Finance 400

A. Penati - G. Pennacchi

Mean Variance Analysis

Consider the utility of an individual who invests her beginning-of-period wealth, W_0, in a particular portfolio of assets. Let $\tilde{r}_p$ be the random rate of return on this portfolio, so that the individual's end-of-period wealth is $\tilde{W} = W_0(1+\tilde{r}_p)$. Denote this individual's end-of-period utility by $U(\tilde{W})$. Given W_0, for notational simplicity we can write $U(\tilde{W}) = U\left(W_0(1+\tilde{r}_p)\right)$as just $U(\tilde{r}_p)$ because $\tilde{W}$ is fully determined by $\tilde{r}_p$.

Now take a Taylor series expansion of $U(\tilde{r}_p)$ around the mean of $\tilde{r}_p$, denoted as $E[\tilde{r}_p]$, and let $U'(\cdot)$, $U''(\cdot)$, and $U^{(n)}(\cdot)$ be the first, second, and n^{th} derivatives of the utility function:

$$U(\tilde{r}_p) = U\left(E[\tilde{r}_p]\right) + \left(\tilde{r}_p - E[\tilde{r}_p]\right) U'\left(E[\tilde{r}_p]\right) + \tfrac{1}{2}\left(\tilde{r}_p - E[\tilde{r}_p]\right)^2 U''\left(E[\tilde{r}_p]\right) + ... \tag{1}$$

$$\tfrac{1}{n!}\left(\tilde{r}_p - E[\tilde{r}_p]\right)^n U^{(n)}\left(E[\tilde{r}_p]\right) + ...$$

If

(a) The utility function is quadratic ($U''' = 0$), then

$$E\left[U(\tilde{r}_p)\right] = U\left(E[\tilde{r}_p]\right) + \tfrac{1}{2}E\left[\left(\tilde{r}_p - E[\tilde{r}_p]\right)^2\right] U''\left(E[\tilde{r}_p]\right) \tag{2}$$

$$= U\left(E[\tilde{r}_p]\right) + \tfrac{1}{2}V[\tilde{r}_p]U''\left(E[\tilde{r}_p]\right)$$

where $V[\tilde{r}_p]$ is the variance of the rate of return on the portfolio.[1]

Alternatively, if $U(\tilde{r}_p)$ is a general concave utility but

(b) Portfolio rates of return are normally distributed, then third, fourth, and all higher central moments are either zero or a function of the variance: $E\left[\left(\tilde{r}_p - E[\tilde{r}_p]\right)^n\right] = 0$, for n odd, and $E\left[\left(\tilde{r}_p - E[\tilde{r}_p]\right)^n\right] = \frac{n!}{(n/2)!}\left(\frac{1}{2}V[\tilde{r}_p]\right)^{n/2}$, for n even. For this case we have

$$E\left[U(\tilde{r}_p)\right] = U\left(E[\tilde{r}_p]\right) + \tfrac{1}{2}V[\tilde{r}_p]U''\left(E[\tilde{r}_p]\right) + 0 + \frac{1}{8}\left(V[\tilde{r}_p]\right)^2 U''''\left(E[\tilde{r}_p]\right) + 0 + ... \tag{2'}$$

[1] The expected value of the second term in the Taylor series, $E\left[\left(\tilde{r}_p - E[\tilde{r}_p]\right) U'\left(E[\tilde{r}_p]\right)\right]$, equals zero.

$$+\frac{1}{(n/2)!}\left(\frac{1}{2}V[\tilde{r}_p]\right)^{n/2}U^{(n)}\left(E[\tilde{r}_p]\right)+...$$

Thus for case (a) or (b), we see that $E\left[U(\tilde{r}_p)\right]$ can be written as a function of only the mean, $E[\tilde{r}_p]$, and the variance, $V[\tilde{r}_p]$, of the portfolio return distribution. Given that expected utility is a function of only the mean and variance of a portfolio's *rate* of return, clearly, it can be written also as a function of the mean and variance of one plus the portfolio's rate of return, which we denote as $\tilde{R}_p \equiv 1+\tilde{r}_p$.

