Finance 400

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Choice Under Uncertainty

Economists typically analyze the price of a good or service by considering the determinants of its supply and demand. The same approach can be taken to price an asset. As a starting point, let us think about how investor demand for an asset might be modeled. In contrast to goods and services, assets do not provide direct, immediate consumption benefits to individuals. Rather, assets are a vehicle for saving. As the components of an investor's financial wealth, assets represent claims on future purchasing power or consumption. The main distinguishing feature of assets is differences in their future payoffs. With the exception of assets that pay a risk-free return, assets' payoffs are random. Thus, a theory of the demand for assets needs to specify investors' preferences for assets with different, uncertain payoffs. In other words, how investors choose between assets that have different probability distributions of returns needs to be modeled.

Let us consider some intuitive criteria that individuals might use to rank their preferences for different risky assets. One possible measure of the attractiveness of an asset is the *expected value* of its payoff. Suppose an asset offers a single random payoff that has a discrete distribution with n outcomes, $(x_1, ..., x_n)$, and corresponding probabilities $(p_1, ..., p_n)$, where $\sum_{i=1}^{n} p_i = 1$. Then the expected value of the payoff is $\bar{x} = \sum_{i=1}^{n} x_i p_i$.

How reasonable is it to think that individuals value risky assets based on the assets' expected values? In 1728, Nicholas Bernoulli clearly demonstrated a weakness of using expected value as the sole measure of preferences with his "St. Petersberg Paradox." He gave the following example. Suppose a particular asset's payoff is determined a game based on a sequence of coin flips. If, on the first flip, the coin comes up "tails," the asset's payoff is zero and the game is over. However, if it comes up "heads," the asset pays \$1 and, in addition, the coin is flipped again. If, on the second flip, "tails" occurs, the game is over, but if "heads" occurs the asset pays an additional \$2 and the coin is flipped once again. If "heads" comes up on the third flip, the asset pays an additional $\$2^2 = \$4$ and so on. Thus, given the coin comes up "heads" on the first n flips, the asset's payment if a "heads" is obtained on the $n + 1^{th}$ flip is $\$2^n$.

How much would an individual pay to acquire this risky asset? In other words, what would someone pay to participate in this coin flipping game? If an individual values the asset's payoff according to its expected value, the price he would be willing to pay is

$$\bar{x} = \tfrac{1}{2}\$1 + \tfrac{1}{4}\$2 + \tfrac{1}{8}\$4 + ... \tag{1}$$

$$= \tfrac{1}{2}(\$1 + \tfrac{1}{2}\$2 + \tfrac{1}{4}\$4 + ...$$

$$= \tfrac{1}{2}(\$1 + \$1 + \$1 + ... = \$\infty$$

Thus, the expected value of this asset is infinite. Clearly, most individuals would pay only a moderate, not infinite, amount to play this game. In 1738, Daniel Bernoulli, a cousin of Nicholas, provided an explanation for the St. Petersberg Paradox by stating that people cared about the expected "utility" of an asset's payoff, not the expected value of its payoff. As an individual's wealth increases, the "utility" that one receives from the additional increase in wealth grows less than proportionally. In the St. Petersberg Paradox, prizes go up at the same rate that the probabilities decline. In order to obtain a finite valuation, the trick would be to allow the "value" or "utility" of prizes to increase slower than the rate probabilities decline.

