Finance 400

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Risk Aversion and Portfolio Choice

Having developed a concept of risk aversion, we now consider the relation between risk aversion and portfolio choice in a single-period context. Let us assume there is a riskless security that pays a rate of return equal to r_f. In addition, for simplicity suppose there is just one risky security that pays a stochastic rate of return equal to $\widetilde{r}$.

Let W_0 be the individual's initial wealth, and let A be the dollar amount that the individual invests in the risky asset at the beginning of the period. Thus, $W_0 - A$ is the initial investment in the riskless security. Denoting the individual's end-of-period wealth as $\tilde{W}$, it satisfies:

$$\tilde{W} = (W_0 - A)(1 + r_f) + A(1 + \tilde{r}) \tag{1}$$

$$= W_0(1 + r_f) + A(\tilde{r} - r_f)$$

We assume that the individual cares only about consumption at the end of this single period. Therefore, maximizing end-of-period consumption is equivalent to maximizing end-of-period wealth. Assuming that the individual is a von Neumann-Morgenstern expected utility maximizer, she chooses her portfolio by maximizing the expected utility of end-of-period wealth:

$$\max_A E[U(\tilde{W})] = \max_A E\left[U\left(W_0(1 + r_f) + A(\tilde{r} - r_f)\right)\right] \tag{2}$$

Taking the first order condition with respect to A, we obtain

$$E\left[U'\left(\tilde{W}\right)(\tilde{r} - r_f)\right] = 0 \tag{3}$$

This condition determines the amount, A, that the individual invests in the risky asset.[1]

[1] The second order condition for a maximum, $E\left[U''\left(\tilde{W}\right)(\tilde{r} - r_f)^2\right] \leq 0$, is satisfied because $U''\left(\tilde{W}\right) \leq 0$ due to the assumed concavity of the utility function.

Consider the special case in which the expected rate of return on the risky asset equals the risk-free rate. In that case $A = 0$ satisfies the first order condition. To see this, note that when $A = 0$, then W and, therefore, $U'(W)$ are non-stochastic and $E\left[U'\left(\tilde{W}\right)(\tilde{r} - r_f)\right] = U'(W)\,E[\tilde{r} - r_f] = 0$. This result is reminiscent of our earlier finding that a risk-averse individual would not choose to accept a fair lottery. Here, the fair lottery is interpreted as a risky asset that has an expected rate of return just equal to the risk-free rate.

Next, consider the case in which $E[\tilde{r}] - r_f > 0$. Clearly, $A = 0$ would not satisfy the first order condition because $E\left[U'\left(\tilde{W}\right)(\tilde{r} - r_f)\right] = U'(W)\,E[\tilde{r} - r_f] > 0$ when $A = 0$. Rather, when $E[\tilde{r}] - r_f > 0$ condition (3) is satisfied only when $A > 0$. To see this, let r^h denote a realization of $\tilde{r}$ such that it exceeds r_f, and let W^h be the corresponding level of $\tilde{W}$. Also let r^l denote a realization of $\tilde{r}$ such that it is lower than r_f, and let W^l be the corresponding level of $\tilde{W}$. Obviously, $U'(W^h)(r^h - r_f) > 0$ and $U'(W^l)(r^l - r_f) < 0$. For $U'\left(\tilde{W}\right)(\tilde{r} - r_f)$ to average to zero for all realizations of $\tilde{r}$, it must be the case that $W^h > W^l$ so that $U'\left(W^h\right) < U'\left(W^l\right)$ due to the concavity of the utility function. This is because since $E[\tilde{r}] - r_f > 0$, there is more probability mass on the r^h realizations than the r^l realizations. Therefore, to make $U'\left(\tilde{W}\right)(\tilde{r} - r_f)$ average to zero, the positive $(r^h - r_f)$ terms need to be given weights, $U'\left(W^h\right)$, that are smaller than the weights, $U'(W^l)$, that multiply the fewer negative $(r^l - r_f)$ realizations. This can only occur if $A > 0$ so that $W^h > W^l$. The implication is that an individual will always hold at least some positive amount of the risk-free asset if its expected rate of return exceeds the risk-free rate.[2]

Now, we can go further and explore the relationship between A and the individual's initial wealth, W_0. Using the envelope theorem, we can differentiate the first order condition to obtain[3]

[2]Related to this is the notion that a risk-averse expected utility maximizer should accept a small lottery with a positive expected return. In other words, such an individual should be close to risk-neutral for small-scale bets. However, M. Rabin and R. Thaler (2001) "Risk Aversion," *Journal of Economic Perspectives* 15, 219-232 claim that individuals frequently reject lotteries (gambles) that are modest in size yet have positive expected returns. From this they argue that concave expected utility is not a plausible model for predicting an individual's choice of small-scale risks.

