STATIC GAMES with incomplete information – Bayesian Games

Lecture 9

Complete vs. Incomplete inform.

- SO FAR we had only GAMES of complete information
  - Preferences of all players, structure and whole environment of the game are common knowledge to everyone

!!!NOW we allow for some level of uncertainty!!!

- BAYESIAN GAMES - Static games with incomplete information:
  - some players may be uncertain about other's players preferences
  - generalize the notion of a strategic game
  - allow us to analyze any situation in which each player is imperfectly informed about some aspect of her environment relevant to her choice of an action

Example 1: Variant of BoS

Imagine a dating (chatting) site and Adam and Radka who started chatting with each other and after a while Adam asked her out. Radka accepted and they settled on that they will go jointly on a concert – either Divokej Bill or Nohavica. Radka had already photo in her profile and she asked Adam to send her his photo.

Further, according to the internet conversation and the fact that Adam already saw her photo and ask her out, it is obvious that Adam likes Radka and would like to meet her.

However, Adam cannot be sure whether Radka would like to meet him, as she still has only bare information about him. So maybe she will not like to meet with him after she receives his photo…

Example 1: Variant of BoS

ADAM would like to meet RADKA, he prefers D.BILL

ADAM is not sure whether RADKA would like to meet him…

→ We model this as having "two types of RADKA"

1st type: RADKA would like to meet ADAM, she prefers NOHAV.

2nd type: RADKA would like to avoid ADAM, she prefers NOHAV.

Radka – likes A

1st type	D.BILL	NOHAV.
D.BILL	**2, 1**	**0, 0**
NOHAV.	**0, 0**	**1, 2**

Radka – dislikes A

2nd type	D.BILL	NOHAV.
D.BILL	**2, 0**	**0, 2**
NOHAV.	**0, 1**	**1, 0**

ADAM

Example 1: Variant of BoS

Suppose ADAM thinks that with probability ½ RADKA wants to meet with him, and with probability ½ she wants to avoid him. (This may come from Adam's experience with similar situations)

That is, ADAM thinks that with probability ½ he is playing the game on the left and with probability ½ he is playing the game on the right.

Radka – likes A

Prob. ½	D.BILL	NOHAV.
D.BILL	**2, 1**	**0, 0**
NOHAV.	**0, 0**	**1, 2**

Radka – dislikes A

Prob. ½	D.BILL	NOHAV.
D.BILL	**2, 0**	**0, 2**
NOHAV.	**0, 1**	**1, 0**

ADAM

Example 1: Variant of BoS

We can think of there being two states, one in which the players' preferences are given in the left table and one in which these payoffs are given in the right table.
Radka knows the state — she knows whether she wishes to meet or avoid Adam, whereas Adam does not; he assigns probability ½ to each state.

Radka – likes A

ADAM: 1st - ½	D.BILL	NOHAV.
D.BILL	2, 1	0, 0
NOHAV.	0, 0	1, 2

Radka – dislikes A

ADAM: 2nd - ½	D.BILL	NOHAV.
D.BILL	2, 0	0, 2
NOHAV.	0, 1	1, 0

Example 1: Variant of BoS

From ADAM's point of view, RADKA has two possible types. ADAM does not know RADKA's type → to choose optimal action rationally he needs to form a belief about the chosen action of each type.

Given these beliefs, he can compute expected utility to each of his action.

ADAM

Radka – likes A

1st - ½	D.BILL	NOHAV.
D.BILL	**2, 1**	**0, 0**
NOHAV.	**0, 0**	**1, 2**

Radka – dislikes A

2nd - ½	D.BILL	NOHAV.
D.BILL	**2, 0**	**0, 2**
NOHAV.	**0, 1**	**1, 0**

Example 1: Variant of BoS

If ADAM believes that 1^{st} type RADKA will choose D.BILL and the 2^{nd} type Radka will choose D.BILL, then his EU:

EU (D.BILL) = ½ * 2 + ½ * 2 = 2

EU (NOHAV.) = ½ * 0 + ½ * 0 = 0

If ADAM believes that 1^{st} type – D.BILL, 2^{nd} type – NOHAV.:

EU (D.BILL) = ½ * 2 + ½ * 0 = 1

EU (NOHAV.) = ½ * 0 + ½ * 1 = ½

Radka – likes A

ADAM

1^{st} - ½	D.BILL	NOHAV.
D.BILL	**2, 1**	**0, 0**
NOHAV.	**0, 0**	**1, 2**

Radka – dislikes A

2^{nd} - ½	D.BILL	NOHAV.
D.BILL	**2, 0**	**0, 2**
NOHAV.	**0, 1**	**1, 0**

Example 1: Variant of BoS

If ADAM believes that 1st type RADKA will choose D.BILL and the 2nd type Radka will choose D.BILL, then his EU:

EU (D.BILL) = ½ * 2 + ½ * 2 = 2

EU (NOHAV.) = ½ * 0 + ½ * 0 = 0

If ADAM believes that 1st type – D.BILL, 2nd type – NOHAV.:

EU (D.BILL) = ½ * 2 + ½ * 0 = 1

EU (NOHAV.) = ½ * 0 + ½ * 1 = ½

Radka – likes A

1st - ½	D.BILL	NOHAV.
D.BILL	**2, 1**	**0, 0**
NOHAV.	**0, 0**	**1, 2**

Radka – dislikes A

2nd - ½	D.BILL	NOHAV.
D.BILL	**2, 0**	**0, 2**
NOHAV.	**0, 1**	**1, 0**

ADAM

Example 1: Variant of BoS

If ADAM believes that 1st type – NOHAV., 2nd type – D.BILL:

EU (D.BILL) = ½ * 0 + ½ * 2 = 1

EU (NOHAV.) = ½ * 1 + ½ * 0 = ½

If ADAM believes that 1st type – NOHAV., 2nd type – NOHAV.:

EU (D.BILL) = ½ * 0 + ½ * 0 = 0

EU (NOHAV.) = ½ * 1 + ½ * 1 = 1

Radka – likes A

1st - ½	D.BILL	NOHAV.
D.BILL	**2, 1**	**0, 0**
NOHAV.	**0, 0**	**1, 2**

Radka – dislikes A

2nd - ½	D.BILL	NOHAV.
D.BILL	**2, 0**	**0, 2**
NOHAV.	**0, 1**	**1, 0**

ADAM

Example 1: Variant of BoS

We can represent the game also in one joint table.
Each column of the table is a pair of actions for the two types of RADKA, the first action of each pair refers to the action of the 1st type the second to the action of the 2nd type.
First number in each cell represent EU of ADAM, second number is payoff of 1st type RADKA and the third one payoff of 2nd type RADKA

1st Radka – likes A **2nd Radka – dislikes A**

ADAM	1st:½, 2nd:½	DB , DB	DB , N	N, DB	N, N
	D.BILL	**2, 1, 0**	**1, 1, 2**	**1, 0, 0**	**0, 0, 2**
	NOHAV.	**0, 0, 1**	**½ , 0, 0**	**½, 2, 1**	**1, 2, 0**

Example 1: Variant of BoS

pure strategy Nash equilibrium of this particular game:
triple of actions, one for ADAM and one for each type of RADKA such that:

- the action of ADAM is optimal, given the actions of the two types of RADKA (and ADAM's belief about the state)
- the action of each type of RADKA is optimal, given the action of ADAM.

1st Radka – likes A **2nd Radka – dislikes A**

ADAM	1st: ½, 2nd: ½	DB , DB	DB , N	N, DB	N, N
	D.BILL	2, 1, 0	1, 1, 2	1, 0, 0	0, 0, 2
	NOHAV.	0, 0, 1	½ , 0, 0	½, 2, 1	1, 2, 0

Example 1: Variant of BoS

In a Nash equilibrium:

ADAM's action is a best response to the pair of actions of the two types of RADKA

the action of each type of RADKA is a best response to the action of ADAM.

Both types of RADKA are independent to each other and, naturally, do not react on each other as they model behavior of one person.

1st Radka – likes A **2nd Radka – dislikes A**

ADAM

1st:½, 2nd:½	DB , DB	DB , N	N, DB	N, N
D.BILL	**2, 1, 0**	**1, 1, 2**	**1, 0, 0**	**0, 0, 2**
NOHAV.	**0, 0, 1**	**½ , 0, 0**	**½, 2, 1**	**1, 2, 0**

Example 1: Variant of BoS

(DB, (DB, N)), where the first component is the action of ADAM and the other component is the pair of actions of the two types of RADKA, is a Nash equilibrium.

1^{st} Radka – likes A **2^{nd} Radka – dislikes A**

ADAM	1^{st}:½, 2^{nd}:½	DB , DB	DB , N	N, DB	N, N
	D.BILL	**2, 1, 0**	$\boxed{\textbf{1, 1, 2}}$	**1, 0, 0**	**0, 0, 2**
	NOHAV.	**0, 0, 1**	**½ , 0, 0**	**½, 2, 1**	**1, 2, 0**

Example 1: Variant of BoS

Suppose that in fact RADKA wishes to meet ADAM. Then we interpret the first equilibrium as follows.
Both ADAM and RADKA chooses D.BILL; ADAM, who does not know if RADKA wants to meet him or avoid him , believes that if RADKA wishes to meet him she will choose D.BILL, and if she wishes to avoid him she will choose NOHAVICA.