The first complete axiomatic development of expected utility is due to von Neumann and Morgenstern (1944), which we now illustrate. Define a *lottery* to be risky payoff (asset) and consider an individual's optimal choice of a lottery from a given set of different lotteries. All lotteries have possible payoffs that are contained in the set $\{x_1, ..., x_n\}$. In general, the elements of this set can be viewed as different, uncertain outcomes. For example, they could be interpreted as particular consumption levels (bundles of consumption goods) that the individual obtains in different states of nature or, more simply, different monetary payments received in different states of the world. A given lottery can be characterized as an ordered set of probabilities $P = \{p_1, ..., p_n\}$, where, of course, $\sum\limits_{i=1}^{n} p_i = 1$ and $p_i \geq 0$. A different lottery is characterized by another set of probabilities, for example, $P^* = \{p_1^*, ..., p_n^*\}$. Let $\succ$, $\prec$, and $\sim$ denote preference and indifference between lotteries. We will show that if an individual's preferences satisfy the following conditions (axioms), then these preferences can be represented by a real-valued utility function defined over a given lottery's probabilities, that is, a function $V(p_1, ..., p_n)$.

Axioms:

1) *Completeness*

For any two lotteries P^* and P, either $P^* \succ P$, or $P^* \prec P$, or $P^* \sim P$.

2) *Transitivity*

If $P^{**} \succeq P^*$and $P^* \succeq P$, then $P^{**} \succeq P$.

3) *Continuity*

If $P^{**} \succeq P^* \succeq P$, there exists some $\lambda \in [0,1]$ such that $P^* \sim \lambda P^{**} + (1-\lambda)P$, where $\lambda P^{**} + (1-\lambda)P$ denotes a "compound lottery," namely with probability λ one receives the lottery P^{**} and with probability $(1-\lambda)$ one receives the lottery P.

These three axioms are analogous to those in standard consumer theory and are needed to establish the existence of a real-valued utility function. The fourth axiom is crucial to expected utility theory.

4) *Independence*

For any two lotteries P and P^*, $P^* \succ P$ if and only if for all $\lambda \in$ (0,1] and all P^{**}:

$$\lambda P^* + (1-\lambda)P^{**} \succ \lambda P + (1-\lambda)P^{**}$$

Further, for any two lotteries P and $P^\dagger$, $P \sim P^\dagger$ if and only if for all $\lambda \in$(0,1] and all P^{**}:

$$\lambda P + (1-\lambda)P^{**} \sim \lambda P^\dagger + (1-\lambda)P^{**}$$

To better understand the meaning of the independence axiom, note that P^* is preferred to P by assumption. Now the choice between $\lambda P^* + (1-\lambda)P^{**}$ and $\lambda P + (1-\lambda)P^{**}$ is equivalent to a toss of a coin that has a probability $(1-\lambda)$ of landing "tails", in which case both compound lotteries are equivalent to P^{**}, and a probability λ of landing "heads," in which case the first compound lottery is equivalent to the single lottery P^* and the second compound lottery is equivalent to the single lottery P. Thus, the choice between $\lambda P^* + (1-\lambda)P^{**}$ and $\lambda P + (1-\lambda)P^{**}$ is equivalent to being asked, prior to the coin toss, if one would prefer P^* to P in the event the coin lands "heads."

It would seem reasonable that should the coin land "heads," we would go ahead with our original preference in choosing P^* over P. The independence axiom assumes that preferences over the two lotteries are independent of the way in which we obtain them.[1] For this reason, it is also known as the "no regret" axiom. However, experimental evidence finds some systematic violations of this independence axiom. See the Machina (1987) *Journal of Economic Perspectives* article and, in particular, the Allais Paradox regarding this point.

The final axiom is similar to the independence and completeness axioms.

5) *Dominance*

Let P^1 be the compound lottery $\lambda_1 P^{\ddagger} + (1 - \lambda_1)P^{\dagger}$ and P^2 be the compound lottery $\lambda_2 P^{\ddagger} + (1 - \lambda_2)P^{\dagger}$. If $P^{\ddagger} \succ P^{\dagger}$, then $P^1 \succ P^2$ if and only if $\lambda_1 > \lambda_2$.

Given preferences characterized by the above axioms, we now show that the choice between any two (or more) arbitrary lotteries is that which has the higher (highest) expected utility.

