Notes on Performance of Linear Factor Models: Alphas and Idiosyncratick Risk

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1 Statistic

Consider some collection $y_t = (y_{t1}, y_{t2}, ..., y_{tn})$ of observable excess returns on n risky portfolios, at some dates $t = 1, 2, ..., T$. Suppose that the primary objective is to find a relatively small number $K << n$ of observed key portfolios whose (excess) returns $x_t = (x_{t1}, ..., x_{tK})$ are strongly related to y_t, both in history $t = 1, 2, ..., T$ and, plausibly, into the future. Two specific objectives are to link time series variation within target portfolios with key portfolios and to link cross section variation between target portfolios with key portfolios.

Suppose that the collection $z_t = (x_t, y_t)$ of random variables is jointly normally distributed, and is independent and identically distributed (IID) over time. Fama and French (1993, FF93 henceforth) are not explicit about whether they view their data as IID and normally distributed, but their econometric style is ideally suited to this case. In this setting, the relationship between x and y is adequately summarized by the linear model:

$$y_{it} = \alpha_i + \beta_i x_t + \varepsilon_{it}, \tag{1}$$

with unobserved portfolio-specific intercept parameters α_i, coefficient K-vectors β_i, and model errors ε_{it} which are normal zero-mean random variables with some variances σ_i^2 and a (collective) variance-covariance matrix V_ε. The linear model (1) nominally presents itself as a rather generic econometric model of panel data, with heterogeneous fixed effects α_i and slopes β_i.

In terms of the regression model (1), FF93 view small regression error magnitude $|\varepsilon_{it}|$ as good, consistent with standard econometrics, and also view small intercept magnitude $|\alpha_i|$ as good, with the idea that small intercepts are more consistent with financial theory. However, Fama and French do not commit themselves to any formal measure, or metric, for assessing overall goodness of fit, and instead rely on a succession of informal commentaries. To cater more specifically to the demands that FF93 place on financial models, one possibility is to try to fit a model so as to minimize a linear combination of intercept (α_i) and variability (σ) of error squares i.e. the root mean squared error (RMSE). We define the i-th square error δ_i^2 as:

$$\delta_i^2 = \alpha_i^2 + \sigma_i^2$$

with squared bias term α_i^2 and variance term σ_i^2. We want to size up the errors δ_i in models of systematic risk. In the aggregate, across assets $i = 1, 2, ..., n$, RMSE is a simple measure of the magnitude of model error:

$$\phi = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\delta_i^2}.$$

RMSE can be expressed as a sum of these two components:

$$\phi = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\alpha_i^2 + \sigma_i^2)}.$$

We further define constants $\chi_1 = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\alpha_i^2}$ and $\chi_2 = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2}$, in which case

$$\phi = \sqrt{\chi_1^2 + \chi_2^2}.$$

The constant χ_1 is the root mean square of biases, and hence represents the model's non-stochastic error magnitude, whereas χ_2 is the root mean square of standard deviations, and is therefore the stochastic error magnitude. Table 1 summarizes our notation for future reference.

RMSE is a simple, ex ante measure of goodness of fit, and for a given sample we can estimate it ex post via fitted intercepts and standard errors. As a model performance criterion

RMSE is novel in econometric terms. Such novelty is necessary, as no standard model performance criterion (mean squared error, information criteria, etc.) adequately formalizes the FF93 perspective on model building. To apply the RMSE value ϕ, and its components, note that all of these constants can be computed if bias α_i and error variance σ_i^2 is known for each asset i. It then suffices to have good estimators of α and σ. To estimate the regression system (1), we suppose that for all n asset returns there are observations at successive times, denoted $t = 1, 2, ..., T$, such that the return probability distribution exhibits independence over time. With assumed normality, the maximum likelihood estimator (MLE) of (α, β) is the same as the ordinary least squares (OLS) estimator applied to each separate instance $i = 1, 2, ..., n$. Denoting these MLEs by $\hat{\alpha}, \hat{\beta}$, which are unbiased estimators, let $\hat{\sigma}_i^2$ be the OLS regression standard error (of the estimate), which is a bias-corrected version of the MLE for σ^2. Plugging in $\hat{\alpha}$ and $\hat{\sigma}$ into formulas in Table 1, we obtain consistent estimates $\hat{\chi_1}, \hat{\chi_2}$, and $\hat{\phi}$. While not necessarily supported by empirical evidence, this set-up is implicitly assumed in FF93.