[3]The envelope theorem applies to the problem of examining how the maximized value of the objective function and the control variable change when one of the model's parameters changes. In our context, define $f(A, W_0) \equiv E\left[U\left(\widetilde{W}\right)\right]$ so that $v(W_0) = \max_A f(A, W_0)$ is the maximized value of the objective function when the control variable, A, is optimally chosen. Then applying the chain rule, we have $\frac{dv(W_0)}{dW_0} = \frac{\partial f(A,W_0)}{\partial A}\frac{dA(W_0)}{dW_0} + \frac{\partial f(A,W_0)}{\partial W_0}$. But since $\frac{\partial f(A,W_0)}{\partial A} = 0$, from the first order condition, this simplifies to just $\frac{dv(W_0)}{dW_0} = \frac{\partial f(A,W_0)}{\partial W_0}$. Again applying the chain rule to the first order condition, one obtains $\frac{\partial(\partial f(A,W_0)/\partial A)}{\partial W_0} = 0 = \frac{\partial^2 f(A,W_0)}{\partial A^2}\frac{dA(W_0)}{dW_0} + \frac{\partial^2 f(A,W_0)}{\partial A \partial W_0}$.

$$E\left[U''(\tilde{W})(\tilde{r}-r_f)(1+r_f)\right]dW_0 + E\left[U''(\tilde{W})(\tilde{r}-r_f)^2\right]dA = 0 \tag{4}$$

or

$$\frac{dA}{dW_0} = \frac{(1+r_f)E\left[U''(\tilde{W})(\tilde{r}-r_f)\right]}{-E\left[U''(\widetilde{W})(\tilde{r}-r_f)^2\right]} \tag{5}$$

The denominator of (5) is positive because concavity of the utility function ensures that $U''(\tilde{W})$ is negative. Therefore, the sign of the expression depends on the numerator, which can be of either sign because realizations of $(\tilde{r}-r_f)$ can turn out to be both positive and negative.

To characterize situations in which the sign of (5) can be determined, let us first consider the case where the individual has absolute risk aversion that is decreasing in wealth. As before, let r^h denote a realization of $\tilde{r}$ such that it exceeds r_f, and let W^h be the corresponding level of $\tilde{W}$. Then for $A \geqslant 0$, we have $W^h \geqslant W_0(1+r_f)$. If absolute risk aversion is decreasing in wealth, this implies

$$R\left(W^h\right) \leqslant R\left(W_0(1+r_f)\right) \tag{6}$$

where, as before, $R(W) = -U''(W)/U'(W)$. Multiplying both terms of (6) by $-U'(W^h)(r^h - r_f)$, which is a negative quantity, the inequality sign changes:

$$U''(W^h)(r^h - r_f) \geqslant -U'(W^h)(r^h - r_f)R\left(W_0(1+r_f)\right) \tag{7}$$

Next, we again let r^l denote a realization of $\tilde{r}$ such that it is lower than r_f and define W^l be the corresponding level of $\tilde{W}$. Then for $A \geqslant 0$, we have $W^l \leqslant W_0(1+r_f)$. If absolute risk aversion is decreasing in wealth, this implies

$$R(W^l) \geqslant R\left(W_0(1+r_f)\right) \tag{8}$$

Multiplying (8) by $-U'(W^l)(r^l - r_f)$, which is positive, so that the sign of (8) remains the

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Re-arranging gives us $\frac{dA(W_0)}{dW_0} = -\frac{\partial^2 f(A,W_0)}{\partial A \partial W_0} / \frac{\partial^2 f(A,W_0)}{\partial A^2}$, which is equation (5).

same, we obtain

$$U''(W^l)(r^l - r_f) \geqslant -U'(W^l)(r^l - r_f)R\left(W_0(1 + r_f)\right) \tag{9}$$

Notice that inequalities (7) and (9) are of the same form. The inequality holds whether the realization is $\tilde{r} = r^h$ or $\tilde{r} = r^l$. Therefore, if we take expectations over all realizations, where $\tilde{r}$ can be either higher than or lower than r_f, we obtain

$$E\left[U''(\tilde{W})(\tilde{r} - r_f)\right] \geqslant -E\left[U'(\tilde{W})(\tilde{r} - r_f)\right] R\left(W_0(1 + r_f)\right) \tag{10}$$

Since the first term on the right-hand-side is just the first order condition, inequality (10) reduces to

$$E\left[U''(\tilde{W})(\tilde{r} - r_f)\right] \geqslant 0 \tag{11}$$

Thus, the first conclusion that can be drawn is that declining absolute risk aversion implies $dA/dW_0 > 0$, that is, the individual invests an increasing amount of wealth in the risky asset for larger amounts of initial wealth. For two individuals with the same utility function but different initial wealths, the more wealthy one invests a greater dollar amount in the risky asset if utility is characterized by decreasing absolute risk aversion. While not shown here, the opposite is true, namely, that the more wealthy individual invests a smaller dollar amount in the risky asset if utility is characterized by increasing absolute risk aversion.

