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Notes on the Demand for Labor
(Orley Ashenfelter)

It is conventional in analyzing the demand for labor to embed the typical firm in a competitive industry so as to show how the demand for labor at the industry level depends on (1) technology, (2) the demand for the product produced by the industry, and (3) the supply of cooperating factors (capital) to the industry. This leads to the rules describing the "derived" demand for labor as set out in Hicks' Theory of Wages (the appendix), and first due apparently to Marshall.

The purpose of studying the demand for labor in this way is to allow a partial equilibrium analysis to carry us a little farther toward a general equilibrium analysis without going so far as to make the problem empirically unmanageable. This is important for the examination of most policies that affect the demand for labor since few of these policies affect individual firms alone. We start with the typical firm; then, for the sake of exposition, embed the firm in an industry where capital is perfectly elastically supplied; and finally take up the case where capital is supplied with finite elasticity.

I.

The cost minimizing firm that produces a level of output q, facing the wage rate, w, and the rental on capita r demands labor, L, and capital k, according to the output constant factor demand functions

(1) $L = L(q,w,r,)$

(2) $k = k(q,w,r,)$.

These results are derived in the usual way by setting up a Lagrangean $V = wL + rk + \lambda[q - f(L, k)]$ and minimizing with respect to L and k for fixed q, w, and r. The solution of the marginal conditions $w = \lambda(\partial f/\partial L)$, $r = \lambda(\partial f/\partial k)$, and $f(L, k) = q$, the production function, for the endogenous variables L, k, and λ as functions of the exogenous variables q, w, and r gives rise to the optimizing demand rules (1) and (2).

Since we will be interested in the effect of small changes in w, r, and q, on the optimum demands it is useful to differentiate (1) and (2) totally to get

(3) $dL = (\partial L/\partial q)dq + (\partial L/\partial w)dw + (\partial L/\partial r)dr$

(4) $dk = (\partial k/\partial q)dq + (\partial k/\partial w)dw + (\partial k/\partial r)dr$.

The student should know that the demand functions (1) and (2) are homogeneous of degree zero in w and r, and that the "cross" effects are equal, that is $\partial L/\partial r = \partial k/\partial w$ (symmetry). Observing that $dz = zd \ln z$ for any variable, replacing the differentials in (3) and (4) accordingly, and dividing (3) by L and (4) by k gives the differentials of the demand functions in logarithmic form:

(5) $d\ln L = \eta_{Lq} d\ln q + \eta_{Lw} d\ln w + \eta_{Lr} d\ln r$

(6) $d\ln k = \eta_{kq} d\ln q + \eta_{kw} d\ln w + \eta_{kr} d\ln r$,

where $\eta_{Lq} = \frac{q}{L}\partial L/\partial q$ is the output elasticity of demand for labor and the other η's are also elasticities. Note carefully that homogeneity of the demand functions implies $\eta_{Lw} = -\eta_{Lr}$ and $\eta_{kr} = -\eta_{kr}$. Likewise,

(7)
$$\begin{aligned} -\eta_{kw} &= -(\partial k/\partial w)(w/k) \\ &= -(\partial L/\partial r)(w/k) \\ &= -(r/L)(\partial L/\partial r)(wL/rk) \end{aligned}$$

$$= -\eta_{Lr}[S/(1-S)]$$
$$= \eta_{Lw}[S/(1-S)],$$

Where S = wL/(wL + rk), the share that labor costs are of total costs, and the first line is a definition, the second line results from the symmetry property ($\partial k/\partial w = \partial L/\partial r$), and the third line is obtained by multiplication by (r/L)(L/r).

It is conventional to define the substitution possibilities between labor and capital in this model by the elasticity of substitution, which is

(8) $$-d\ln(L/k)/d\ln(w/r)\Big|_{dq=0} = \sigma$$

and is defined for dlnq = 0. σ is the (negative of the) proportionate change in the labor/capital ratio that results from a small proportionate change in the ratio of the wage to the rental rate on capital, output being kept constant. Several points should be recognized in connection with this "elasticity of substitution." First, it is not inherently a technical definition. The elasticity of substitution is interesting because of its behavioral implications as a measure of the price responsiveness of firms. Of course, because $w = \lambda(\partial f/\partial L)$ and $r = \lambda(\partial f/\partial k)$ σ may also be defined as

(8a) $$\sigma = - d\ln(L/k)/d\ln(f_L / f_k),$$

where I have put $\partial f/\partial L = f_L$ for the marginal product of labor and $\partial f/\partial k = f_k$ for the marginal product of capital. Sure enough, (8a) reflects only the variables of the production function, but of what interest would this particular function be if we did not suppose that firms were cost minimizers? Of course, (8a) does tell us, from production data alone, what the price responsiveness of a firm would be if it were a cost minimizer. A second point to note is that $\sigma(w,r,q)$ in (8) is not a constant but a function.

