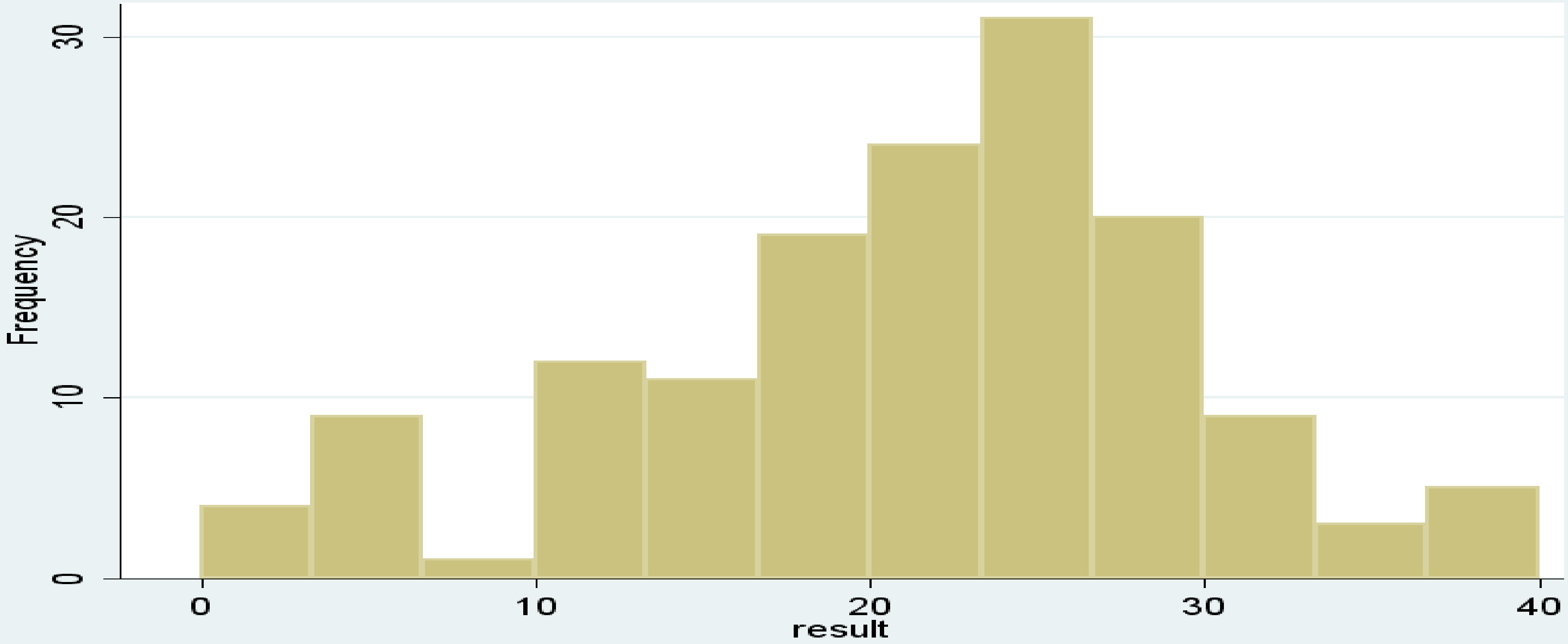
DYNAMIC GAMES with simultaneous moves

Lecture 8

Midterm

- Results: !!! already adjusted to maximum of 40 points !!!

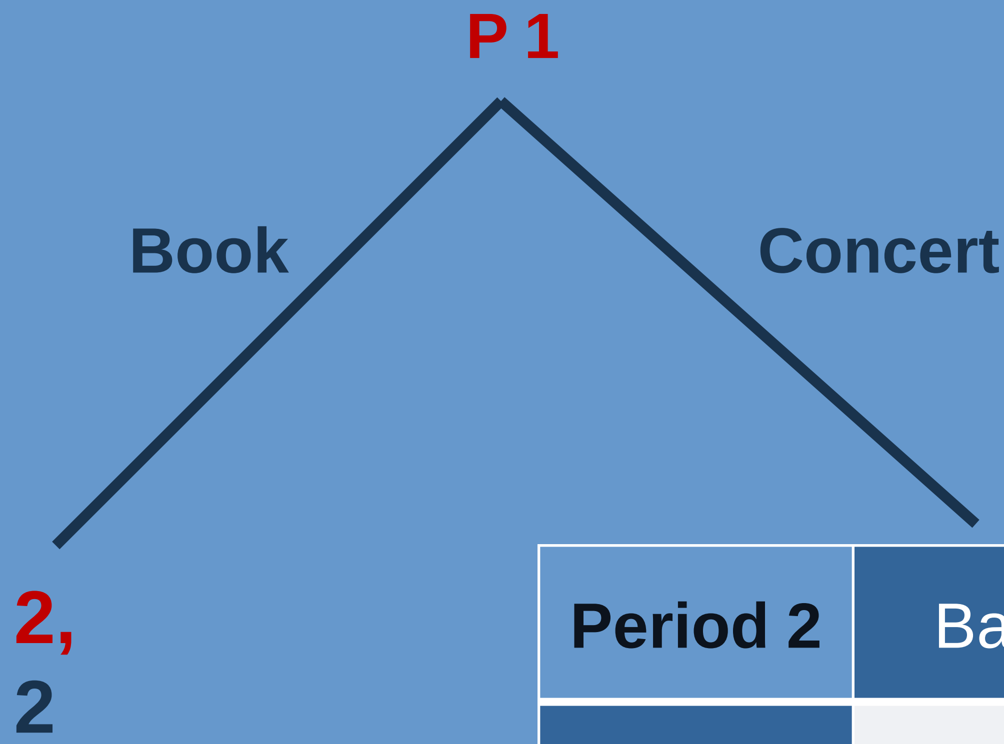


- According to syllabus you may opt for make-up midterm just before your final exam. This make-up midterm will be, however, a little bit harder…

Simultaneous and sequential

- SO FAR we had either:
  - STATIC GAMES – simultaneous moves
  - DYNAMIC GAMES – sequential moves
- NOW we allow for both:
  - Combination of simultaneous and sequential moves

Example 1: Variant of BoS



P 2

P 1

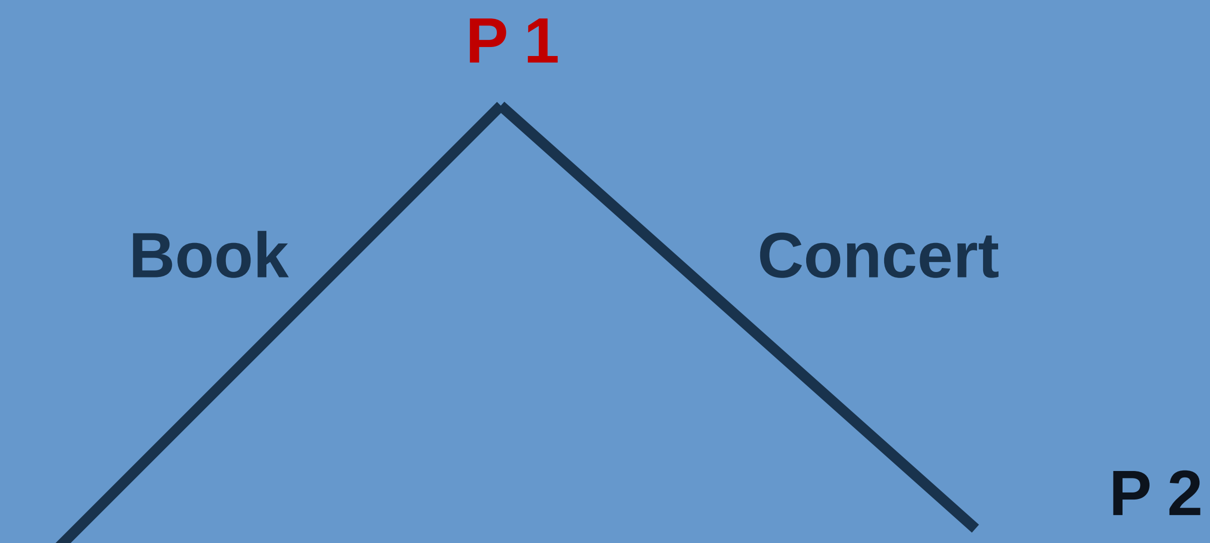
Period 2	Bach	Strav.
Bach	3, 1	0, 0
Strav.	0, 0	1, 3

Example 1: Variant of BoS

First, person 1 decides whether to stay home and read a book or to attend a concert. If she reads a book, the game ends.