Before we analyze portfolio choice decisions assuming that the individual cares only about the mean and variance of her portfolio's return (or mean and variance of her end-of-period wealth), we should pause to ask whether case (a) or case (b) is realistic. If not, then there is little justification in assuming that the first two moments of the portfolio return distribution are the only ones that matter to the individual investor. Clearly, the assumption of quadratic utility, case (a), is problematic. As mentioned earlier, marginal utility for quadratic utility is non-negative only for levels of wealth below the "bliss point." There is also a potential problem with case (b). The assumption that asset returns are normally distributed over a finite period of time implies that assets can take on negative values.[2] For some assets, such as stockholders' equity, negative values are inconsistent with limited-liability.

It turns out, however, that the assumption of normal rates of return can be modified if we generalize the model to have multiple periods and assume that asset rates of return follow continuous-time stochastic processes. In that context, one can assume that assets' rates of return are *instantaneously* normally distributed, which implies that if their means and variances are constant over infinitesimal intervals, then over any finite internal asset values are lognormally distributed. This turns out to be a better way of modeling limited liability assets because the lognormal distribution bounds these assets' values to be no less than zero. As we shall see when we later consider multi-period models, the results derived here assuming a single-period, discrete-time model continue to hold, under particular conditions, in the more-realistic multi-period context.

Therefore, let us proceed by assuming that the individual's utility function, U, is a general

[2]This because realizations from the normal distribution have no lower (or upper) bound.

concave utility function and that $\tilde{R}_p$ (and, hence, $\tilde{r}_p$) is normally distributed with probability density function $f(\bar{R}_p, \sigma_p^2)$, where we use the short-hand notation $\bar{R}_p \equiv E[\tilde{R}_p]$ and $\sigma_p^2 \equiv V[\tilde{R}_p]$. Expected utility can then be written:

$$E\left[U\left(\widetilde{R}_p\right)\right] = \int_{-\infty}^{\infty} U(R) f(\bar{R}_p, \sigma_p^2) dR \tag{3}$$

We first analyze an individual's indifference curves in portfolio mean-variance space. An indifference curve represents the combinations of portfolio mean and variance that would give an individual the same level of expected utility. Define $\widetilde{x} \equiv \frac{\tilde{R}_p - \bar{R}_p}{\sigma_p}$. Then

$$E\left[U\left(\widetilde{R}_p\right)\right] = \int_{-\infty}^{\infty} U(\bar{R}_p + x\sigma_p) n(x) dx \tag{4}$$

where $n(x)$ is the standardized normal probability density function, that is, the normal density having a zero mean and unit variance. Taking the partial derivative with respect to $\bar{R}_p$:

$$\frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial \bar{R}_p} = \int_{-\infty}^{\infty} U' n(x) dx > 0 \tag{5}$$

since U' is always greater than zero. Next, take the partial derivative of (4) with respect to σ_p^2:

$$\frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial \sigma_p^2} = \frac{1}{2\sigma_p} \frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial \sigma_p} = \frac{1}{2\sigma_p} \int_{-\infty}^{\infty} U' x n(x) dx \tag{6}$$

While U' is always positive, x ranges between $-\infty$ and $+\infty$. Because x has a standard normal distribution, which is symmetric, for each positive realization there is a corresponding negative realization with the same probability density. For example, take the positive and negative pair $+x_i$ and $-x_i$. Then $n(+x_i) = n(-x_i)$. Comparing the integrand of equation (6) for equal absolute realizations of x, we can show

$$U'(\bar{R}_p + x_i\sigma_p) x_i n(x_i) + U'(\bar{R}_p - x_i\sigma_p)(-x_i) n(-x_i) \tag{7}$$

$$= U'(\bar{R}_p + x_i\sigma_p) x_i n(x_i) - U'(\bar{R}_p - x_i\sigma_p) x_i n(x_i)$$