In (8) it is a function of w, r, and q, while in (8a) it is a function of L and k. Only by mere chance, perhaps in an econometrician's dreamland, would σ be constant. Of course, it might be convenient to assume that σ was a constant in some applications, though certainly not in others. As one would expect, there is a simple relationship between σ and the elasticity of demand for labor. To see this, put $d\ell nq = 0$ on the right-hand side of (5) and (6) and subtract the latter from the former to get

$$(9) \quad dlnL - dlnk = dln(L/k)$$
$$= \eta_{Lw}\, dlnw + \eta_{Lr} dlnr - \eta_{kw} dlnw - \eta_{kr} dlnr$$
$$= \eta_{Lw}[dlnw - dlnr] - \eta_{kw}[dlnw - dlnr]$$
$$= (\eta_{Lw} - \eta_{kw})[dlnw - dlnr]$$
$$= (\eta_{Lw} - \eta_{kw})[dln(w/r)].$$

where the third line follows from the fact that $\eta_{Lw} = -\eta_{Lr}$ and $\eta_{kw} = -\eta_{kr}$ the homogeneity result. Dividing both sides of (9) by $d\ell n(w/r)$ then shows that

$$(10) \quad \sigma = \eta_{kw} - \eta_{Lw}$$
$$= -\eta_{Lw}[S/(l-S)l - \eta_{Lw}$$
$$= -\eta_{Lw}/(1-S),$$

where the second line follows from substituting (7). (10) says that the elasticity of substitution is merely the negative of the output-constant wage rate elasticity of demand for labor divided by capital costs as a share of total costs.

II.

To proceed to the analysis of the industry we must consider two other relationships. One of these is the demand function for industry output

(11) $q = q(p)$,

where p is the price of output. In logarithmic (elasticity form) this is

(12) $d\ln q = -\eta d\ln p$.

Notice that two implicit assumptions are being made here: The prices of other products remain constant as does total income. These assumptions are maintained throughout, and to the extent they were empirically invalid one would want to take account of that fact in using the following analysis in practical matters.

Second, we assume that firms operate under constant returns to scale so that average and marginal costs are equal. Moreover, we assume that new entry always keeps output price equal to average costs. In this case it is easy to show that output price is always a function of input prices only: the industry supply function is perfectly elastic. Moreover, we may always write

(13) $d\ln p = S\, d\ln w + (1-S) d\ln r$.

It is now easy to see what happens when the wage faced by all firms in the industry goes up. On the one hand, from (13) the output price that firms must receive goes up, and from (12) the demand for the output of firms declines. Presumably this will induce from (5) and (6) a decline in the demand for labor. In addition, an increase in the wage relative to "the cost of capital, r, will induce capital labor substitution from (8), even if output remained constant. The sum of these "output" and "substitution" effects is the full effect of 'the wage change on the demand for labor.

To see this more formally, assume $d\ln r = 0$ --capital is perfectly elastically supplied -- and substitute (13) into (12) and the result into (5) to get

(14) $d\ln L = \eta_{Lq} [-\eta d\ln p) + \eta_{Lw} d\ln w$

$= \eta_{Lq}$ [-ηS dlnw] + η_{Lw} dlnw

= (η_{Lw}-ηS) dlnw,

where the third line follows from the fact that η_{Lq} = 1 under constant returns to scale. A more conventional formula follows by substituting from (10) into (14) to get

(15) dlnL = -[σ(l-S) + ηS]dlnw.

This shows that the elasticity of demand for labor is merely

(16) ∂lnL/∂lnW/$_{dr=0}$ = -[σ(1-S) +ηS]

a weighted average of the elasticity of substitution and the elasticity of product demand.

Formula (16) is the simplest version of the "elasticity of derived demand" from which three of Marshall's four rules may be derived. Two of these are that the elasticity of demand is greater the greater is the ease of substituting capital for labor and the greater is the elasticity of demand for the product. Likewise, by differentiating (16) with respect to S it is easy to see that the elasticity of demand for labor is greater for larger S so long as $\eta > \sigma$, the elasticity of product demand is greater than the elasticity of substitution. It is nice to be unimportant in the production process so long as substitution away from the product by consumers is easier than substitution away from labor by producers.

III.

Before turning to the case where the supply of capital is taken to be other than infinitely elastic it is useful to consider the many-factor analogue to the preceding case. The importance of this case is that it provides a handle for the case where there may be many types of labor involved in the production process, skilled, semi-skilled, and so on. Treating each type of labor as a separate factor is one way to handle this "many types of labor" case.