1st Radka – likes A **2nd Radka – dislikes A**

ADAM	1st:½, 2nd:½	DB , DB	DB , N	N, DB	N, N
	D.BILL	$\underline{2}$, $\underline{1}$, 0	$\boxed{\underline{1}, \underline{1}, \underline{2}}$	$\underline{1}$, 0, 0	0, 0, $\underline{2}$
	NOHAV.	0, 0, $\underline{1}$	½ , 0, 0	½, $\underline{2}$, $\underline{1}$	$\underline{1}$, $\underline{2}$, 0

Example 2: Variant of BoS

Again imagine the same situation, but now Radka does not have a picture in her profile …
So imagine again a dating(chatting) site and Adam and Radka who started chatting with each other and after a while they settled on that they will go jointly on a concert – either Divokej Bill or Nohavica. Further they asked each other to send the other person their own photo.
In this situation, Adam cannot be sure whether Radka would like meet him, as she still has only bare information about him. So maybe she will not like to meet with him after she receives his photo…
However, now also Radka cannot be sure whether Adam would like to meet her, as he still has also only bare information about her. So maybe he will not like to meet with her…

Example 2: Variant of BoS

Further, suppose ADAM thinks that with probability ½ RADKA wants to meet with him, and with probability ½ she wants to avoid him.
(This may come from Adam's experience with similar situations)

And, suppose RADKA thinks that with probability 2/3 ADAM wants to meet with her, and with probability 1/3 he wants to avoid her.

(This may also come from Radka's experience with similar situations – she may be more attractive to men than Adam is to women).

As before, assume that each player knows her own preferences about the other person and also about the preferred concert.

Example 2: Variant of BoS

Before we had only two states, one in which the Radka likes Adam and one in which Radka does not like Adam.

We can represent the situation where both ADAM and RADKA are unsure about the preferences of the other person as having “two types of ADAM” (likes Radka, dislikes Radka) and “two types of RADKA (likes Adam, dislikes Adam).

Therefore instead of two states, we are now having four hypothetical states:

1: Adam likes Radka, Radka likes Adam – A1st R1st
2: Adam likes Radka, Radka dislikes Adam – A1st R2nd
3: Adam dislikes Radka, Radka likes Adam – A2nd R1st
4: Adam dislikes Radka, Radka dislikes Adam – A2nd R2nd

Example 2: Variant of BoS

1st ADAM

Radka – likes A

A1st R1st	D.BILL	NOHAV.
D.BILL	**2, 1**	**0, 0**
NOHAV.	**0, 0**	**1, 2**

Radka – dislikes A

A1st R2nd	D.BILL	NOHAV.
D.BILL	**2, 0**	**0, 2**
NOHAV.	**0, 1**	**1, 0**

2nd ADAM

Radka – likes A

A2nd R1st	D.BILL	NOHAV.
D.BILL	**0, 1**	**2, 0**
NOHAV.	**1, 0**	**0, 2**

Radka – dislikes A

A2nd R2nd	D.BILL	NOHAV.
D.BILL	**0, 0**	**2, 2**
NOHAV.	**1, 1**	**0, 0**

Example 2: Variant of BoS

The fact that ADAM does not know RADKA's preferences means that he cannot distinguish between states 1 and 2, or between states 3 and 4. (he is of course aware about his preferences…)

four hypothetical states:

1:	ADAM likes RADKA,	RADKA likes ADAM –	beliefs 1/2
2:	ADAM likes RADKA,	RADKA dislikes ADAM –	beliefs 1/2
3:	ADAM dislikes RADKA,	RADKA likes ADAM –	beliefs 1/2
4:	ADAM dislikes RADKA,	RADKA dislikes ADAM –	beliefs 1/2

We can model the players' information by assuming that each player receives a signal before choosing an action. ADAM receives the same (good) signal, say nice photo of RADKA, in states 1 and 2, and a different (bad) signal, say not so nice photo of RADKA, in states 3 and 4. (He did not receive any signal whether RADKA likes him or not…)

Example 2: Variant of BoS

1st ADAM

½ **Radka – likes A**

A1st R1st	D.BILL	NOHAV.
D.BILL	**2, 1**	**0, 0**
NOHAV.	**0, 0**	**1, 2**

½ **Radka – dislikes A**

A1st R2nd	D.BILL	NOHAV.
D.BILL	**2, 0**	**0, 2**
NOHAV.	**0, 1**	**1, 0**

2nd ADAM

½ **Radka – likes A**

A2nd R1st	D.BILL	NOHAV.
D.BILL	**0, 1**	**2, 0**
NOHAV.	**1, 0**	**0, 2**

½ **Radka – dislikes A**

A2nd R2nd	D.BILL	NOHAV.
D.BILL	**0, 0**	**2, 2**
NOHAV.	**1, 1**	**0, 0**

Example 2: Variant of BoS

The fact that RADKA does not know ADAM's preferences means that she cannot distinguish between states 1 and 3, or between states 2 and 4. (she is of course aware about her preferences…)

four hypothetical states:

1: ADAM likes RADKA,	RADKA likes ADAM –	beliefs 2/3
2: ADAM likes RADKA,	RADKA dislikes ADAM –	beliefs 2/3
3: ADAM dislikes RADKA,	RADKA likes ADAM –	beliefs 1/3
4: ADAM dislikes RADKA,	RADKA dislikes ADAM –	beliefs 1/3

We can model the players' information by assuming that each player receives a signal before choosing an action. RADKA receives the same (good) signal, say nice photo of ADAM, in states 1 and 3, and a different (bad) signal, say not so nice photo of ADAM, in states 2 and 4. (She did not receive any signal whether ADAM likes her or not…)

Example 2: Variant of BoS

1st ADAM

2/3 **Radka – likes A**

A1st R1st	D.BILL	NOHAV.
D.BILL	**2, 1**	**0, 0**
NOHAV.	**0, 0**	**1, 2**

2/3 **Radka – dislikes A**

A1st R2nd	D.BILL	NOHAV.
D.BILL	**2, 0**	**0, 2**
NOHAV.	**0, 1**	**1, 0**

2nd ADAM

1/3 **Radka – likes A**

A2nd R1st	D.BILL	NOHAV.
D.BILL	**0, 1**	**2, 0**
NOHAV.	**1, 0**	**0, 2**

1/3 **Radka – dislikes A**

A2nd R2nd	D.BILL	NOHAV.
D.BILL	**0, 0**	**2, 2**
NOHAV.	**1, 1**	**0, 0**

Example 2: Variant of BoS

ADAM's beliefs:

Type 1st ADAM:

1: ADAM likes RADKA, RADKA likes ADAM – beliefs 1/2

2: ADAM likes RADKA, RADKA dislikes ADAM – beliefs 1/2

Type 2nd ADAM:

3: ADAM dislikes RADKA, RADKA likes ADAM – beliefs 1/2

4: ADAM dislikes RADKA, RADKA dislikes ADAM – beliefs 1/2

RADKA's beliefs:

Type 1st RADKA:

1: ADAM likes RADKA, RADKA likes ADAM – beliefs 2/3

3: ADAM dislikes RADKA, RADKA likes ADAM – beliefs 1/3

Type 2nd RADKA:

2: ADAM likes RADKA, RADKA dislikes ADAM – beliefs 2/3

4: ADAM dislikes RADKA, RADKA dislikes ADAM – beliefs 1/3

Example 2: Variant of BoS

1st ADAM

1/2 2/3 Radka – likes A

A1st R1st	D.BILL	NOHAV.
D.BILL	**2, 1**	**0, 0**
NOHAV.	**0, 0**	**1, 2**

1/2 2/3 Radka – dislikes A

A1st R2nd	D.BILL	NOHAV.
D.BILL	**2, 0**	**0, 2**
NOHAV.	**0, 1**	**1, 0**

2nd ADAM

1/2 1/3 Radka – likes A

A2nd R1st	D.BILL	NOHAV.
D.BILL	**0, 1**	**2, 0**
NOHAV.	**1, 0**	**0, 2**

1/2 1/3 Radka – dislikes A

A2nd R2nd	D.BILL	NOHAV.
D.BILL	**0, 0**	**2, 2**
NOHAV.	**1, 1**	**0, 0**

Example 2: Variant of BoS

When finding Nash equilibria of this game, **we start with computing expected utility** of playing DB or N for ADAM and RADKA, given their beliefs about the particular state of the game (1-4) after receiving their signal (photo – good or bad).

ADAM received good signal → 1^{st} type ADAM likes RADKA

If 1^{st} ADAM believes that 1^{st} type RADKA plays – D.BILL, 2^{nd} type RADKA – D.BILL:

EU (D.BILL) = ½ * 2 + ½ * 2 = 2

EU (NOHAV.) = ½ * 0 + ½ * 0 = 0

If 1^{st} ADAM believes that 1^{st} type RADKA plays – D.BILL, 2^{nd} type RADKA – NOHAV.:

EU (D.BILL) = ½ * 2 + ½ * 0 = 1

EU (NOHAV.) = ½ * 0 + ½ * 1 = ½

Example 2: Variant of BoS

When finding Nash equilibria of this game, **we start with computing expected utility** of playing DB or N for ADAM and RADKA, given their beliefs about the particular state of the game (1-4) after receiving their signal (photo – good or bad).