$$= x_i n(x_i)\left[U'(\bar{R}_p + x_i\sigma_p) - U'(\bar{R}_p - x_i\sigma_p)\right] < 0$$

because

$$U'(\bar{R} + x_i\sigma_p) < U'(\bar{R} - x_i\sigma_p) \tag{8}$$

due to the assumed concavity of U, that is, $U'' < 0$. Thus, comparing $U'x_i n(x_i)$ for each positive and negative pair, we conclude that

$$\frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial\sigma_p^2} = \frac{1}{2\sigma_p}\int_{-\infty}^{\infty} U'xn(x)dx < 0 \tag{9}$$

which is the intuitive result that higher portfolio variance, without higher portfolio expected return, reduces a risk-averse individual's expected utility.[3]

To obtain an indifference curve, take the total differential

$$dE\left[U\left(\widetilde{R}_p\right)\right] = \frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial\sigma_p^2}d\sigma_p^2 + \frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial\bar{R}_p}d\bar{R}_p = 0 \tag{10}$$

The slope of each indifference curve is then

$$\frac{d\bar{R}_p}{d\sigma_p^2} = -\frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial\sigma_p^2}\Big/\frac{\partial E\left[U\left(\widetilde{R}_p\right)\right]}{\partial\bar{R}_p} > 0 \tag{11}$$

Indifference curves are typically drawn in mean - standard deviation space, rather than mean - variance space, because standard deviations of returns are in the same unit of measurement as returns or interest rates (rather than squared returns). In Figure 1, the arrow indicates an increase in utility. The curves are convex due to the concavity of the utility function.[4]

The Efficient Frontier

The individual's optimal choice of portfolio mean and variance is determined by the point where one of these indifference curves is tangent to the set of means and standard deviations for all feasible portfolios, what we might describe as the "risk-expected return production

[3]Note that this result depends on the individual's utility function being concave. If the individual had convex utility, that is, was risk-loving, then the derivative in (9) would be positive.

[4]The proof of this is left as an exercise.

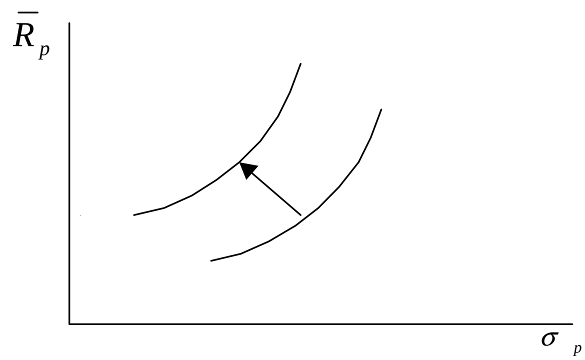

Figure 0-1: Indifference Curves

possibility set." This set represents all possible ways of combining various *individual* assets to generate alternative combinations of *portfolio* mean and variance (or standard deviation). This set includes inefficient portfolios (those in the interior of the opportunity set) as well as efficient portfolios (those on the "frontier" of the set). Efficient portfolios are those that make best use of the benefits of diversification.

To illustrate the effects of diversification, consider the following simple example. Suppose there are two assets, assets A and B, that have expected returns $\bar{R}_A$ and $\bar{R}_B$ and variances of σ_A^2 and σ_B^2, respectively. Further the correlation between their returns is given by ρ. Let us assume that $\bar{R}_A < \bar{R}_B$ but $\sigma_A^2 < \sigma_B^2$. Now form a portfolio with a proportion w invested in asset A and a proportion $1 - w$ invested in asset B. The expected return on this portfolio is

$$\overline{R}_p = w\bar{R}_A + (1-w)\bar{R}_B \tag{12}$$

The expected return of a portfolio is a simple weighted average of the expected returns of the individual financial assets. Expected returns are not fundamentally transformed by combining individual assets into a portfolio. The standard deviation of the return on the portfolio is

$$\sigma_p = \left[w^2\sigma_A^2 + 2w(1-w)\sigma_A\sigma_B\rho + (1-w)^2\sigma_B^2\right]^{\frac{1}{2}} \tag{13}$$

In general, portfolio risk, as measured by the portfolio's return standard deviation, is a nonlinear function of the individual assets' risks (standard deviations). Thus, risk is altered in a relatively complex way when individual assets are combining into a portfolio.

