the third line of (29) the expectations are evaluated assuming C_1 is lognormally distributed. Hence, with power utility and lognormally distributed consumption, we have

$$\left|\frac{E\left[r_i\right]-r_f}{\sigma_{r_i}}\right| \leq \left(1-\gamma\right)\sigma_c \tag{30}$$

For example, let r_i be the return on a broadly diversified portfolio of U.S. stocks, such as the S&P500. Over the last 75 years, this portfolio's annual real return in excess of the risk-free interest rate has averaged 8.3 percent, suggesting $E\left[r_i\right]-r_f=.083$. The portfolio's annual standard deviation has been approximately $\sigma_{r_i} = 0.17$, implying a Sharpe ratio of $\frac{E[r_i]-r_f}{\sigma_{r_i}} = 0.49$. Assuming a "representative agent" and using per capita U.S. consumption data to estimate the standard deviation of consumption growth, researchers have come up with annualized estimates of σ_c between 0.01 and 0.0386.[3] Thus, even if a diversified portfolio of U.S. stocks was an efficient portfolio of risky assets, so that (30) held with equality, it would imply a coefficient of relative risk aversion of $\gamma = 1 - \left(\frac{E[r_i]-r_f}{\sigma_{r_i}}\right)/\sigma_c$ between -11.7 and -48.[4] Since reasonable levels of risk aversion are often taken to be in the range of -2 to -4, the inequality (30) appears to not hold for U.S. stock market data and standard specifications of utility. In other words, consumption appears to be too smooth relative to the premium that investors demand for holding stocks. This phenomenon has been referred to as the "equity premium puzzle."[5] Attempts to explain this puzzle have involved using different specifications of utility and questioning whether the ex-post sample mean of U.S. stock returns is a good estimate of the a priori expected return on U.S. stocks.[6]

III. Market Completeness, Arbitrage, and State Pricing

The notion that assets can be priced using a stochastic discount factor is attractive because the discount factor, m_{01}, is independent of the asset being priced: it can be used to price any asset no matter what its risk. We derived this discount factor from a consumption - portfolio choice problem and, in this context, showed that it equaled the marginal rate of substitution between current and end-of-period consumption. However, the usefulness of this approach is in doubt since empirical evidence using aggregate consumption data and standard specifications of utility appears inconsistent with the discount factor equaling the marginal rate of substitution. Fortunately, a general pricing relationship of the form $P_i = E_0[m_{01}R_i]$ can be shown to hold without assuming that m_{01} represents a marginal rate of substitution. In other words, we need not assume a consumption- portfolio choice structure to derive this relationship. Instead, our

---

[3] See John Y. Campbell (1999) "Asset Prices, Consumption, and the Business Cycle," in John Taylor and Michael Woodford, eds. *Handbook of Macroeconomics* 1, North-Holland, Amsterdam and Stephen G. Cecchetti, Pok-Sam Lam, and Nelson C. Mark (1994) "Testing Volatility Restrictions on Intertemporal Rates of Substitution Implied by Euler Equations and Asset Returns," *Journal of Finance* 49, 123-52.

[4] If the stock portfolio was less than efficient, so that a strict inequality held in (30), the magnitude of the risk-aversion coefficient would need to be even higher.

[5] See Rajnish Mehra and Edward Prescott (1985) "The Equity Premium: A Puzzle," *Journal of Monetary Economics* 15, 145-61.

[6] Jeremy J. Siegel and Richard H. Thaler (1997) "Anomalies: The Equity Premium Puzzle," *Journal of Economic Perspectives* 11, 191-200 review this literature.

derivation can be based on the notions of market completeness and the absence of arbitrage. With these alternative assumptions, one can show that a law of one price holds and that a stochastic discount factor exists.

To illustrate, suppose once again that an individual can freely trade in n different assets. We also assume that there are a finite number of end-of-period "states of the world," with state s having probability π_s. Let R_{si} be the cashflow generated by one share (unit) of asset i in state s. Also assume that there are k states of the world and n assets. The following vector describes the payoffs to financial asset i:

$$R_i = \begin{bmatrix} R_{1i} \\ \vdots \\ R_{ki} \end{bmatrix}. \tag{31}$$

Thus, the per-share cashflows of the universe of all assets can be represented by the $k \times n$ matrix

$$R = \begin{bmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & \ddots & \vdots \\ R_{k1} & \cdots & R_{kn} \end{bmatrix}. \tag{32}$$

We will assume that $n = k$ and that R is of full rank. A necessary condition for there to be a complete market (and unique state prices) is that $n \geq k$. If $n > k$, there are some "redundant" assets, that is, assets whose cashflows in the k states are linear combinations of others. In this case we could reduce the number of assets to k by combining them into k linearly independent (portfolios of) assets without loss of generality.

Suppose an individual wishes to divide her wealth among the k assets so that she can obtain target levels of wealth in each of the states. Let W denote this kx1 vector of target wealth levels:

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_k \end{bmatrix}. \tag{33}$$

where W_s is the level of wealth in state s. To obtain this set of target wealth levels, at the initial date the individual needs to purchase shares in the k assets, which we denote as $w = [w_1 \ \ldots \ w_k]'$.

w must satisfy

$$Rw = W \tag{34}$$

or

$$w = R^{-1}W. \tag{35}$$

Hence, we see that any arbitrary levels of wealth in the k states can be attained because the assets' payoffs span the k states, that is, markets are complete. Given an absence of arbitrage opportunities, the price of the new security (contingent claim) represented by the payoff W must equal the cost of creating it. If $P = [P_1 \, \ldots \, P_k]'$ is the kx1 vector of beginning-of-period per share prices of the k assets, then the amount of initial wealth required to produce the target level of wealth given in (33) is simply $P'w$.