ADAM received good signal → 1^{st} type ADAM likes RADKA

If 1^{st} ADAM believes that 1^{st} type RADKA plays – NOHAV., 2^{nd} type RADKA – D.BILL:

EU (D.BILL) = ½ * 0 + ½ * 2 = 1

EU (NOHAV.) = ½ * 1 + ½ * 0 = ½

If 1^{st} ADAM believes that 1^{st} type RADKA plays – NOHAV., 2^{nd} type RADKA – NOHAV.:

EU (D.BILL) = ½ * 0 + ½ * 0 = 0

EU (NOHAV.) = ½ * 1 + ½ * 1 = 1

Example 2: Variant of BoS

When finding Nash equilibria of this game, **we start with computing expected utility** of playing DB or N for ADAM and RADKA, given their beliefs about the particular state of the game (1-4) after receiving their signal (photo – good or bad).

ADAM received bad signal → 2^{nd} type ADAM dislikes RADKA

If 2^{nd} ADAM believes that 1^{st} type RADKA plays – D.BILL, 2^{nd} type RADKA – D.BILL:

EU (D.BILL) = ½ * 0 + ½ * 0 = 0

EU (NOHAV.) = ½ * 1 + ½ * 1 = 1

If 2^{nd} ADAM believes that 1^{st} type RADKA plays – D.BILL, 2^{nd} type RADKA – NOHAV.:

EU (D.BILL) = ½ * 0 + ½ * 2 = 1

EU (NOHAV.) = ½ * 1 + ½ * 0 = ½

Example 2: Variant of BoS

When finding Nash equilibria of this game, **we start with computing expected utility** of playing DB or N for ADAM and RADKA, given their beliefs about the particular state of the game (1-4) after receiving their signal (photo – good or bad).

ADAM received bad signal → 2nd type ADAM dislikes RADKA

If 2nd ADAM believes that 1st type RADKA plays – NOHAV., 2nd type RADKA – D.BILL:

EU (D.BILL) = ½ * 2 + ½ * 0 = 1

EU (NOHAV.) = ½ * 0 + ½ * 1 = ½

If 2nd ADAM believes that 1st type RADKA plays – NOHAV., 2nd type RADKA – NOHAV.:

EU (D.BILL) = ½ * 2 + ½ * 2 = 2

EU (NOHAV.) = ½ * 0 + ½ * 0 = 0

Example 2: Variant of BoS

When finding Nash equilibria of this game, **we start with computing expected utility** of playing DB or N for ADAM and RADKA, given their beliefs about the particular state of the game (1-4) after receiving their signal (photo – good or bad).

RADKA received good signal → 1^{st} type RADKA likes ADAM

If 1^{st} RADKA believes that 1^{st} type ADAM plays – D.BILL, 2^{nd} type ADAM – D.BILL:

EU (D.BILL) = 2/3 * 1 + 1/3* 1 = 1

EU (NOHAV.) = 2/3 * 0 + 1/3 * 0 = 0

If 1^{st} RADKA believes that 1^{st} type ADAM plays – D.BILL, 2^{nd} type ADAM – NOHAV.:

EU (D.BILL) = 2/3 * 1 + 1/3 * 0 = 2/3

EU (NOHAV.) = 2/3 * 0 + 1/3 * 2 = 2/3

Example 2: Variant of BoS

When finding Nash equilibria of this game, **we start with computing expected utility** of playing DB or N for ADAM and RADKA, given their beliefs about the particular state of the game (1-4) after receiving their signal (photo – good or bad).

RADKA received good signal → 1^{st} type RADKA likes ADAM

If 1^{st} RADKA believes that 1^{st} type ADAM plays – NOHAV., 2^{nd} type ADAM – D.BILL:

EU (D.BILL) = 2/3 * 0 + 1/3 * 1 = 1/3

EU (NOHAV.) = 2/3 * 2 + 1/3* 0 = 4/3

If 1^{st} RADKA believes that 1^{st} type ADAM plays – NOHAV., 2^{nd} type ADAM – NOHAV.:

EU (D.BILL) = 2/3 * 0 + 1/3 * 0 = 0

EU (NOHAV.) = 2/3 * 2 + 1/3 * 2 = 2

Example 2: Variant of BoS

When finding Nash equilibria of this game, **we start with computing expected utility** of playing DB or N for ADAM and RADKA, given their beliefs about the particular state of the game (1-4) after receiving their signal (photo – good or bad).