Let us consider some special cases regarding the correlation between the two assets. Suppose $\rho = 1$, so that the two assets are perfectly positively correlated. Then assuming that $w\sigma_A + (1-w)\sigma_B > 0$, the portfolio standard deviation equals

$$\begin{aligned} \sigma_p &= \left[w^2\sigma_A^2 + 2w(1-w)\sigma_A\sigma_B + (1-w)^2\sigma_B^2\right]^{\frac{1}{2}} \\ &= w\sigma_A + (1-w)\sigma_B \end{aligned} \tag{14}$$

which is a simple weighted average of the individual assets' standard deviations. Since rearranging (14) implies $w = -(\sigma_p - \sigma_B)/(\sigma_B - \sigma_A)$, and we can substitute in for w in (12) to obtain

$$\bar{R}_p = \bar{R}_B + \left[\frac{\sigma_p - \sigma_B}{\sigma_B - \sigma_A}\right](\bar{R}_B - \bar{R}_A) \tag{15}$$

This relationship between portfolio risk and expected return is a positively sloped line in $\bar{R}_p$, σ_p space. It goes through the points $(\bar{R}_A, \sigma_A)$ and $(\bar{R}_B, \sigma_B)$ when $w = 1$ and $w = 0$, respectively.

Next, suppose $\rho = -1$, so that the assets are perfectly negatively correlated. Then

$$\sigma_p = \left[(w\sigma_A - (1-w)\sigma_B)^2\right]^{\frac{1}{2}} \tag{16}$$

One can see that there is a w such that portfolio risk can be eliminated. σ_p equals zero when

$$w = \frac{\sigma_B}{\sigma_A + \sigma_B} \tag{17}$$

Hence there is a portfolio with positive proportions of both assets A and B that produces a riskless return, equal to $\bar{R}_{p|_{\sigma_p=0}} = (\sigma_B\bar{R}_A + \sigma_A\bar{R}_B)/(\sigma_A + \sigma_B)$. When $\rho = -1$, we see from (16) that when $(w\sigma_A - (1-w)\sigma_B) \geq 0$, $\sigma_p = w(\sigma_B + \sigma_A) - \sigma_B$. Since $\bar{R}_p = \bar{R}_B + w(\bar{R}_A - \bar{R}_B)$, substituting for w give us

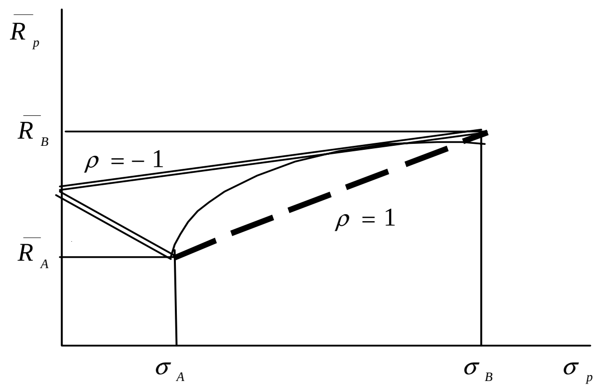

Figure 0-2: Efficient Frontier for Two Risky Assets

$$\bar{R}_p = \bar{R}_B + - \left[\frac{\sigma_B + \sigma_p}{\sigma_A + \sigma_B}\right] (\bar{R}_B - \bar{R}_A) \tag{18}$$

which is a negatively sloped line in $\bar{R}_p$, σ_p space and goes through the point $(\bar{R}_A, \sigma_A)$ when $w = 1$. In equation (16), when $w\sigma_A - (1-w)\sigma_B \leq 0$, $\sigma_p = \sigma_B - w(\sigma_A + \sigma_B)$. In this case

$$\bar{R}_p = \bar{R}_B + \left[\frac{\sigma_p - \sigma_B}{\sigma_A + \sigma_B}\right] (\bar{R}_B - \bar{R}_A) \tag{19}$$

which is, a positively sloped line in $\bar{R}_p$, σ_p space and goes through the point $(\bar{R}_B, \sigma_B)$ when $w = 0$. Figure 2 summarizes these risk - expected return constraints.