If she decides to attend a concert then, as in BoS, she and person 2 independently choose Bach or Stravinsky, not knowing the other person's choice. Both people prefer to attend the concert of their favorite composer in the company of the other person to the outcome in which person 1 stays home and reads a book, and prefer this outcome to attending the concert of their less preferred composer in the company of the other person; the worst outcome for both people is that they attend different concerts.

Example 1: Variant of BoS



2, 2

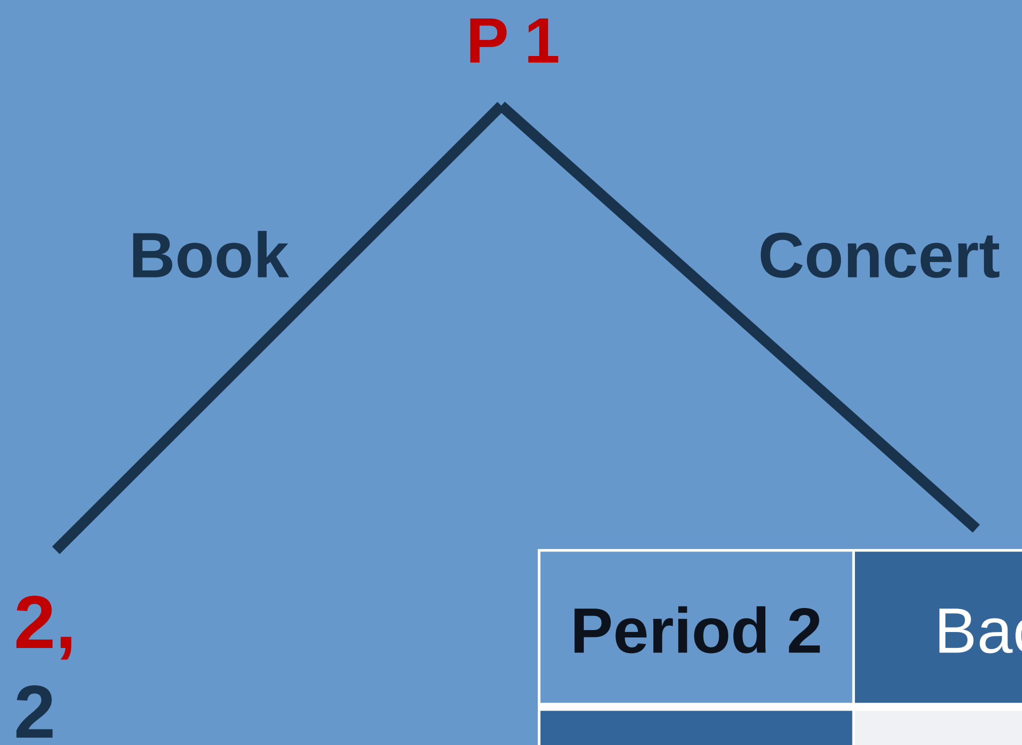
P 1

Period 2	Bach	Strav.
Bach	**3, 1**	**0, 0**
Strav.	**0, 0**	**1, 3**

Dynamic games with simultaneous moves

- Set of players:
  - Two persons
- Terminal histories:
  - All possible sequences of actions in the game
  - *Book,(Concert, (B, B)), (Concert, (B, S)), (Concert, (S, B)), (Concert, (S, S))*
- Player function
- Set of actions for each players' turn
- Preferences for the players

Example 1: Variant of BoS



P 2

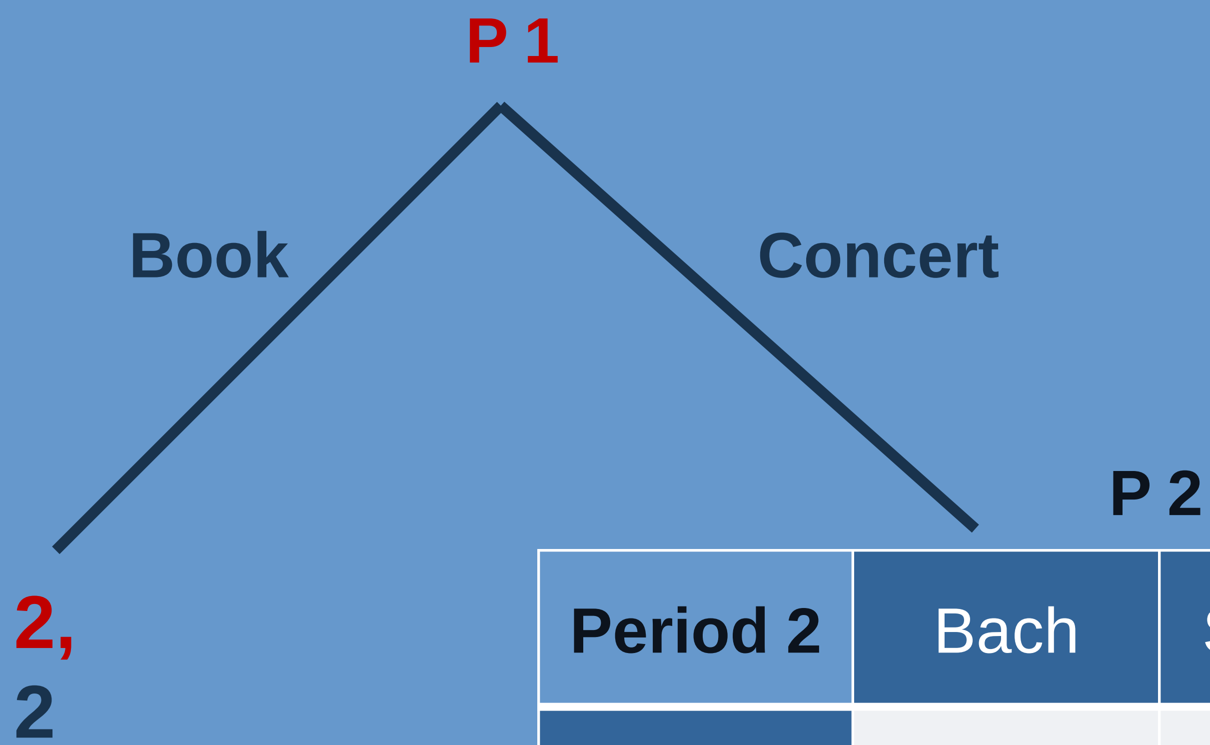
P 1

Period 2	Bach	Strav.
Bach	3, 1	0, 0
Strav.	0, 0	1, 3

Dynamic games with simultaneous moves

- Set of players:
- Terminal histories:
- Player function:
  - set of players who take an action after history h
  - P(Ø) = Person 1 ; P(Concert) = {Person 1, Person 2}
- Set of actions for each players' turn:
  - the set of actions available to player i after the history h
  - The set of player 1's actions at the initial history ∅ is $A_1(\emptyset)$ = *{Concert,Book} , after the history Concert is A_1(Concert) ={B, S}; the set of player 2's actions after the history Concert is A_2(Concert) = {Bach, Stravinsky}.*
- Preferences for the players