RADKA received bad signal → 2^{nd} type RADKA dislikes ADAM

If 2^{nd} RADKA believes that 1^{st} type ADAM plays – D.BILL, 2^{nd} type ADAM – D.BILL:

EU (D.BILL) = 2/3 * 0 + 1/3 * 0 = 0

EU (NOHAV.) = 2/3 * 2 + 1/3 * 2 = 2

If 2^{nd} RADKA believes that 1^{st} type ADAM plays – D.BILL, 2^{nd} type ADAM – NOHAV.:

EU (D.BILL) = 2/3 * 0 + 1/3 * 1 = 1/3

EU (NOHAV.) = 2/3 * 2 + 1/3 * 0 = 4/3

Example 2: Variant of BoS

When finding Nash equilibria of this game, **we start with computing expected utility** of playing DB or N for ADAM and RADKA, given their beliefs about the particular state of the game (1-4) after receiving their signal (photo – good or bad).

RADKA received bad signal → 2^{nd} type RADKA dislikes ADAM

If 2^{nd} RADKA believes that 1^{st} type ADAM plays – NOHAV., 2^{nd} type ADAM – D.BILL:

EU (D.BILL) = 2/3 * 1 + 1/3 * 0 = 2/3

EU (NOHAV.) = 2/3 * 0 + 1/3 * 2 = 2/3

If 2^{nd} RADKA believes that 1^{st} type ADAM plays – NOHAV., 2^{nd} type ADAM – NOHAV.:

EU (D.BILL) = 2/3 * 1 + 1/3 * 1 = 1

EU (NOHAV.) = 2/3 * 0 + 1/3 * 0 = 0

Example 2: Variant of BoS

First number in each cell represents EU of 1st type ADAM, second number is EU of 2nd ADAM, third one is EU of 1st type RADKA and fourth one is EU of 2nd RADKA

1st Radka – likes A | **2nd Radka – dislikes A**

1st ADAM / **2nd ADAM**

	DB , DB	DB , N	N, DB	N, N
DB, DB	2, 0, 1, 0	1, 1, 1, 2	1, 1, 0, 0	0, 2, 0, 2
DB, N	2, 1, 2/3, 1/3	1, ½, 2/3, 4/3	1, ½, 2/3, 1/3	0, 0, 2/3, 4/3
N, DB	0, 0, 1/3, 2/3	½, 1, 1/3, 2/3	½, 1, 4/3, 2/3	1, 2, 4/3, 2/3
N, N	0, 1, 0, 1	½,½, 0, 0	½, ½, 2, 1	1, 0, 2, 0

Example 2: Variant of BoS

As in the previous example, to study the equilibria of this model we consider the players' plans of action before they receive their signals (the photo).

That is, each player plans an action for each of the two possible signals (good or bad) he or she may receive.
We may think of there being four players: the "two types of ADAM" and the "two types of RADKA".
Nash equilibrium consists of four actions, one for each of these players, such that the action of each type of ADAM and RADKA is optimal, given RADKA's and ADAM's belief about the state after observing his signal (for example Adam's beliefs ½ -1 ½ - 2 if he receives good signal) , and given the actions of each type of the other original player.

Example 2: Variant of BoS

pure strategy Nash equilibrium of this particular game:
set of four actions, one for each type of ADAM and one for each type of RADKA such that:

- the action of each type of ADAM is optimal, given the actions of the two types of RADKA (and each type of ADAM's belief about the state)

- the action of each type of RADKA is optimal, given the action of each type of ADAM. (and each type of RADKA's belief about the state)

Example 2: Variant of BoS

In a Nash equilibrium:

the action of each type of ADAM is a best response to the pair of actions of the two types of RADKA

the action of each type of RADKA is a best response to the action of ADAM.

Each type of RADKA or ADAM are independent to each other and, naturally, do not react on each other as they model behavior of one person.

Example 2: Variant of BoS

First number in each cell represents EU of 1st type ADAM, second number is EU of 2nd ADAM, third one is EU of 1st type RADKA and fourth one is EU of 2nd RADKA

1st Radka – likes A **2nd Radka – dislikes A**

1st ADAM / **2nd ADAM**

	DB , DB	DB , N	N, DB	N, N
DB, DB	**2, 0, 1, 0**	**1, 1, 1, 2**	**1, 1, 0, 0**	**0, 2, 0, 2**
DB, N	**2, 1, 2/3, 1/3**	**1, ½, 2/3, 4/3**	**1, ½, 2/3, 1/3**	**0, 0, 2/3, 4/3**
N, DB	**0, 0, 1/3, 2/3**	**½, 1, 1/3, 2/3**	**½, 1, 4/3, 2/3**	**1, 2, 4/3, 2/3**
N, N	**0, 1, 0, 1**	**½,½, 0, 0**	**½, ½, 2, 1**	**1, 0, 2, 0**

Example 1: Variant of BoS

((DB, DB), (DB, N)) and ((N,DB), (N, N)), are Nash equilibria
The first component gives the actions of “two types of ADAM”
The second component gives the action of “two types of RADKA”