We use an updated version of the FF93 monthly data for the sample 1926:7-2008:9. The dependent variables to be explained are excess stock returns on 25 portfolios, sorted on size and (independently) on book-to-market equity (BE/ME). The portfolios are constructed by FF93 as follows. The quintile breakpoints for size in a given year are based on market capitalization of NYSE stocks in June of the same year. The quintiles for book-to-market ratios are calculated using NYSE stocks using BE and ME from December of the previous year. The portfolios are then formed using stocks from NYSE, AMEX and NASDAQ, for which there is a positive book equity (from COMPUSTAT) available from December of the previous year and market equity available in June of the given year and December of the previous year. Finally, value-weighted monthly portfolio returns are computed starting in July of the current year and ending in June of the following year (stock prices are from CRSP). The excess returns are calculated using the one-month Treasury bill rate (from

Ibbotson Associates).

The explanatory variables are the market excess return plus the two additional empirically motivated factors – SMB and HML – related to size and book-to-market ratios. The market excess return is defined as the value-weight return on all NYSE, AMEX, and NASDAQ stocks (from CRSP) minus the one-month Treasury bill rate (from Ibbotson Associates). The two latter factors are constructed from six portfolios, again sorted on size and book-to-market equity. The algorithm to construct these portfolios is the same as above, with size breakpoint being the median and book-to-market equity breakpoints being respectively the 30th and the 70th NYSE percentiles.

To construct the SMB factor, all available stocks are divided into two groups based on median market equity (size), *Small* and *Big*. For the HML factor, the stocks are grouped by their book-to-market equity ratios (BE/ME), and the breakpoints are the 30th and the 70th BE/ME percentiles, resulting in three BE/ME categories: *High, Medium* and *Low*. High BE/ME is consistently associated with low earnings on assets (the so called value stocks) and vice versa (the growth stocks). The returns on SMB and HMB are respectively calculated as

$$\begin{aligned} SMB &= 1/3\,(Small\ High + Small\ Medium + Small\ Low) \\ &- 1/3\,(Big\ High + Big\ Medium + Big\ Low). \end{aligned} \tag{2}$$

and

$$\begin{aligned} HML &= 1/2\,(Small\ High + Big\ High) \\ &- 1/2\,(Small\ Low + Big\ Low). \end{aligned} \tag{3}$$

2 Results

We first calculate the three statistics for the full sample i.e. 1926:7-2008:9. χ_1 statistic in Table 2 suggests that the Fama-French model may not be better then CAPM when the intercepts are concerned. What is interesting is that this result is due to high alpha's for the lowest book-to-market/small size and the highest book-to-market/small size groups - see Table 3. We calculated the statistics for each month using five-year data windows- see Figure 1. The Fama-French model tends to do better in periods of higher volatility in the sense of the alphas closer to zero.