Example 1: Variant of BoS



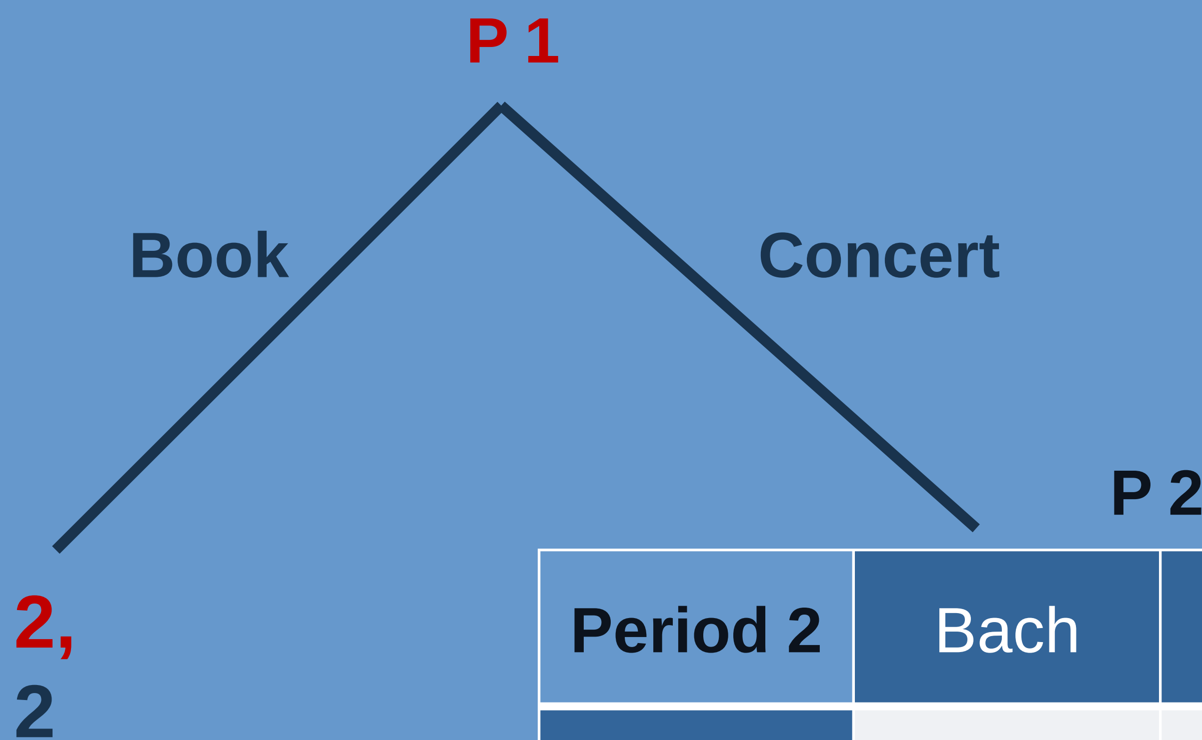
Period 2	Bach	Strav.
Bach	3, 1	0, 0
Strav.	0, 0	1, 3

P 1 (row player)

Dynamic games with simultaneous moves

- Set of players:
- Terminal histories:
- Player function:
- Set of actions for each players' turn:
- Preferences for the players:
  - Preferences over terminal histories
  - Preferences over outcomes of terminal histories
  - Again represented by utility (payoff) function
  - Person 1: u_1 for which u_1(*Concert, (B, B)*) = 3, u_1(Book) = 2, u_1 *(Concert, (S, S))* = 1 *and* u_1*(Concert,(B,S))*=u_1*(Concert,(S,B))=0*

Example 1: Variant of BoS



P 1

Period 2	Bach	Strav.
Bach	3, 1	0, 0
Strav.	0, 0	1, 3

Strategies

- strategy (as in simple case without simultaneous moves) specifies the action the player chooses for every history after which it is her turn to move

- Definition: A **strategy of player i in an dynamic game with** simultaneous moves is a function that assigns to each history h after which i is one of the players whose turn it is to move (i.e. P(h) = i, where P is the player function) an action in $A_i(h)$ (the set of actions available to her after h)

Example 1: Variant of BoS

Strategies:

Player 1:

(Book, Bach)

(Book, Stravinsky)

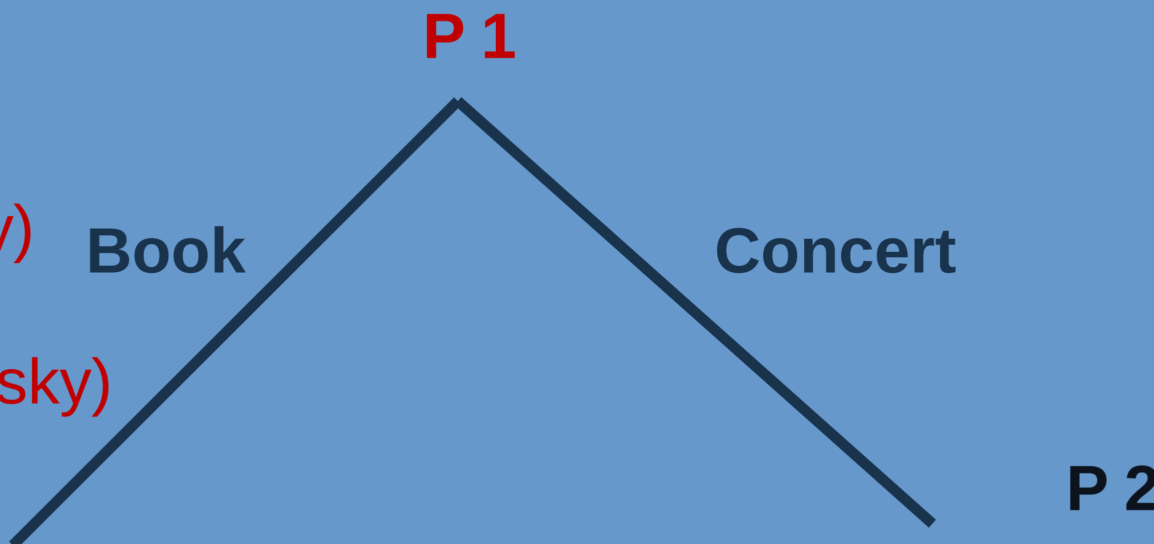
(Concert, Bach)

(Concert, Stravinsky)

Player 2:

Bach

Stravinsky

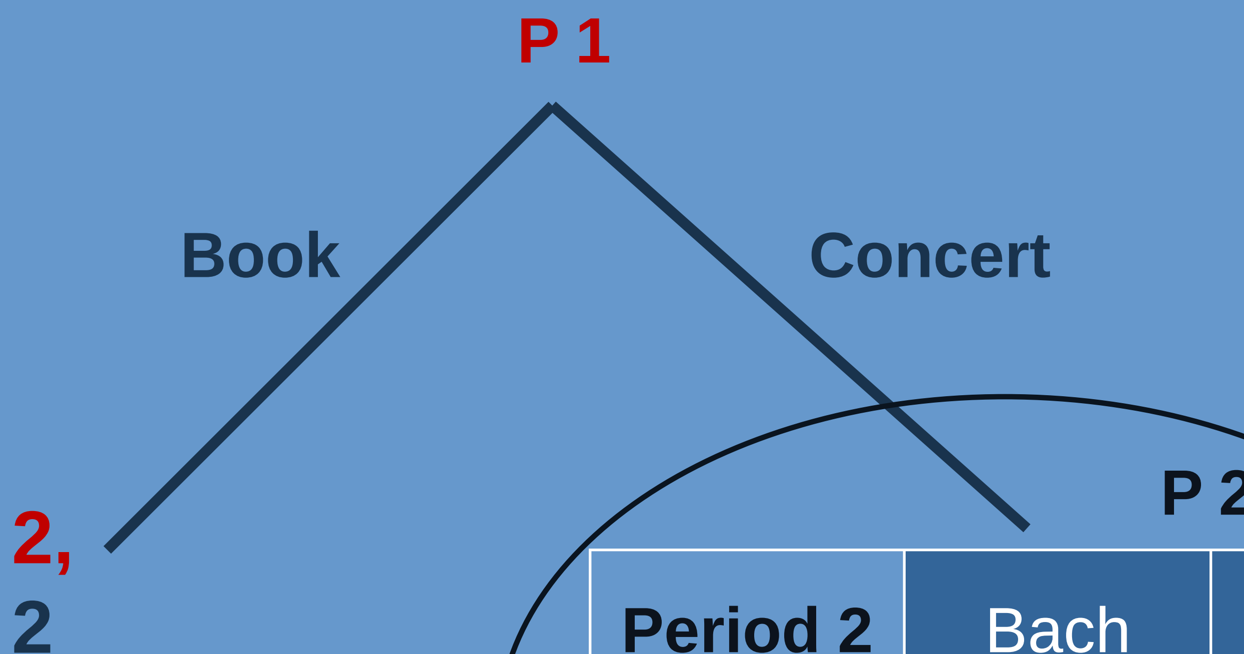


2, 2

Period 2	Bach	Strav.
Bach	**3, 1**	**0, 0**
Strav.	**0, 0**	**1, 3**

!!!Player 1 plays at start and after concert → each strategy has to specify his turn in each of these two states!!!

Example 1: Variant of BoS



Period 2	Bach	Strav.
Bach	3, 1	0, 0
Strav.	0, 0	1, 3

SPNE – strategy profile: NE in every subgame

we have two subgames here:

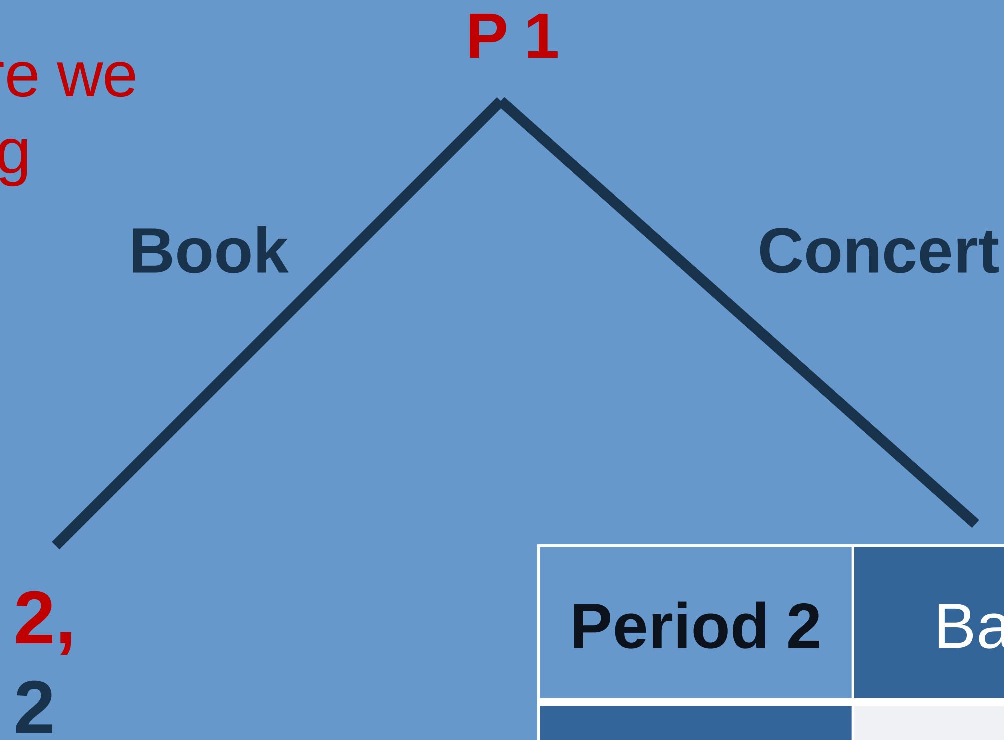
One is the whole game

Second one is circled →

Example 1: Variant of BoS

SPNE

Similar as before we start with solving the subgame:



P 2

P 1

Period 2	Bach	Strav.
Bach	3, 1	0, 0
Strav.	0, 0	1, 3

SPNE

- We cannot simply find an optimal action for the player whose turn it is to move at the start of each subgame, given the players' behavior in the remainder of the game.
- We need to find a list of actions for the players who move at the start of each subgame, with the property that each player's action is optimal given the other players' simultaneous actions and the players' behavior in the remainder of the game.

Example 1: Variant of BoS

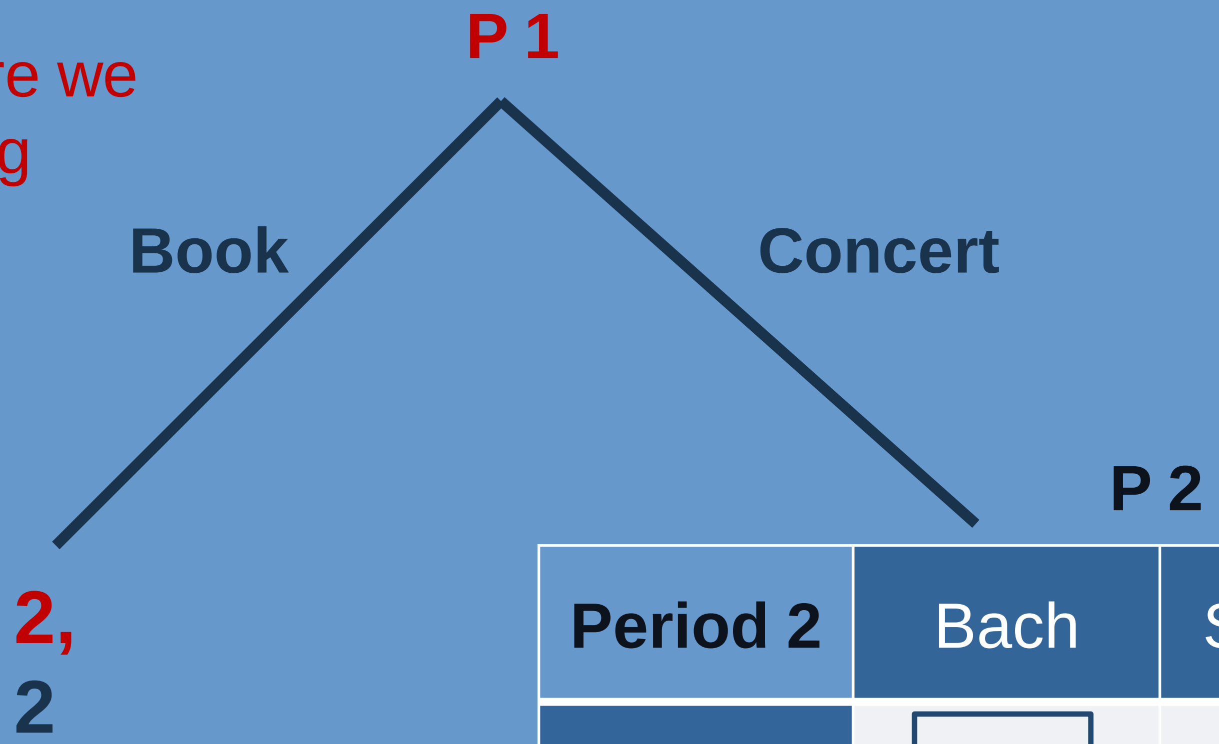
SPNE

Similar as before we start with solving the subgame:

two possible outcomes (NE):

(Bach,Bach)

(Strav.,Strav.)