In each of these examples a Nash equilibrium is a list of actions, one for each type of each player, such that the action of each type of each player is a best response to the actions of all the types of the other player, given the player’s beliefs about the state after she or he observes the signal. The actions planned by the various types of player *i are not relevant to the decision problem of any type of player i,* but there is no harm in taking them, as well as the actions of the types of the *other* player, as given when player *i is choosing an action.*

Bayesian Games

A strategic game with imperfect information is called a "Bayesian game" and consists of:

- Set of players
- Set of states

And for each player:

- Set of actions
- Set of signals that she may receive and a signal function that associates a signal with each state
- for each signal that she may receive, a **belief about the states consistent with** the signal (a probability distribution over the set of states with which the signal is associated)
- vNM preferences over pairs ***(a, ω), where a is an action profile and*** *ω is a state*

Bayesian Games

- Set of players
  - ADAM and RADKA in both previous examples
- Set of states
  - complete description of one collection of the players' relevant characteristics, including both their preferences and their information (for example state 2: 1st ADAM 2nd RADKA)
  - For every collection of characteristics that some player believes to be possible, there must be a state.
  - In example 1 – the reason to have 2 states with two types of Radka is that ADAM believes that RADKA may like him with prob. ½ or may not like him with prob. ½
  - Two states in the first example, four states in example 2

Bayesian Games

And for each player:

- Set of actions
  - Divokej Bill or Nohavica for each player in both examples
- Set of signals that she may receive and a signal function that associates a signal with each state
  - At the start of the game a state is realized
    - Whether ADAM likes and RADKA likes is determined
  - The players do not observe this state – they receives a signal that may give them some information about state
  - In example 1 Radka receives signal such that she knows exactly the state, but ADAM does not
  - In example 2 they receive signal such that ADAM knows that the state is 1-2 or 3-4 and RADKA 1-3 or 2-4

Bayesian Games

- Set of signals that she may receive and a signal function that associates a signal with each state
  - signal is a deterministic function of the state: for each state a definite signal is received – it actually defines type of player t_i
  - Denote the signal which player i receives in state ω by $\tau_i(\omega)$. The function τ_i is called player i's signal function
  - Signal and signal function define the amount of information players have
  - If $\tau_i(\omega)$ different for each value of ω → player i knows, given her signal, the state that has occurred → perfectly informed about all the players' relevant characteristics.
  - For example Radka in the first example is perfectly informed – she knows ADAM's preferences and also her after receiving signal – the photo..

Bayesian Games

- Set of signals that she may receive and a signal function that associates a signal with each state
  - Signal and signal function define the amount of information players have
  - If $\tau_i(\omega)$ same for each value of ω → player i has no information about the state.
  - For example ADAM in the first example is not informed at all – he does not know RADKA's preferences
    - In example 1 Radka receives signal such that she knows exactly the state, but ADAM do not
    - In example 2 they receive signal such that ADAM knows that the state is 1-2 or 3-4 ($\tau_A(1) = \tau_A(2) <> \tau_A(3) = \tau_A(4)$) and RADKA 1-3 or 2-4 ($\tau_R(1) = \tau_R(3) <> \tau_R(2) = \tau_R(4)$)

Bayesian Games

- for each signal that she may receive, a **belief about the states consistent with** the signal (a probability distribution over the set of states with which the signal is associated)
  - Each type of each player holds a belief about the likelihood of the states consistent with her signal
  - If, for example, $t_i = \tau_i(\omega_1) = \tau_i(\omega_2)$, then type t_i of player i assigns probabilities to states ω_1 and ω_2 and zero to all others
  - 1st type ADAM assigns probability ½ to state 1 and 2 and zero probability to 3 and 4 after receiving the signal 1= *good* (good 1= $\tau_A(1) = \tau_A(2)$ <> bad 2= $\tau_A(3) = \tau_A(4)$)
  - 2nd type RADKA assigns probability 2/3 to state 2 and 1/3 to state 4 and zero probabilities to 1 and 3 after receiving the signal *2 = bad* (good 1=$\tau_R(1) = \tau_R(3)$ <> bad 2=$\tau_R(2) = \tau_R(4)$)

Bayesian Games

- vNM preferences over pairs ***(a, ω), where a is an action profile and*** *ω is a state*
  - the expected utility represents the player's preferences among lotteries over the set of such pairs
  - Given player's signal and beliefs, he can compare pairs ***(a, ω)*** **by comparing his expected utility**
  - 2nd type RADKA assigns probability 2/3 to state 2 and 1/3 to state 4 and zero probabilities to 1 and 3 after receiving the signal *2 = bad* (1=$\tau_R(1) = \tau_R(3)$ <> 2=$\tau_R(2) = \tau_R(4)$)
  - **here utility function gives her payoffs to each pair: (DB,DB, 1) = 1, …, (N, DB, 2) = 1, … (DB, DB, 4) = 0**
- If 2nd RADKA believes that 1st type ADAM plays – NOHAV., 2nd type ADAM – D.BILL: EU (D.BILL) = 2/3 * 1 + 1/3 * 0 = 2/3