In general, when $-1 < \rho < 1$, if the portfolio expected return is between $\bar{R}_A$ and $\bar{R}_B$, the efficient frontier will be a concave function contained in the triangle of the above diagram. Maximum benefits from diversification occur where the individual's indifference curve is tangent to the frontier. Note that there is no need to have $\rho < 0$ to obtain diversification benefits, only $\rho < 1$. We now set out to prove the above assertions for the general case of N assets.

Mathematics of the efficient frontier

The problem we wish to solve is the following: Given the expected returns and the covariance matrix of N individual assets, find the set of portfolio weights that minimizes the variance of the portfolio for each feasible portfolio expected return. The locus of these points is the efficient portfolio frontier.

Let $\bar{R} = (\bar{R}_1 \, \bar{R}_2 \, ... \, \bar{R}_n)'$ be an $N \times 1$ vector of the expected returns of the N assets. Let V be the $N \times N$ covariance matrix of the returns on the N assets. V is assumed to be of full rank.[5] Also, let $w = (w_1 \, w_2 \, ... \, w_n)'$ be an $N \times 1$ vector of portfolio proportions, such that w_i is the proportion of total portfolio wealth invested in the i^{th} asset. Thus, the expected return on the portfolio is given by

$$\bar{R}_p = w'\bar{R} \tag{20}$$

and the variance of the portfolio return is given by

$$\sigma_p^2 = w'Vw \tag{21}$$

The constraint that the portfolio proportions must sum to 1 can be written as $w'e = 1$ where e is defined to be an $N \times 1$ vector of ones. The problem can be stated as a quadratic optimization exercise:

$$\min_w L = \tfrac{1}{2}w'Vw + \lambda \left[\overline{R}_p - w'\bar{R}\right] + \gamma[1 - w'e] \tag{22}$$

The first order conditions are:

$$\frac{\partial L}{\partial w} = Vw - \lambda\bar{R} - \gamma e = 0 \tag{23a}$$

$$\frac{\partial L}{\partial \lambda} = \overline{R}_p - w'\bar{R} = 0 \tag{23b}$$

$$\frac{\partial L}{\partial \gamma} = 1 - w'e = 0 \tag{23c}$$

Solving (23a), the optimal portfolio weights satisfy

$$w^* = \lambda V^{-1}\bar{R} + \gamma V^{-1}e \tag{24}$$

[5] This implies that there are no redundant assets among the N assets. An asset would be redundant if its return was an exact linear combination of the the returns on other assets. If such an asset exists, it can be ignored since whether or not it is held will not affect the efficient portfolio frontier.

Pre-multiplying equation (24) by $\bar{R}'$, we have

$$\overline{R}_p = \bar{R}'w^* = \lambda \bar{R}'V^{-1}\bar{R} + \gamma \bar{R}'V^{-1}e \tag{25}$$

Pre-multiplying equation (24) by e' we have

$$1 = e'w^* = \lambda e'V^{-1}\bar{R} + \gamma e'V^{-1}e \tag{26}$$

Equations (25) and (26) are two linear equations in two unknowns, the Lagrange multipliers λ and γ. The solution is

$$\lambda = \frac{\delta \overline{R}_p - \alpha}{\varsigma\delta - \alpha^2} \tag{27a}$$

$$\gamma = \frac{\varsigma - \alpha \overline{R}_p}{\varsigma\delta - \alpha^2} \tag{27b}$$