Figure 1: Model Errors for Five-year Samples

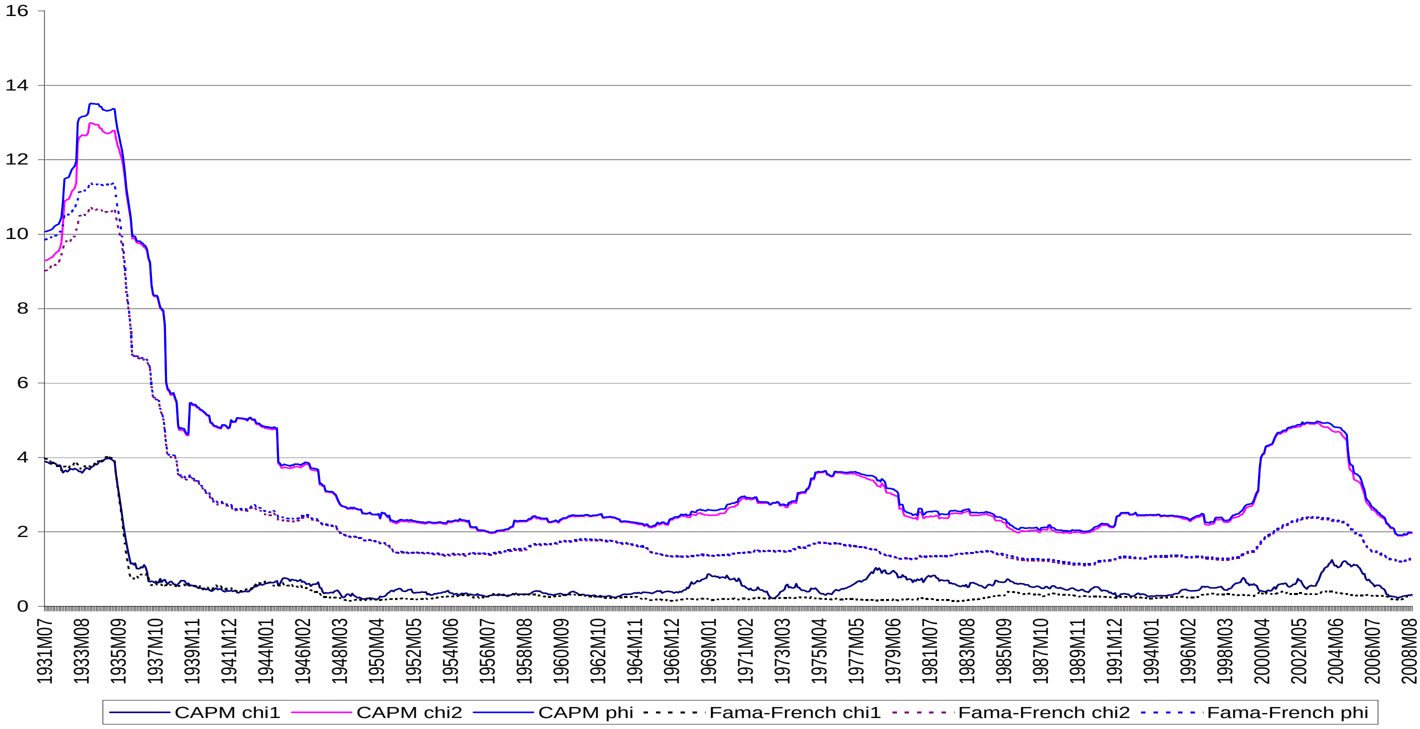

Table 1: Measures of Model Error

Measure	Symbol	Formula
non-stochastic error	χ_1	$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\alpha_i^2}$
stochastic error	χ_2	$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2}$
total error (RMSE)	ϕ	$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\alpha_i^2+\sigma_i^2)}$

Table 2: Sample Errors of CAPM and Fama-French Models, Size/Value Portfolios, 1926:07-2008:09

Model	χ_1	χ_2	ϕ
CAPM	0.32	4.66	4.67
Fama-French	0.34	3.44	3.46

Note:

χ_1, χ_2, ϕ are defined in Table 1.

Table 3: Alphas and Idiosyncratic Risk

	Book-to-market quintile									
	Low	2	3	4	5	Low	2	3	4	5
Size quintile										
					CAPM					
			α_i^2					σ_i^2		
small	0.35	0.02	0.02	0.13	0.27	73.33	50.29	29.20	25.41	35.22
2	0.05	0.02	0.08	0.10	0.13	18.53	14.77	12.54	13.62	21.48
3	0.02	0.03	0.07	0.08	0.06	10.79	5.92	6.82	8.88	18.02
4	0.00	0.00	0.03	0.05	0.03	5.12	3.89	5.31	8.79	19.81
big	0.00	0.00	0.00	0.00	1.07	2.43	2.44	4.75	10.19	134.61
					Fama-French					
			α_i^2					σ_i^2		
small	0.75	0.17	0.02	0.00	0.01	54.18	22.28	12.03	5.65	6.19
2	0.06	0.00	0.01	0.00	0.00	6.74	4.40	3.40	2.83	3.61
3	0.03	0.01	0.01	0.00	0.01	4.17	3.29	3.62	3.39	4.84
4	0.01	0.00	0.00	0.00	0.04	2.73	3.07	3.69	3.96	6.72
big	0.01	0.00	0.00	0.05	1.74	1.44	2.08	3.10	3.63	125.63