Nash Equilibrium

- denote the probability assigned by the belief of type t_A of player A to state ω by $Pr(\omega|t_A)$.
- Denote the action taken by each type t_B (depends on signal) of each player B by $a(B, t_B)$.
- Player B's signal in state ω is $\tau_B(\omega)$, so her action in state ω is $a(B, \tau_B(\omega))$.
- For each state ω, denote by $A(\omega)$ the action profile in which each player B chooses the action $a(B, \tau_B(\omega))$.
- Then the expected payoff of type t_A of player A when she chooses the action a_i is

$$\sum_{\omega\in\Omega} P(\omega|t_A)\, u_A((a_i, A(\omega)), \omega)$$

Nash Equilibrium

DEFINITION: **Nash equilibrium of a Bayesian game is**

Nash equilibrium of the strategic game (with vNM preferences) defined as follows:

- Players: The set of all pairs (A, t_A) where A is a player in the Bayesian game and t_A is one of the signals A may receive
- Actions: The set of actions of each player (A, t_A) is the set of actions of player A in the Bayesian game
- Preferences: Preferences over action profiles for each type of player (A, t_A) are given by the expected utility, given the beliefs about the state after receiving the signal t_A(formula on the previous slide)

$$\sum_{\omega\in\Omega} P(\omega|t_A)\, u_A((a_i, A(\omega)), \omega)$$

How to find Nash Equilibrium

- Given the BAYESIAN GAME
  1) Find all types of players – what signal may each player receive?

     Example 1 – 2 types of RADKA → 3 players

     Example 2 – 2 types of ADAM and RADKA → 4 players
  2) What are the beliefs of each player type after receiving the signal?

     Example 1 – ADAM→ ½ and ½ . 1st Radka – 1 and 0

     2nd Radka – 0 and 1
  3) Given the beliefs, compute EU for each possible action of each type, given the action profile of the other players' types

How to find Nash Equilibrium

- Given the BAYESIAN GAME
  3) Given the beliefs, compute EU for each possible action of each type, given the action profile of the other players' types

  Example 2: If 2nd RADKA believes that 1st type ADAM plays – NOHAV., 2nd type ADAM – D.BILL: EU (D.BILL) = 2/3 * 1 + 1/3 * 0 = 2/3

  3) Given computed all the EU for all types of all players and all possible action profiles, find NE of this game → such that no type of player A have any incentive to deviate given the actions of all other players' types

  Example 1 and 2 – large table jointly for all 3 players (Ex1) or for all 4 players (Ex2)

Example 1: Variant of BoS

We can represent the game also in one joint table.
Each column of the table is a pair of actions for the two types of RADKA, the first action of each pair refers to the action of the 1st type the second to the action of the 2nd type.
First number in each cell represent EU of ADAM, second number is payoff of 1st type RADKA and the third one payoff of 2nd type RADKA

1st Radka – likes A **2nd Radka – dislikes A**

ADAM	1st:½, 2nd:½	DB , DB	DB , N	N, DB	N, N
	D.BILL	2, 1, 0	1, 1, 2	1, 0, 0	0, 0, 2
	NOHAV.	0, 0, 1	½ , 0, 0	1, 2, 1	1, 2, 0

Example 2: Variant of BoS

First number in each cell represents EU of 1st type ADAM, second number is EU of 2nd ADAM, third one is EU of 1st type RADKA and fourth one is EU of 2nd RADKA

1st Radka – likes A **2nd Radka – dislikes A**

1st ADAM / **2nd ADAM**

	DB , DB	DB , N	N, DB	N, N
DB, DB	**2, 0, 1, 0**	**1, 1, 1, 2**	**1, 1, 0, 0**	**0, 2, 0, 2**
DB, N	**2, 1, 2/3, 1/3**	**1, ½, 2/3, 4/3**	**1, ½, 2/3, 1/3**	**0, 0, 2/3, 4/3**
N, DB	**0, 0, 1/3, 2/3**	**½, 1, 1/3, 2/3**	**½, 1, 4/3, 2/3**	**1, 2, 4/3, 2/3**
N, N	**0, 1, 0, 1**	**½,½, 0, 0**	**½, ½, 2, 1**	**1, 0, 2, 0**

Summary

- Static games with incomplete information – Bayesian games
- Nash Equilibrium of Bayesian games

- Gibbons 3-3.2; Osborne 9-9.4

NEXT WEEK:

Bayesian games – examples and illustrations