Finance 400

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An Application of Mean Variance Analysis: Cross Hedging

Consider the following one-period model of an individual or institution that is required to buy or sell a commodity in the future and would like to hedge the risk of such a transaction by taking positions in futures (or other financial securities) markets. Assume that this financial operator is committed at the beginning of the period, date 0, to sell y units of a risky commodity at the end of the period, date 1, at the then-prevailing spot price p_1. For example, a commitment to sell could be due to the operator producing a commodity that is non-storable.[1] If $y < 0$, this represents a commitment to buy $-y$ units, which could arise if the commodity is a necessary input in the operator's production process.[2] What is important is that, as of date 0, y is deterministic while p_1 is stochastic.

There are n financial securities (for example, futures contracts) in the economy. Denote the date 0 price of the i^{th} financial security as p^f_{i0}. Its date 1 price is p^f_{i1}, which is uncertain as of date 0. Let f_i denote the amount of the i^{th} security sold short at date 0. Thus, $f_i < 0$ indicates a purchase of, or long position in, the security.

Define the $nx1$ quantity and price vectors $f \equiv [f_1 \ldots f_n]'$, $p^f_0 \equiv [p^f_{10} \ldots p^f_{n0}]'$, and $p^f_1 \equiv [p^f_{11} \ldots p^f_{n1}]'$. Also define $p^f \equiv p^f_1 - p^f_0$ as the $n \times 1$ vector of security price changes. This is the profit at date 1 from having taken unit long positions in each of the securities (futures contracts) at date 0. Thus we can write the operator's profit from its security position as $-p^{f\prime}f$, where this expression includes a minus sign because we defined $f_i > 0$ as a short position. Also define the first and second moments of the date 1 prices of the spot commodity and the financial securities:

$E[p_1] = \bar{p}_1$, $var[p_1] = \sigma_{00}$, $E[p^f_{i1}] = \bar{p}^f_i$, $cov[p^f_{i1}, p^f_{j1}] = \sigma_{ij}$, and the $(n+1) \times (n+1)$ covariance matrix of the spot commodity and financial securities is

$$\Sigma = \begin{bmatrix} \sigma_{00} & \Sigma_{01} \\ \Sigma'_{01} & \Sigma_{11} \end{bmatrix}$$

[1] For example, the operator could be a producer of an agricultural good, such as corn, wheat, or soybeans.
[2] An example of this case would be a utility that generates electricity from oil.

where σ_{00} is a scalar, Σ_{11} is an $n \times n$ matrix, and Σ_{01} is an $1 \times n$ vector.

We assume that the operator cares about only the mean and variance of end-of-period profit, π. For simplicity, let us assume that y is fixed and, therefore, is not a decision variable at date 0. Then the end-of-period profit (wealth) of the financial operator is given by

$$\pi = p_1 y - p^{f\prime} f$$

What the operator must decide is the date 0 positions to take in the financial securities that maximize the following objective function:

$$\max_f E[\pi] - \tfrac{1}{2}\alpha var[\pi] \tag{1}$$

As was shown earlier, this objective function results from maximizing expected utility of wealth when portfolio returns are normally distributed.[3] Substituting in for the operator's profit, we have

$$\max_f \bar{p}_1 y - \bar{p}^{f\prime} f - \tfrac{1}{2}\alpha \left[y^2 \sigma_{00} + f' \Sigma_{11} f - 2y\Sigma_{01} f \right] \tag{2}$$

The first order conditions are

$$-\bar{p}^f - \alpha \left[\Sigma_{11} f - y\Sigma_{01}' \right] = 0 \tag{3}$$

Thus, the optimal positions in financial securities are

$$f = -\frac{1}{\alpha}\Sigma_{11}^{-1}\bar{p}^f + y\Sigma_{11}^{-1}\Sigma_{01}' \tag{4}$$

$$= \frac{1}{\alpha}\Sigma_{11}^{-1}\left(p_0^f - \bar{p}_1^f\right) + y\Sigma_{11}^{-1}\Sigma_{01}'$$

Let us first consider the case of $y = 0$. This can be viewed as the situation faced by a pure "speculator." If $n = 1$ and $\bar{p}_1^f < p_0^f$, the speculator shorts the security, while if $\bar{p}_1^f > p_0^f$, the

[3] In this case, α is the coefficient of relative risk aversion.

speculator takes a net long position in the security. The net long position is tempered by the volatility of the security, σ_{11}, and the speculator's level of risk aversion, α. However, for the general case of $n > 1$, an expected price decline or rise is not sufficient to determine whether a speculator takes a short or long position in a particular security. All of the elements in Σ_{11}^{-1} need be considered, since a position in a given security may have particular diversification benefits.

For the general case of $y \neq 0$, the situation faced by a "hedger," the demand for financial securities is similar to that of a pure speculator in that it also depends on price expectations. In addition, there are hedging components to the demand for financial assets, call them f^h :

$$f^h \equiv y\Sigma_{11}^{-1}\Sigma_{01}' \tag{5}$$

This is the solution to the problem $\min_f var(\pi)$. Thus, even for a hedger, it is never optimal to minimize volatility (risk) unless risk aversion is infinitely large. Even a risk-averse, expected utility maximizing hedger should behave somewhat like a speculator in that securities' expected returns matter. In this sense, it may not be useful to categorize traders in financial markets as either hedgers or speculators. From definition (5), note that when $n = 1$ the pure hedging demand per unit of the commodity sold, f^h/y, simplifies to

$$\frac{f^h}{y} = \frac{cov(p_1, p_1^f)}{var(p_1^f)} \tag{6}$$

For the general case, $n > 1$, the elements of the vector $\Sigma_{11}^{-1}\Sigma_{01}'$equal the coefficients $\beta_1, ..., \beta_n$ in the multiple regression model

$$\Delta p_1 = \beta_0 + \beta_1 \Delta p_1^f + \beta_2 \Delta p_2^f + ... + \beta_n \Delta p_n^f + \varepsilon \tag{7}$$

where $\Delta p_1 \equiv p_1 - p_0$, $\Delta p_i^f \equiv p_{1i}^f - p_{0i}^f$, and ε is a mean-zero error term. An implication of (7) is that an operator might estimate the "hedge ratios," f^h/y, by performing a statistical regression using an historical times series of the $n \times 1$ vector of security price changes. In fact, this is a standard way that practitioners use to estimate hedge ratios.