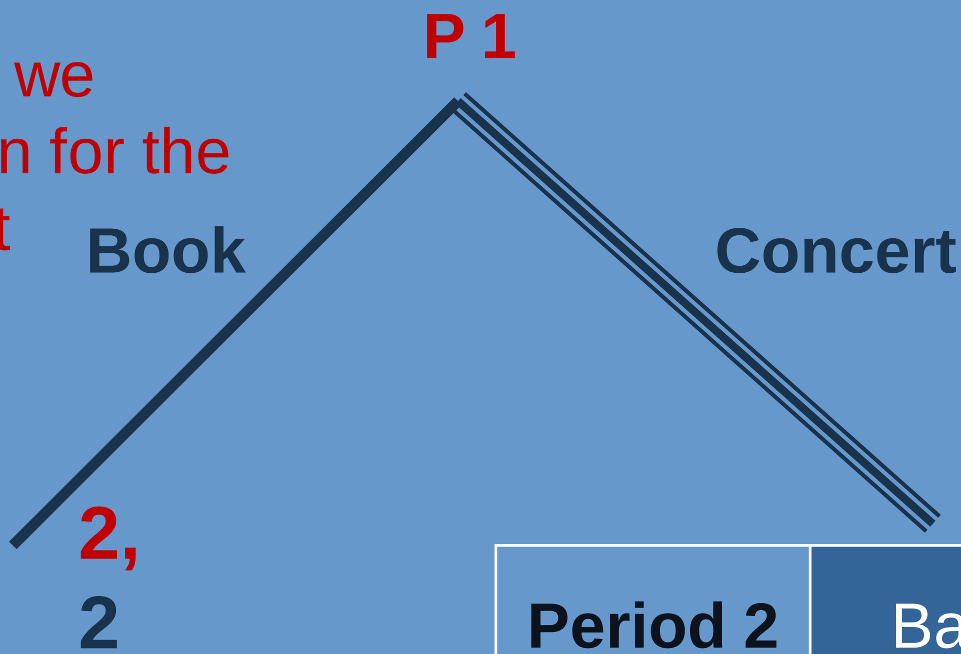
P 1

Period 2	Bach	Strav.
Bach	**3**, **1**	0, 0
Strav.	0, 0	**1**, **3**

Example 1: Variant of BoS

SPNE

Similar as before we find optimal action for the player at the start of the game for each possible combination of optimal actions (here for each outcome)



P 1

Period 2	Bach	Strav.
Bach	**3, 1**	**0, 0**
Strav.	**0, 0**	**1, 3**

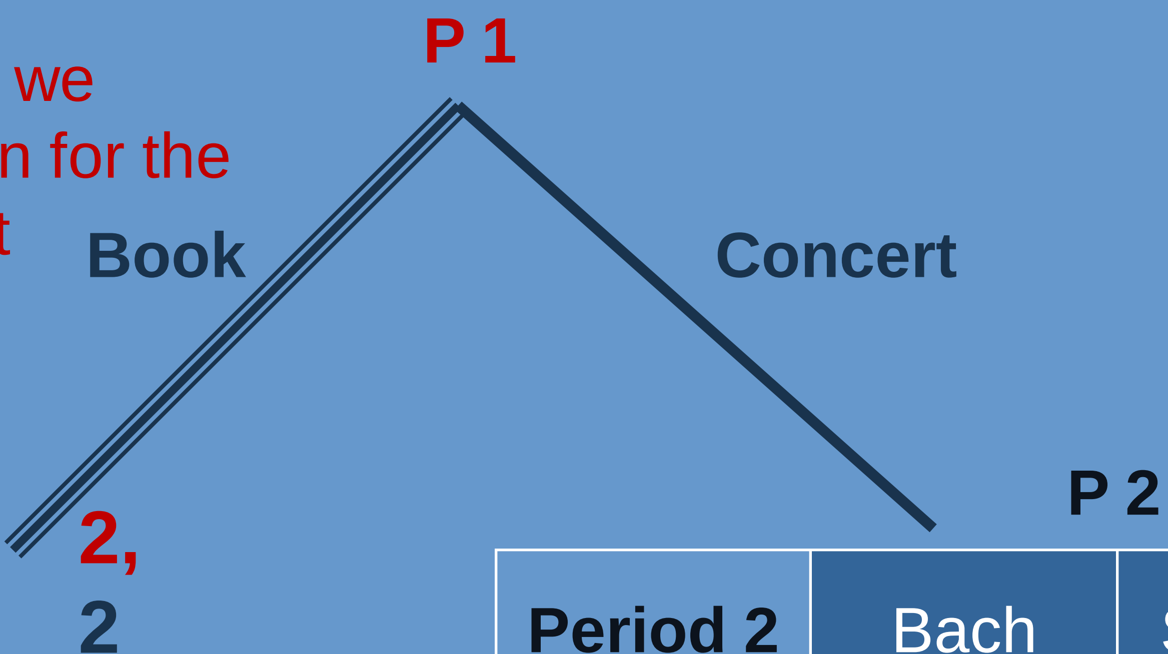
(Bach,Bach) → Concert

SPNE: ((Concert, Bach),Bach)

Example 1: Variant of BoS

SPNE

Similar as before we find optimal action for the player at the start of the game for each possible combination of optimal actions (here for each outcome)



(Strav.,Strav.) → Book

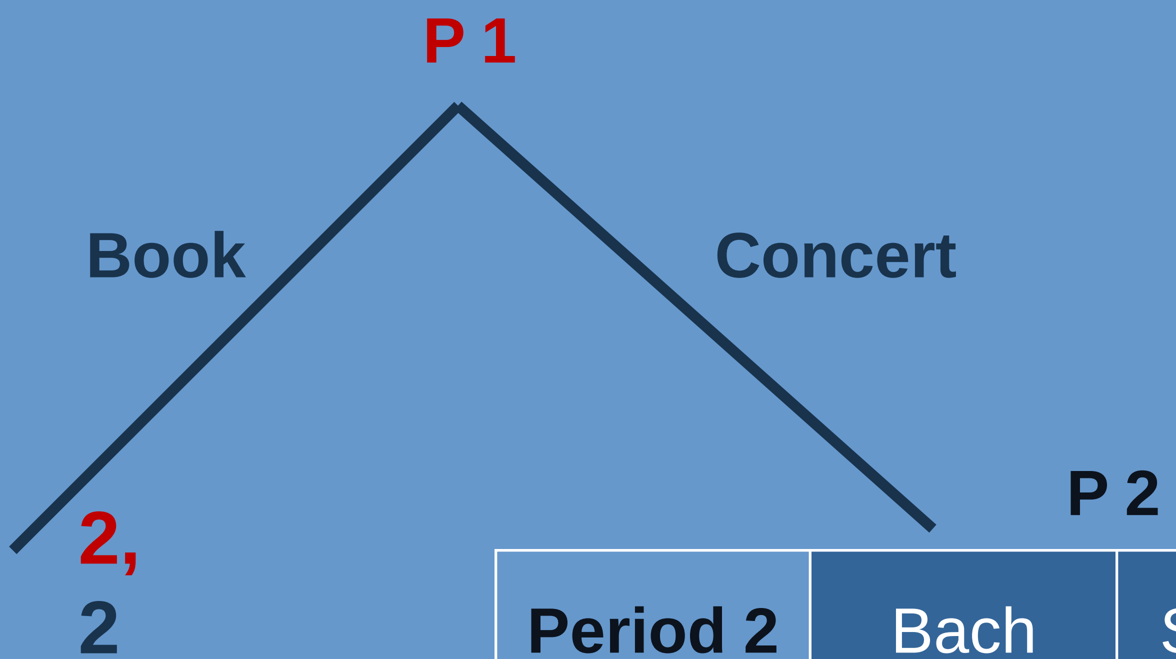
P 1

Period 2	Bach	Strav.
Bach	**3**, **1**	**0**, **0**
Strav.	**0**, **0**	**1**, **3**

SPNE: ((Book, Strav.),Strav.)