where $\alpha \equiv \bar{R}'V^{-1}e = e'V^{-1}\bar{R}$, $\varsigma \equiv \bar{R}'V^{-1}\bar{R}$, and $\delta \equiv e'V^{-1}e$. Note that the denominators of λ and γ, given by $\varsigma\delta - \alpha^2$, are guaranteed to be positive when V is of full rank. This can be shown using the Cauchy-Schwartz inequality which states that correlations must be less than unity in absolute value. Substituting for λ and γ in equation (24), we have

$$w^* = \frac{\delta \overline{R}_p - \alpha}{\varsigma\delta - \alpha^2} V^{-1}\bar{R} + \frac{\varsigma - \alpha \overline{R}_p}{\varsigma\delta - \alpha^2} V^{-1}e \tag{28}$$

Collecting terms in $\bar{R}_p$ gives

$$w^* = h\overline{R}_p + g \tag{29}$$

where $h \equiv \dfrac{\delta V^{-1}\bar{R} - \alpha V^{-1}e}{\varsigma\delta - \alpha^2}$ and $g \equiv \dfrac{\varsigma V^{-1}e - \alpha V^{-1}\bar{R}}{\varsigma\delta - \alpha^2}$.

Equation (29) is both a necessary and sufficient condition for portfolio efficiency. Given $\overline{R}_p$, a portfolio must have weights satisfying (29) to be efficient.

Having solved for the optimal portfolio weights and given $\overline{R}_p$, the variance of the efficient portfolio is

$$\sigma_p^2 = w^{*\prime} V w^* = (h\overline{R}_p + g)' V (h\overline{R}_p + g) = \frac{\delta \overline{R}_p^2 - 2\alpha \overline{R}_p + \varsigma}{\varsigma\delta - \alpha^2} \tag{30}$$

which is a parabola in σ_p^2, $\overline{R}_p$ space, or a hyperbola in σ_p, $\overline{R}_p$ space. Equation (30) is therefore the function for the efficient frontier. In σ_p, $\overline{R}_p$ space, this is the upper arc of the hyperbola and, thus, is a concave function. See Figure 3.

Separation property

We now state and prove a fundamental result:

> Any frontier portfolio can be duplicated by any of two other frontier portfolios; and an individual will be indifferent between choosing among the N financial assets, or choosing a combination of just two efficient portfolios.

The proof is as follows. Let $\bar{R}_{1p}$ and $\bar{R}_{2p}$ be the expected returns on any two distinct frontier portfolios. Let $\bar{R}_{3p}$ be the expected return on a third frontier portfolio. Now consider investing a proportion of wealth, x, in the first frontier portfolio and the remainder, $(1-x)$, in the second frontier portfolio. Clearly, a value for x can be found that makes the expected return on this "composite" portfolio equal to that of the third frontier portfolio.[6]

$$\bar{R}_{3p} = x\bar{R}_{1p} + (1-x)\bar{R}_{2p} \tag{31}$$

In addition, because portfolios 1 and 2 are frontier portfolios, we can write their portfolio proportions as a linear function of their expected returns. Specifically, we have $w^1 = g + h\bar{R}_{1p}$ and $w^2 = g + h\bar{R}_{2p}$ where w^i is the $N \times 1$ vector of optimal portfolio weights for frontier portfolio i. Now create a new portfolio with an $N \times 1$ vector of portfolio weights given by

$$\begin{aligned} xw^1 + (1-x)w^2 &= x(g + h\bar{R}_{1p}) + (1-x)(g + h\bar{R}_{2p}) \\ &= g + h(x\bar{R}_{1p} + (1-x)\bar{R}_{2p}) \\ &= g + h\bar{R}_{3p} = w^3 \end{aligned} \tag{32}$$

[6] x may be any positive or negative number.

where, in the last line of (32) we have substituted in equation (31). Based on the portfolio weights of the composite portfolio, $xw^1 + (1-x)w^2$,equalling $g + h\bar{R}_{3p}$, which is the portfolio weights of the third frontier portfolio, w^3, this composite portfolio replicates the third frontier portfolio. Hence, any arbitrary efficient portfolio can be replicated by two others.