The completeness axiom's ordering on lotteries naturally induces an ordering on the set of outcomes. Without loss of generality, suppose that the outcomes are ordered such that $x_n \succeq x_{n-1} \succeq ... \succeq x_1$. This follows from the completeness axiom for the case of n degenerate or "primitive" lotteries where the i^{th} primitive lottery is defined to return outcome x_i with probability 1 and all of the other outcomes have probability zero. Note that this ordering may not necessarily coincide with ranking the elements of x strictly by the size of their monetary payoffs, as the state of nature for which x_i is the outcome may differ from the state of nature for which x_j is the outcome, and these states of nature may have different effects on how an individual values the same monetary outcome. For example, x_i may be received in a state of nature when the economy is depressed, and monetary payoffs may be highly valued in this state of nature. In contrast, x_j may be received in a state of nature characterized by high economic expansion, and monetary payments may not be as highly valued.

From the continuity axiom, we know that for each x_i, there exists a $U_i \in [0, 1]$ such that

[1] In the context of standard consumer choice theory, λ would be interpreted as the amount (rather than probability) of a particular good or bundle of goods consumed (say C) and $(1 - \lambda)$ as the amount of another good or bundle of goods consumed (say C^{**}). In this case, it would not be reasonable to assume that the choice of these different bundles is independent. This is due to some goods being substitutes or complements with other goods. Hence, the validity of the independence axiom is linked to outcomes being uncertain (risky), that is, the interpretation of λ as a probability rather than a deterministic amount.

$$x_i \sim U_i x_n + (1 - U_i)x_1 \tag{2}$$

and for $i = 1$, this implies $U_1 = 0$ and for $i = n$, this implies $U_n = 1$. The values of the U_i weight the most and least preferred outcomes such that the individual is just indifferent between a combination of these polar payoffs and the payoff of x_i. The U_i can adjust for both differences in monetary payoffs and differences in the states of nature during which the outcomes are received.

Now consider a given arbitrary lottery, $P = \{p_1, ..., p_n\}$. This can be considered a compound lottery over the n primitive lotteries, where the i^{th} primitive lottery which pays x_i with certainty is obtained with probability p_i. By the independence axiom, and using equation (2), the individual is indifferent between the compound lottery, P, and the following lottery given on the right-hand-side of the equation below:

$$p_1x_1 + ... + p_nx_n \sim p_1x_1 + ... + p_{i-1}x_{i-1} + p_i\left[U_ix_n + (1 - U_i)x_1\right] + p_{i+1}x_{i+1} + ... + p_nx_n \tag{3}$$

where we have used the indifference relation in equation (2) to substitute for x_i on the right hand side of (3). By repeating this substitution for all i, $i = 1, ..., n$, we see that the individual will be indifferent between P, given by the left hand side of (3), and

$$p_1x_1 + ... + p_nx_n \sim \left(\sum_{i=1}^{n} p_iU_i\right) x_n + \left(1 - \sum_{i=1}^{n} p_iU_i\right) x_1 \tag{4}$$

Now define $\Lambda \equiv \sum_{i=1}^{n} p_iU_i$. Thus, we see that lottery P is equivalent to a compound lottery consisting of a Λ probability of obtaining x_n and a $(1 - \Lambda)$ probability of obtaining x_1. In a similar manner, we can show that any other arbitrary lottery $P^* = \{p_1^*, ..., p_n^*\}$ is equivalent to a compound lottery consisting of a Λ^* probability of obtaining x_n and a $(1 - \Lambda^*)$ probability of obtaining x_1, where $\Lambda^* \equiv \sum_{i=1}^{n} p_i^*U_i$.

Thus, we know from the dominance axiom that $P^* \succ P$ if and only if $\Lambda^* > \Lambda$, which implies $\sum_{i=1}^{n} p_i^*U_i > \sum_{i=1}^{n} p_iU_i$. So defining an expected utility function as

$$V(p_1,...,p_n) = \sum_{i=1}^{n} p_i U_i \tag{5}$$

will imply that $P^* \succ P$ if and only if $V(p_1^*,...,p_n^*) > V(p_1,...,p_n)$.