Example 1: Variant of BoS

SPNE



P 1

Period 2	Bach	Strav.
Bach	3, 1	0, 0
Strav.	0, 0	1, 3

We have 2 SPNE:

SPNE: ((Book, Strav.),Strav.)

((Concert, Bach),Bach)

Example 2: Bank runs

Two investors have each deposited 50k CZK with a bank at too optimistic interest rate of 30%.

The bank has invested these deposits (100k CZK) in a long-term project, however, only projects with maximum 20% return were available for investment.

If the bank is forced to liquidate its investment before the project matures, the bank would go bankrupt and a total of 80k CZK can be recovered.

If the bank allows the investment to reach maturity, however, the project will pay out a total of 120k CZK.

Example 2: Bank runs

P 2

P 1

Period 1	withdraw	don't
withdraw	40, 40	50, 30
don't	30, 50	next

P 2

P 1

Period 2	withdraw	don't
withdraw	60, 60	65, 55
don't	55, 65	60, 60

Example 2: Bank runs

P 2 (columns), P 1 (rows)

Period 1	withdraw	don't
withdraw	**40**, 40	**50**, 30
don't	**30**, 50	**next**

Similar as before – at first find optimal actions in subgame (Period 2):
Here – (withdraw, withdraw)

P 2 (columns), P 1 (rows)

Period 2	withdraw	don't
withdraw	**60**, **60**	**65**, 55
don't	55, **65**	60, 60

Example 2: Bank runs

P 2

P 1

Period 1	withdraw	Don't
withdraw	40, 40	50, 30
Don't	30, 50	60, 60

Given the optimal actions in Period 2
(withdraw, withdraw)
Find the optimal actions in Period 1

Example 2: Bank runs

P 2

P 1

Period 1	withdraw	don't
withdraw	**40**, **40**	**50**, 30
don't	**30**, **50**	**60**, **60**

The first equilibrium can be described as bank run. The model does not predict when bank runs will occur, but does show that they can occur *as* an equilibrium phenomenon.

Therefore this game has two SPNE:

First one: ((withdraw, withdraw);(withdraw, withdraw))

leading to the outcome **40**, **40**

Second one: ((don't, withdraw);(don't, withdraw))

leading to the outcome **60**, **60**

Example 3: International tariffs

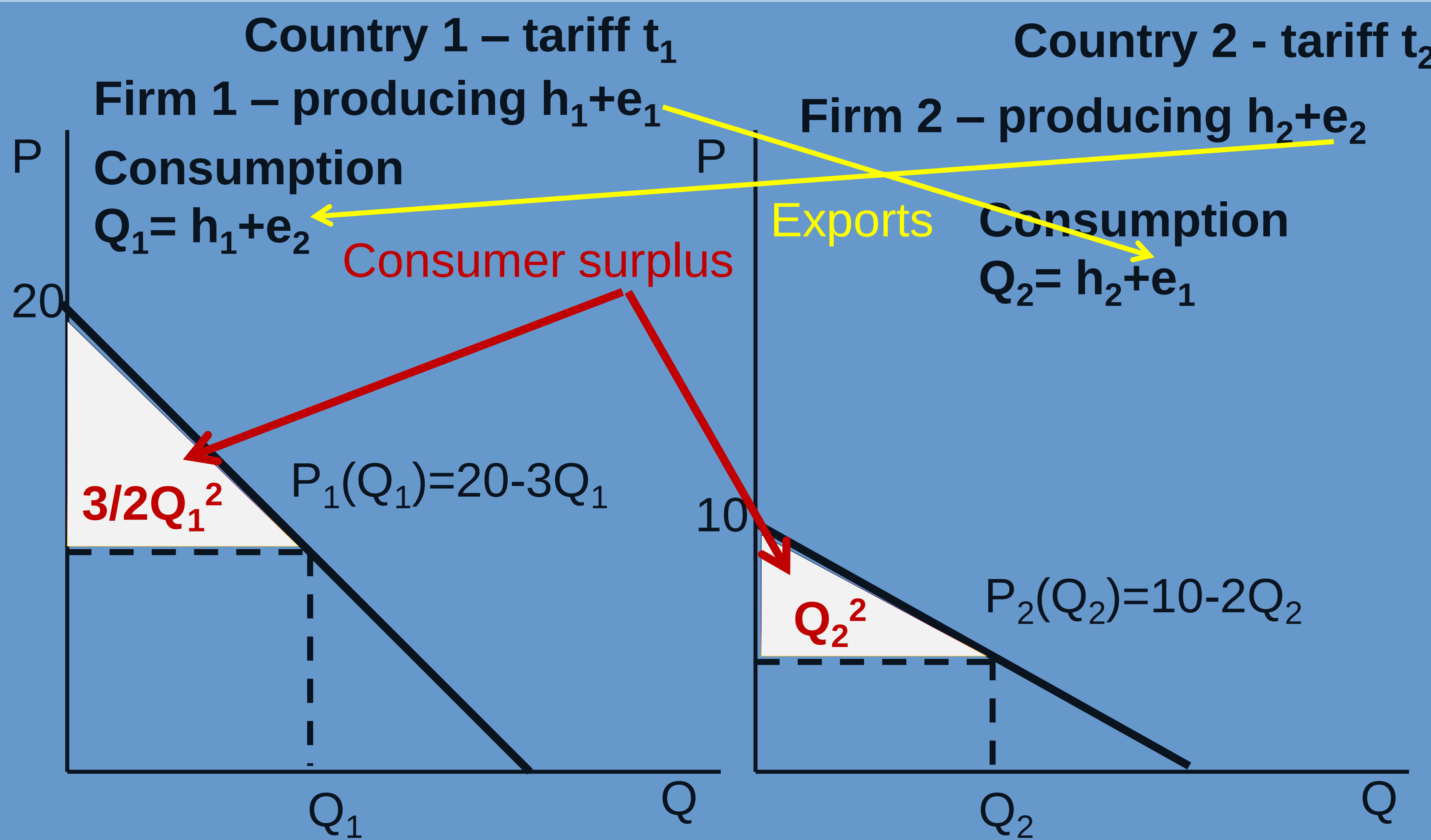
Consider two countries, denoted x=1,2. Each country has a government that chooses a tariff rate on the imports.

There are two firms – one in each country. Each firm produces output for both home consumption (h_x) and export (e_x).

Consumers buy on the home market from either the home firm or the foreign firm. If total quantity on the market in country x is Q_x, then market-clearing price is $P_x(Q_x)=a_x-b_xQ_x$. Where $Q_x=h_x+e_y$.

Particularly: $P_1(Q_1)=20-3Q_1$ and $P_2(Q_2)=10-2Q_2$

Example 3: International tariffs



Example 3: International tariffs

Each firm has constant marginal cost, c=2, and no fixed costs. Thus, the total cost of production for firm x is $C_x(h_x,e_x)=2(h_x+e_x)$.

The firms also incur tariff costs on exports: if firm x exports e_x to country y when government y has set the tariff rate t_y, then firm x must pay $t_y e_x$ (i.e. $t_y = 3$ tariff → $3e_x$ are additional costs for exports).