The Efficient Frontier with a Riskless Asset

Thus far, we have assumed that all assets are risky. Introducing the possibility of investing in a riskless asset changes the problem in a fundamental manner. Assume that there is a riskless asset with return R_f. Let w continue to be the $N \times 1$ vector of portfolio proportions invested in the risky assets. Now, however, the constraint $1 = w'e$ does not apply. We can impose the restriction that the portfolio weights for all $N+1$ assets sum to one by writing the expected return on the portfolio as

$$\bar{R}_p = R_f + w'(\bar{R} - R_f e) \tag{33}$$

The variance of the return on the portfolio continues to be given by $w'Vw$. Thus, the individual's optimization problem is changed to:

$$\min_{w} L = \tfrac{1}{2} w'Vw + \lambda \left\{ \overline{R}_p - \left[R_f + w'(\bar{R} - R_f e) \right] \right\} \tag{34}$$

In a manner similar to the previous derivation, the first order conditions lead to the solution

$$w^* = kV^{-1}(\bar{R} - R_f e) \tag{35}$$

where $k \equiv \dfrac{\overline{R}_p - R_f}{\varsigma - 2\alpha R_f + \delta R_f^2}$.[7] Thus, the amount optimally invested in the riskless asset is $1 - e'w^*$. Note that since k is linear in $\overline{R}_p$, so is w^*, the same as in the previous case of no riskless asset. The variance of the portfolio now takes the form

$$\sigma_p^2 = w^{*\prime} V w^* = \frac{(\overline{R}_p - R_f)^2}{\varsigma - 2\alpha R_f + \delta R_f^2} \tag{36}$$

or

[7] The proof of this is left to the reader.

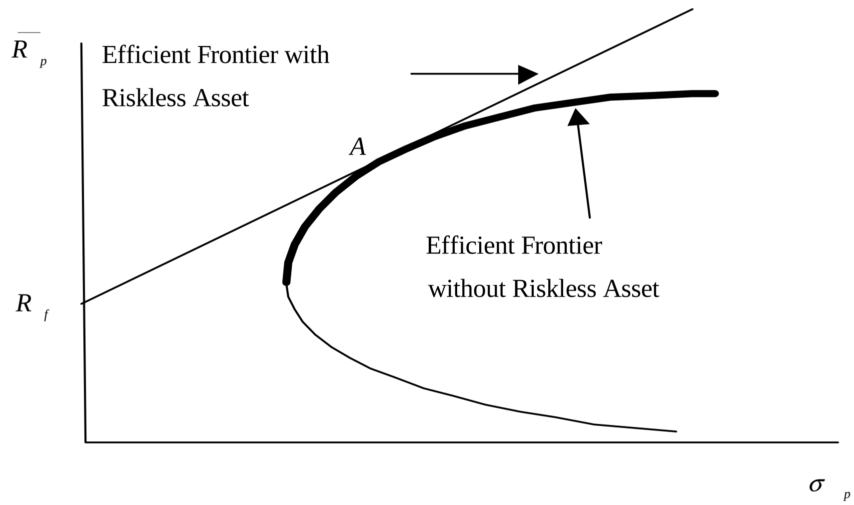

Figure 0-3: Efficient Frontier

$$\sigma_p = \frac{\overline{R}_p - R_f}{\left(\varsigma - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}}} \tag{37}$$

Hence, the efficient frontier is *linear* in σ_p, $\overline{R}_p$ space. This case of including a riskless asset can be compared with the previous case of only risky assets in Figure 3.

At point A, the ray from R_f is tangent to the hyperbola given by equation (30), the efficient frontier for the case of only risky assets. This ray running through R_f and A is now the new efficient frontier when holding a riskless asset is permitted. As one can see, it weakly dominates the case with no riskless asset. To the left of point A, the individual holds a positive amount of (invests in) the riskless asset. To the right of point A, the individual holds a negative amount of (borrows) the riskless asset.