The utility function given in equation (5) is known as a von Neumann - Morgenstern utility function. Note that it is linear in the probabilities and is unique up to a linear transformation. This implies that the utility function is "cardinal," unlike the ordinal utility functions of standard consumer theory.[2] For example, if $U_i = U(x_i)$, an individual's choice over lotteries will be the same under the transformation $aU(x_i) + b$, but not a non-linear transformation that changes the "shape" of $U(x_i)$.

The von Neumann-Morgenstern expected utility framework may only partially explain the phenomenon illustrated by the St. Petersberg Paradox. Suppose utility is given by the square root of a monetary payoff, that is, $U_i = U(x_i) = \sqrt{x_i}$. This is a monotonically increasing, concave function of x, which we here assume is simply a monetary amount. Then the expected utility (value) of the payoff of the St. Petersberg asset is

$$V = \sum_{i=1}^{n} p_i U_i = \sum_{i=1}^{\infty} \frac{1}{2^i}\sqrt{2^{i-1}} = \sum_{i=2}^{\infty} 2^{-\frac{i}{2}} \tag{6}$$

$$= 2^{-\frac{2}{2}} + 2^{-\frac{3}{2}} + ...$$

$$= \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^i - 1 - \frac{1}{\sqrt{2}} = \frac{1}{2-\sqrt{2}} \cong 1.7071$$

which is finite. The asset would be worth \$1.71 to a square-root utility maximizer.

However, the reason that this is not a complete resolution of the paradox is that one can always construct a "super St. Petersberg paradox" where even expected utility is infinite. Note that in the regular St. Petersberg paradox, the probability of winning declines at rate 2^n while the winning payoff increases at rate 2^n. In a super St. Petersberg paradox, we can make the winning payoff increase at a rate $x_n = U^{-1}(2^n)$ and expected utility would no longer be finite.

[2] An ordinal utility function preserves preference orderings for *any* strictly increasing transformation, not just linear ones.

If we take the example of square-root utility, let the winning payoff be $x_n = 2^{2n-2}$, that is, $x_1 = 1$, $x_2 = 4$, $x_3 = 16$, etc. In this case, the value placed on the asset by a square-root expected utility maximizer is

$$V = \sum_{i=1}^{n} p_i U_i = \sum_{i=1}^{\infty} \frac{1}{2^i} \sqrt{2^{2i-2}} = \infty \tag{7}$$

Should we be concerned by the fact that if we let the prizes grow quickly enough, we can get infinite valuations for any chosen form of expected utility function? Maybe not. One could argue that St. Petersberg games are unrealistic, particularly ones where the payoffs are assumed to grow rapidly. The reason is that any person offering this asset has finite wealth (even Bill Gates). This would set an upper bound on the amount of prizes that could feasibly be paid, making expected utility, and even the expected value of the payoff, finite.

The von Neumann-Morgenstern expected utility approach can be generalized to the case of a continuum of outcomes and lotteries having continuous probability distributions. For example, if outcomes are a possibly infinite number of purely monetary payoffs or consumption levels denoted by the variable x, a subset of the real numbers, then a generalized version of equation (5) is

$$V(F) = \int U(x)\, dF(x) \tag{8}$$

where $F(x)$ is a given lottery's cumulative distribution function over the payoffs, x.[3] Hence, the generalized lottery represented by the distribution function F is analogous to our previous lottery represented by the discrete probabilities $P = \{p_1, ..., p_n\}$. For a given lottery, expected utility defined over the random payoff $\widetilde{x}$ can also be represented as

$$E[U(\widetilde{x})] = \int U(x)\, dF(x) \tag{9}$$

[3]When the random payoff, x, is absolutely continuous, then expected utility can be written in terms of the probability density function, $f(x)$, as $V(f) = \int U(x) f(x)\, dx$.