The timing is as follows: First, the governments simultaneously choose tariff rates, t_1 and t_2. Second, the firms observe the tariff rates and simultaneously choose quantities for home consumption and for export: (h_1,e_1) and (h_2,e_2).

Example 3: International tariffs

The payoffs are profits for the firms:

$\pi_x = [a_x - b_x(h_x + e_y)]h_x \quad + \quad [a_y - b_y(e_x + h_y)]e_x$

price at home* h_x + price abroad*e_x

$- c(h_x + e_x) \qquad -t_y e_x$

- costs of production – costs of exports

$\pi_1 = [20 - 3(h_1 + e_2)]h_1 + [10 - 2(e_1 + h_2)]e_1$

$- 2(h_1 + e_1) \quad -t_2 e_1$

$\pi_2 = [10 - 2(h_2 + e_1)]h_2 + [20 - 3(e_2 + h_1)]e_2$

$- 2(h_2 + e_2) \quad -t_1 e_2$

Example 3: International tariffs

And payoff for country is total welfare of the country, where total welfare of country x consist of consumers' surplus enjoyed by consumers in the country, profit of firm x and tariff revenue collected by the government x.

$$W_x = \tfrac{1}{2} b_x Q_x^2 \quad + \quad \pi_x \quad + \quad t_x e_y$$

consumer surplus + profit of firm + revenue of g.

$$W_1 = 3/2\, Q_1^2 + \pi_1 + t_1 e_2$$

$$W_2 = Q_2^2 + \pi_2 + t_2 e_1$$

Example 3: International tariffs

Period 1: Governments are simultaneously choosing t_1 and t_2

Period 2: Firms when observed t_1 and t_2 are simultaneously choosing quantities for home consumption and for export: (h_1,e_1) and (h_2,e_2)

When searching for SPNE we start with subgame

– here Period 2 and we have to find all optimal quantities of both firms i.e. (h_1^*,e_1^*) and (h_2^*,e_2^*) when playing simultaneously

→ searching for NE of this subgame

Example 3: International tariffs

We will start with best response h_1 and e_1 to the action of the firm 2 (h_2 and e_2) when it observes tariffs t_1 and t_2.

The profit π has two additive parts so we can find optimal (best response) h_1 and e_1 separately:

$$\pi_1 = [20 - 3(h_1+e_2)]h_1 + [10 - 2(e_1+h_2)]e_1 - 2(h_1+e_1) - t_2 e_1$$

$$\pi_1 = [20 - 3(h_1+e_2) - 2]h_1 \quad + \quad [10 - 2(e_1+h_2) - 2 - t_2]e_1$$

$$\max_{(h_1,e_1)} \pi_1 = \max_{(h_1)} [20 - 3(h_1+e_2) - 2]h_1 + \max_{(e_1)} [10 - 2(e_1+h_2) - 2 - t_2]e_1$$

Example 3: International tariffs

We will start with best response h_1 and e_1 to the action of the firm 2 (h_2 and e_2) when it observes tariffs t_1 and t_2.

The profit π has two additive parts so we can find optimal (best response) h_1 and e_1 separately:

$$\max_{(h_1)} [20 - 3(h_1+e_2)-2]h_1 = [18 - 3e_2 - 3h_1]h_1$$

$$= (18 - 3e_2)h_1 - 3h_1^2$$

taking derivative with respect to h_1 (assuming $18 > 3e_2$)

$$18 - 3e_2 - 6h_1 = 0$$

$$h_1 = (18 - 3e_2)/6 \qquad [(a_x - c - b_x e_y)/2b_x]$$

Example 3: International tariffs

We will start with best response h_1 and e_1 to the action of the firm 2 (h_2 and e_2) when it observes tariffs t_1 and t_2.

The profit π has two additive parts so we can find optimal (best response) h_1 and e_1 separately:

$$\max_{(e_1)} [10 - 2(e_1+h_2)-2-t_2]e_1 = [8 - 2h_2 - t_2 - 2e_1]e_1$$

$$= (8 - 2h_2 - t_2)e_1 - 2e_1^2$$

taking derivative with respect to e_1 (assuming $8 - t_2 > 2h_2$)

$$8 - 2h_2 - t_2 - 4e_1 = 0$$

$$e_1 = (8-2h_2 - t_2)/4 \qquad [(a_y-c-b_yh_y-t_y)/2b_y]$$

Example 3: International tariffs

Now continue with best response h_2 and e_2 to the action of the firm 1 (h_1 and e_1) when it observes tariffs t_1 and t_2.

The profit π has two additive parts so we can find optimal (best response) h_2 and e_2 separately:

$$\pi_2 = [10 - 2(h_2+e_1)]h_2 + [20 - 3(e_2+h_1)]e_2 - 2(h_2+e_2) - t_1 e_2$$

$$\pi_2 = [10 - 2(h_2+e_1) - 2]h_2 \quad + \quad [20 - 3(e_2+h_1) - 2 - t_1]e_2$$

$$\max_{(h_2,e_2)} \pi_2 = \max_{(h_2)} [10 - 2(h_2+e_1) - 2]h_2 + \max_{(e_2)} [20 - 3(e_2+h_1) - 2 - t_1]e_2$$

Example 3: International tariffs

Now continue with best response h_2 and e_2 to the action of the firm 1 (h_1 and e_1) when it observes tariffs t_1 and t_2.

The profit π has two additive parts so we can find optimal (best response) h_2 and e_2 separately:

$$\max (h_2)\ [10 - 2 (h_2+e_1)-2]h_2 = [8- 2e_1-2h_2]h_2$$

$$= (8- 2e_1)\ h_2-2h_2^2$$

taking derivative with respect to h_2 (assuming $8 > 2e_1$)

$$8 – 2e_1 - 4 h_2 = 0$$

$$h_2 = (8-2e_1)/4 \qquad [(a_y-c-b_ye_x)/2b_y]$$

Example 3: International tariffs

Now continue with best response h_2 and e_2 to the action of the firm 1 (h_1 and e_1) when it observes tariffs t_1 and t_2.

The profit π has two additive parts so we can find optimal (best response) h_2 and e_2 separately:

$$\max (e_2)\ [20 - 3\,(e_2+h_1)-2-t_1]e_2 = [18- 3h_1-t_1-3e_2]e_2$$

$$= (18- 3h_1-t_1)\ e_2-3e_2^2$$

taking derivative with respect to e_2 (assuming $18-t_1 > 3h_1$)

$$18 – 3h_1 - t_1 - 6\,e_2 = 0$$

$$e_2 = (18-3h_1-t_1)/6 \quad [(a_x-c-b_xh_x-t_x)/2b_x]$$

Example 3: International tariffs

We have best responses of both players. To find NE of this subgame: chosen actions of player 2 have to be best response to player 1 and vice versa.

Firm 1: $h_1 = (18 - 3e_2)/6$ $e_1 = (8-2h_2 - t_2)/4$

Firm 2: $h_2 = (8-2e_1)/4$ $e_2 = (18-3h_1-t_1)/6$

By plugging e_2 to first equation we get:

$h_1 = (18 - 3e_2)/6 \rightarrow (18-6h_1)/3 = e_2 = (18-3h_1-t_1)/6$ /.6

$36-12h_1 = 18-3h_1-t_1$

$18+t_1 = 9h_1 \rightarrow h_1^* = (18+t_1)/9$

$3e_2 = 18-6h_1 = 18 – 6(18+t_1)/9 = (3*18- 2*18-2t_1)/3$ /:3

$e_2 = (3*18- 2*18-2t_1)/9 \rightarrow e_2^* = (18-2t_1)/9$

Example 3: International tariffs

We have best responses of both players. To find NE of this subgame: chosen actions of player 2 have to be best response to player 1 and vice versa.