A Specific Example with Negative Exponential Utility

To illustrate our results, let us assume a specific form for an individual's utility function. This will enable us to determine the individual's preferred efficient portfolio, that is, the point of tangency between the individual's highest indifference curve and the efficient frontier.

As before, let $\tilde{W}$ be the individual's end-of-period wealth and assume that she maximizes expected negative exponential utility.

$$U(\tilde{W}) = -e^{-b\tilde{W}} \tag{38}$$

where b is the individual's coefficient of absolute risk aversion. Now define $b_r \equiv bW_0$, which is the individual's coefficient of relative risk aversion at initial wealth W_0. Equation (38) can be re-written:

$$U(\tilde{W}) = -e^{-b_r \tilde{W}/W_0} = -e^{-b_r \tilde{R}_p} \tag{39}$$

where $\tilde{R}_p$ is the total return (one plus the rate of return) on the portfolio.

In this problem, we assume that initial wealth can be invested in a riskless asset and N risky assets. As before, denote the return on the riskless asset as R_f and the returns on the N risky assets as the $N \times 1$ vector $\tilde{R}$. Also as before, let $w = (w_1 \dots w_N)'$ be the vector of portfolio weights for the N risky assets. The risky assets' returns are assumed to have a joint normal distribution where $\bar{R}$ is the $N \times 1$ vector of expected returns on the N risky assets and V is the $N \times N$ covariance matrix of returns. Thus, the expected return on the portfolio can be written $\bar{R}_p \equiv R_f + w'(\bar{R} - R_f e)$ and the variance of the return on the portfolio is $\sigma_p^2 \equiv w'Vw$.

Now recall the properties of the lognormal distribution. If $\tilde{x}$ is a normally distributed random variable, for example, $\tilde{x} \sim \Phi(\mu, \sigma^2)$, then $\tilde{z} = e^{\widetilde{x}}$ is lognormally distributed. The expected value of $\widetilde{z}$ is

$$E[\tilde{z}] = e^{\mu + \frac{1}{2}\sigma^2} \tag{40}$$

From (39), we see that if $\tilde{R}_p = w'\tilde{R}$ is normally distributed, then $U\left(\tilde{W}\right)$ is lognormally distributed. Using equation (40), we have

$$E\left[U\left(\widetilde{W}\right)\right] = -e^{-b_r\left[R_f + w'(\bar{R} - R_f e)\right] + \frac{1}{2}b_r^2 w'Vw} \tag{41}$$

The individual chooses portfolio weights by maximizing expected utility:

$$\max_w E\left[U\left(\widetilde{W}\right)\right] = -e^{-b_r\left[R_f + w'(\bar{R} - R_f e)\right] + \frac{1}{2}b_r^2 w'Vw} \tag{42}$$

Because the expected utility function is monotonic in its exponent, the maximization problem in (42) is equivalent to

$$\max_{w} E\left[U\left(\widetilde{W}\right)\right] = w'(\bar{R} - R_f e) - \tfrac{1}{2} b_r w' V w \tag{43}$$

The N first order conditions are

$$\bar{R} - R_f e - b_r V w = 0 \tag{44}$$

Solving for w, we obtain

$$w^* = \frac{1}{b_r} V^{-1} (\bar{R} - R_f e) \tag{45}$$

Thus, we see that the individual's optimal portfolio choice depends on b_r, her coefficient of relative risk aversion, and the expected returns and covariances of the assets. The greater the individual's relative risk aversion, b_r, the smaller the proportion of wealth invested in the risky assets. In fact, multiplying both sides of (45) by W_0, we see that the absolute amount of wealth invested in the risky assets is

$$W_0 w^* = \frac{1}{b} V^{-1} (\bar{R} - R_f e) \tag{46}$$

Thus the individual with constant absolute risk aversion, b, invests a fixed dollar amount in the risky assets, independent of her initial wealth. As wealth increases, each additional dollar is invested in the risk-free asset.