Thus far, we have not said anything about the *proportion* of initial wealth invested in the risky asset. To analyze this issue, we need the concept of relative risk aversion. Define

$$\eta \equiv \frac{dA}{dW_0}\frac{W_0}{A} \tag{12}$$

which is the elasticity measuring the proportional increase in the risky asset for an increase in initial wealth. Adding $1 - \frac{A}{A}$ to the right hand side of (12) gives

$$\eta = 1 + \frac{(dA/dW_0)W_0 - A}{A} \tag{13}$$

Substituting the expression dA/dW_0 from equation (5), we have

$$\eta = 1 + \frac{W_0(1+r_f)E\left[U''(\tilde{W})(\tilde{r}-r_f)\right] + AE\left[U''(\tilde{W})(\tilde{r}-r_f)^2\right]}{-AE\left[U''(\tilde{W})(\tilde{r}-r_f)^2\right]} \tag{14}$$

Collecting terms in $U''(\tilde{W})(\tilde{r}-r_f)$, this can be re-written as

$$\eta = 1 + \frac{E\left[U''(\tilde{W})(\tilde{r}-r_f)\{W_0(1+r_f) + A(\tilde{r}-r_f)\}\right]}{-AE\left[U''(\tilde{W})(\tilde{r}-r_f)^2\right]} \tag{15}$$

$$= 1 + \frac{E\left[U''(\tilde{W})(\tilde{r}-r_f)\tilde{W}\right]}{-AE\left[U''(\tilde{W})(\tilde{r}-r_f)^2\right]}$$

The denominator is always positive. Therefore, we see that the elasticity, η, is greater than one, so that the individual invests proportionally more in the risky asset with an increase in wealth, if $E\left[U''(\tilde{W})(\tilde{r}-r_f)\tilde{W}\right] \geqslant 0$. Can we relate this to the individual's risk aversion? The answer is yes and the derivation is almost exactly the same as that just given.

Consider the case where the individual has *relative* risk aversion that is decreasing in wealth. Let r^h denote a realization of $\tilde{r}$ such that it exceeds r_f, and let W^h be the corresponding level of $\tilde{W}$. Then for $A \geqslant 0$, we have $W^h \geqslant W_0(1+r_f)$. If relative risk aversion, $R_r(W) \equiv WR(W)$, is decreasing in wealth, this implies

$$W^h R(W^h) \leqslant W_0(1+r_f)R\left(W_0(1+r_f)\right) \tag{16}$$

Multiplying both terms of (16) by $-U'(W^h)(r^h - r_f)$, which is a negative quantity, the inequality sign changes:

$$W^h U''(W^h)(r^h - r_f) \geqslant -U'(W^h)(r^h - r_f)W_0(1+r_f)R\left(W_0(1+r_f)\right) \tag{17}$$

Next, let r^l denote a realization of $\tilde{r}$ such that it is lower than r_f, and let W^l be the corresponding level of $\tilde{W}$. Then for $A \geqslant 0$, we have $W^l \leqslant W_0(1+r_f)$. If relative risk aversion is decreasing in wealth, this implies

$$W^l R(W^l) \geqslant W_0(1+r_f)R\left(W_0(1+r_f)\right) \tag{18}$$

Multiplying (18) by $-U'(W^l)(r^l - r_f)$, which is positive, so that the sign of (18) remains the same, we obtain

$$W^l U''(W^l)(r^l - r_f) \geqslant -U'(W^l)(r^l - r_f)W_0(1+r_f)R\left(W_0(1+r_f)\right) \tag{19}$$

Notice that inequalities (17) and (19) are of the same form. The inequality holds whether the realization is $\tilde{r} = r^h$ or $\tilde{r} = r^l$. Therefore, if we take expectations over all realizations, where $\tilde{r}$ can be either higher than or lower than r_f, we obtain

$$E\left[\tilde{W}U''(\tilde{W})(\tilde{r} - r_f)\right] \geqslant -E\left[U'(\tilde{W})(\tilde{r} - r_f)\right] W_0(1+r_f)R(W_0(1+r_f)) \tag{20}$$

Since the first term on the right-hand-side is just the first order condition, inequality (20) reduces to

$$E\left[\tilde{W}U''(\tilde{W})(\tilde{r} - r_f)\right] \geqslant 0 \tag{21}$$

Thus, we see that an individual with decreasing relative risk aversion has $\eta > 1$ and invests proportionally more in the risky asset as wealth increases. The opposite is true for increasing relative risk aversion: $\eta < 1$ so that this individual invests proportionally less in the risky asset as wealth increases.