Consider a special case of a security that has a payoff of 1 in state s and 0 in all other states. We refer to such a security as a primitive or "elementary" security. Specifically, elementary security "s" has the vector of cashflows

$$e_s = \begin{bmatrix} W_1 \\ \vdots \\ W_s \\ \vdots \\ W_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}. \tag{36}$$

Let p_s be the beginning of period price or value of this primitive security s. Then as we just showed, its price in terms of the payoffs and prices of the original k assets must equal

$$p_s = P'R^{-1}e_s \tag{37}$$

It is insightful to now consider how any other security (contingent claim) can be valued in terms of these elementary or basis securities. Note that the portfolio composed of the sum of all primitive securities will give a cashflow of 1 unit with certainty. The price of this portfolio defines the risk free interest rate, r_f, by the relation:

$$\sum_{s=1}^{k} p_s = \frac{1}{1+r_f}. \tag{38}$$

In general, let there be some multi-cashflow asset, a, whose cashflow paid in state s is R_{sa}. In the absence of arbitrage, its price, P_a, must equal

$$P_a = \sum_{s=1}^{k} p_s \, R_{sa} \tag{39}$$

Next define $m_s \equiv p_s/\pi_s$ to be the price of elementary security s divided by the probability that state s occurs. Then (39) can be written as

$$\begin{aligned} P_a &= \sum_{s=1}^{k} \pi_s \frac{p_s}{\pi_s} \, R_{sa} \\ &= \sum_{s=1}^{k} \pi_s m_s \, R_{sa} \\ &= E\left[m \, R_a\right] \end{aligned} \tag{40}$$

implying that the stochastic discount factor equals the prices of the elementary securities normalized by their state probabilities. Hence, in a complete market, a unique stochastic discount factor exists.

An alternative formula for pricing assets can be developed. Define $\widehat{\pi}_s \equiv p_s(1+r_f)$ as the price of elementary security s times one plus the risk-free interest rate. Then

$$\begin{aligned} P_a &= \sum_{s=1}^{k} p_s \, R_{sa} \\ &= \frac{1}{1+r_f} \sum_{s=1}^{k} p_s \left(1+r_f\right) \, R_{sa} \\ &= \frac{1}{1+r_f} \sum_{s=1}^{k} \widehat{\pi}_s \, R_{sa} \\ &= \frac{1}{1+r_f} \widehat{E}\left[R_a\right] \end{aligned} \tag{41}$$

where $\widehat{E}[\cdot]$ denotes the expectation operator evaluated using the "pseudo" probabilities $\widehat{\pi}_s$ rather than the true or "physical" probabilities π_s. Note that since $m_s \equiv p_s/\pi_s$, $\widehat{\pi}_s$ can be written as

$$\widehat{\pi}_s = (1 + r_f)\, m_s \pi_s \tag{42}$$

so that the pseudo probability transforms the physical probability by multiplying by the product of the stochastic discount factor and the risk-free growth factor. To see what this entails, suppose that in each state the stochastic discount factor equaled the risk-free discount factor, that is, $m_s = \frac{1}{1+r_f}$. In that case, the pseudo probability would equal the physical probability, and $P_a = E[mR_a] = E[R_a]/(1 + r_f)$. Because the price equals the expected payoff discounted at the risk-free rate, the asset is priced as if investors are risk-neutral. Hence, $\widehat{\pi}_s$ is referred to as the "risk-neutral" probability and $\widehat{E}[\cdot]$ is referred to as the risk-neutral expectations operator. If the stochastic discount factor is interpreted as the marginal rate of substitution, then we see that $\widehat{\pi}_s$ is higher than π_s in states where the marginal utility of consumption is high (or the level of consumption is low). Thus, the risk-neutral probability places extra probability weight on "bad" states and less probability weight on "good" states.

The complete markets or "State Preference" framework can be generalized to an infinite number of states and primitive securities. Basically, this is done by defining probability densities of states and replacing the summations in expressions like (38) and (39) with integrals. For example, let states be indexed as all possible points on the real line between 0 and 1, that is, the state $s \in (0, 1)$. Also let $p(s)$ be the price (density) of a primitive security that pays 1 unit in state s. Further, define $R_a(s)$ as the cashflow paid by security a in state s. Then, analogous to (38) we can write

$$\int_0^1 p(s)\, ds \;=\; \frac{1}{1 + r_f} \tag{43}$$

and instead of (39) we can write the price of security a as

$$P_a \;=\; \int_0^1 p(s)\, R_a(s)\, ds. \tag{44}$$

In some cases, namely where markets are intertemporally complete, State Preference Theory can be extended to allow assets' cashflows to occur at different dates in the future. This generalization is sometimes referred to as Time State Preference Theory. See Stewart C. Myers (1968) "A Time-State Preference Model of Security Valuation," *Journal of Financial and Quantitative Analysis* 3, 1-34. To illustrate, suppose that assets can pay cashflows at both date 1 and date 2 in the future. Let s_1 be a state at date 1 and let s_2 be a state at date 2. States at date 2 can depend on which states were reached at date 1.

For example, suppose there are two events at each date, economic recession (r) or expansion (e). Then we could define $s_1 \in \{r_1, e_1\}$ and $s_2 \in \{r_1 r_2, r_1 e_2, e_1 r_2, e_1 e_2\}$. By assigning suitable probabilities and primitive security state prices for assets that pay cashflows of 1 unit in each of these six states, we can sum (or integrate) over both time and states at a given date to obtain prices of complex securities. Thus, when primitive security prices exist at all states for all future dates, we are essentially back to a single period complete markets framework, and the analysis is the same as that derived previously.