It should be obvious that the generalization will be pretty much trivial, so long as we stick with the constant returns case. First, equations (5) and (6), the logarithmic differentials of the factor demand equations, have immediate analogies as

(17) $\quad dlknX_i = \eta_{iq}\, dlnq + \sum_{j=1}^{m} n_{ij}\; dlnw_j \qquad\qquad (i = 1,...,m),$

where there are m factors, η_{iq} is the elasticity of demand for the ith factor with respect to output., and η_{ij} is the output-constant elasticity of demand for the ith input with respect to the jth price. The demand function for output (12) is the same as before, and constant returns to scale merely implies that we write the output-price/input-price relationship as

(18) $\quad dlnp = \sum_{j=1}^{m} s_j\; dlnw_j,$

where $\sum S_j = 1$, S_j is the share of the jth inputs costs in total costs, and w_j is the price of the jth factor.

Now substitute, as before, (18) into (12) and the result into (17; remembering that $\eta_{iq} = 1$ under constant returns, to get

(19) $\quad dlnX_i = \sum_j (\eta_{ij} - s_j\eta)\, dlnw_j.$

It follows immediately that

(20) $\quad \hat{\eta}_{ij} = \partial lnX_i/\partial lnw_j - S_j\eta;$

the industry elasticity of demand, $\hat{\eta}_{ij}$, is the sum of an output-constant substitution effect, η_{ij}, and a scale effect, $S_j\eta$, which is the same for all factors i. It is sometimes popular to re-write (20) as

(21) $\quad \hat{\eta}_{ij} = \partial lnX_i/\partial lnw_j = S[\sigma_{ij} - n],$

where $\sigma_{ij} = \eta_{ij}/S_j$. σ is called the Allen partial elasticity of substitution between the ith and jth factors.* $\sigma_{ij} > 0$ implies that two factors are substitutes, of course, while $\sigma_{ij} < 0$ implies they are complements. It is not entirely clear why these Allen partial elasticities of substitution play such an enormous role in the literature. It is probably because they lead to a terribly convenient form for the restrictions on them that may be deduced from the theory of the firm. In particular, the student should be able to show why the matrix $[\sigma_{ij}]$ is symmetric and negative semi-definite. That is, the homogeneity restriction takes on the form

(22) $\sum_j S_j \sigma_{ij} = 0 = \sum \eta_{ij}$ for all i,

and we must also have

(23) $\sigma_{ij} = \sigma_{ji}$

IV.

Finally, we may turn to <u>the</u> case where capital is supplied with elasticity δ. By direct analogy with equations (19), for the two-factor case corresponding to the first two sections of these notes, we first have

(5a) $d\ln L = [\eta_{Lw} - S\eta]d\ln w + [\eta_{Lr} - (1-S)\eta]d\ln r$

(6a) $d\ln k = [\eta_{kw} - S\eta]d\ln w + [\eta_{kr} - (1-S)\eta]d\ln r$.

In effect, the factor-price relationship and the demand function for output have been used to substitute the induced output effects out of equations (5) and (6) to get (5a) and (6a). With capital supply adjusting also we have a third relationship that must be satisfied, k = k(r), the capital supply function. In logarithmic differentials this is

(24) $d\ln k = \delta d\ln r$.

Now, we shall require that as we change w in equation (5a), r changes so as to keep the changes in the supply [from (24)] and the demand [from (6a)] for capital in continuous equilibrium. But for this to be true r must change in a certain way as w changes. In particular, equating (6a) and (24) and solving for dlnr, we have

(25) $\mathrm{dlnr} = \{(\eta_{kw} - S\eta)/[\delta-\eta_{kr} + (l-S)\eta]\}\mathrm{dlnw}.$

This expression shows how r must change to maintain capital market in equilibrium as w changes. Finally, substituting (25) into (5a) and dividing by $d\ell nw$ gives the derived elasticity of demand

(26) $\mathrm{dlnL/dlnw} = (\eta_{Lw} - S\eta) + [\eta_{Lr} - (1-S)\eta][\eta_{kw}-S\eta]/[\delta-\eta_{kr} + (1-S)\eta].$

For those with the patience the terms η_{Lw}, η_{Lr}, η_{kw}, and η_{kr}, may all be substituted out using the relationships in (7) and (10) to obtain the Hicksian formula in terms of σ, η, S, and δ:

(27) $$\mathrm{dlnL/dlnw} = \frac{\sigma(\eta+\delta)+S\delta(\eta-\sigma)}{\eta+\delta-\delta(\eta-\sigma)}$$

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*This is to be **found in R.G.D. Allen's** **Mathematical Analysis for** Economists, 1939, p. 508.