Firm 1: $h_1 = (18 - 3e_2)/6$ $e_1 = (8-2h_2 - t_2)/4$

Firm 2: $h_2 = (8-2e_1)/4$ $e_2 = (18-3h_1-t_1)/6$

By plugging e_1 to h_2 equation we get:

$h_2 = (8 - 2e_1)/4 \rightarrow (8-4h_2)/2 = e_1 = (8-2h_2-t_2)/4$ /.4

$16-8h_2 = 8-2h_2-t_2$

$8+t_2 = 6h_2 \rightarrow h_2^* = (8+t_2)/6$

$2e_1 = 8-4h_2 = 8 – 4(8+t_2)/6 = (3*8- 2*8-2t_2)/3$ /:2

$e_1 = (3*8- 2*8-2t_2)/6 \rightarrow e_1^* = (8-2t_2)/6$

Example 3: International tariffs

We have best responses of both players. To find NE of this subgame: chosen actions of player 2 have to be best response to player 1 and vice versa.

Firm1: $h_1^* = (18+t_1)/9$ $e_1^* = (8-2t_2)/6$

$[=(a_x-c+t_x)/3b_x]$ $[=(a_y-c-2t_y)/3b_y]$

Firm 2: $h_2^* = (8+t_2)/6$ $e_2^* = (18-2t_1)/9$

$[=(a_y-c+t_y)/3b_y]$ $[=(a_x-c-2t_x)/3b_x]$

As we can see in equilibrium with increasing t (tariff) the home production increases, however, the exports decreases at a faster rate.

Example 3: International tariffs

Having solved the subgame in Period 2 we can now turn to the Period 1 and the simultaneous choice of tariffs of both governments. They are maximizing the total welfare – sum of consumer surplus, profit of home firm and government revenue.

$W_1 = 3/2\ Q_1^2 + \pi_1 + t_1 e_2$ $h_1^* = (18+t_1)/9;\ e_1^* = (8-2t_2)/6$

$W_2 = Q_2^2 + \pi_2 + t_2 e_1$ $h_2^* = (8+t_2)/6\ ;\ e_2^* = (18-2t_1)/9$

$Q_1^* = h_1^* + e_2^* = (18+t_1)/9 + (18-2t_1)/9 = (36-t_1)/9$

$Q_2^* = h_2^* + e_1^* = (8+t_2)/6 + (8-2t_2)/6 = (16-t_2)/6$

$\pi_1^* = ([18-3(h_1+e_2)]h_1 =)(18+t_1)^2/27 + ([8-2(e_1+h_2)-t_2]e_1 =)(8-2t_2)^2/18$

$\pi_2^* = ([8-2(h_2+e_1)]h_2 =)(8+t_2)^2/18 + ([18-3(e_2+h_1)-t_1]e_2 =)(18-2t_1)^2/27$

Example 3: International tariffs

Lets find best response of country 1 to action of country 2 – tariff t_2

$W_1 = 3/2\,(36-t_1)^2/81+(18+t_1)^2/27+(8-2t_2)^2/18 + t_1(18-2t_1)/9$

Taking derivative with respect to t_1

$-3(36-t_1)/81+2(18+t_1)/27+ 0 +(18-4t_1)/9=0$

$-12/9+t_1/27+12/9+ 2t_1/27+ 0 +18/9-4t_1/9=0$

$18/9-9t_1/27=0$

$2=t_1/3$

$t_1=6$ $[=(a_x-c)/3]$

Example 3: International tariffs

Lets find best response of country 2 to action of country 1 – tariff t_1

$W_2 = (16-t_2)^2/36+(8+t_2)^2/18+(18-2t_1)^2/27 + t_2(8-2t_2)/6$

Taking derivative with respect to t_1

$-2(16-t_2)/36+2(8+t_2)/18 + 0 + (8-4t_2)/6=0$

$-16/18+t_2/18+16/18+ 2t_2/18+0+8/6-4t_2/6=0$

$8/6-9t_2/18=0$

$4/3=t_2/2$

$t_2=8/3$ $[=(a_y-c)/3]$

Example 3: International tariffs

Best responses of country 1 and country 2 are:

$t_1=6$ $[=(a_x-c)/3]$

$t_2=8/3$ $[=(a_y-c)/3]$

Best responses are not dependent on the other country's choice. In other words both countries have dominant strategies ($t_1=6$, $t_2=8/3$)

If we plug these values to optimal choices of the firms we get: $h_1^* = (18+t_1)/9 = 24/9$; $e_1^* = (8-2t_2)/6 = 4/9$

$h_2^* = (8+t_2)/6 = 16/9$; $e_2^* = (18-2t_1)/9 = 6/9$

Example 3: International tariffs

With tariffs we have:

$h_1^* = 24/9;\ e_1^* = 4/9;\ h_2^* = 16/9;\ e_2^* = 6/9$

$Q_1^* = h_1^* + e_2^* = 30/9 \qquad Q_2^* = h_2^* + e_1^* = 20/9$

$P_1^* = 20\text{-}3\ Q_1^* = 10 \qquad P_2^* = 10\text{-}2Q_2^* = 50/9$

$\pi_1^* = (18+t_1)^2/27 + (8-2t_2)^2/18 = 24*24/27 + 8/3*8/3\ /18 = 21.7$

$\pi_2^* = (8+t_2)^2/18 + (18-2t_1)^2/27 = 32/3 * 32/3\ /18 + 6*6/27 = 7.7$

$W_1 = 3/2\ Q_1^2 + \pi_1 + t_1 e_2 = 3/2\ (30/9)^2 + 21.7 + 6*6/9 = 42.4$

$W_2 = Q_2^2 + \pi_2 + t_2 e_1 = (20/9)^2 + 7.7 + 8/3*4/9 = 13.8$

Example 3: International tariffs

Without any tariffs we would get

$h_1^* = 18/9$; $e_1^* = 12/9$; $h_2^* = 12/9$; $e_2^* = 18/9$

$Q_1^* = h_1^* + e_2^* = 36/9$ $\quad Q_2^* = h_2^* + e_1^* = 24/9$

$P_1^* = 20 - 3\,Q_1^* = 8$ $\quad P_2^* = 10 - 2Q_2^* = 42/9$

$\pi_1^* = (18+t_1)^2/27 + (8-2t_2)^2/18 = 18*18/27 + 8*8/18 = 15.6$

$\pi_2^* = (8+t_2)^2/18 + (18-2t_1)^2/27 = 8*8/18 + 18*18/27 = 15.6$

$W_1 = 3/2\,Q_1^2 + \pi_1 + t_1 e_2 = 3/2\,(36/9)^2 + 15.6 + 0 = 39.6$

$W_2 = Q_2^2 + \pi_2 + t_2 e_1 = (24/9)^2 + 15.6 + 0 = 22.7$

$Q_1^* = 36/9 > 30/9 = Q_1^*$ $\quad Q_2^* = 24/9 > 20/9 = Q_2^*$

Example 3: International tariffs

Without any tariffs, prices will be lower and consumer surpluses will be higher. Also the sum of welfare of both countries would be higher.

$Q_1^* = 36/9 > 30/9 = Q_1^*$ $\quad$ $Q_2^* = 24/9 > 20/9 = Q_2^*$

$P_1^* = 8 < 10 = P_1^*$ $\quad$ $P_2^* = 42/9 < 50/9 = P_2^*$

$\pi_1^* = 15.6 < 21.7 = \pi_1^*$ $\quad$ $\pi_2^* = 15.6 > 7.7 = \pi_2^*$

$W_1 = 39.6 < 42.4 = W_1$ $\quad$ $W_2 = 22.7 > 13.8 = W_2$

$$W_1 + W_2 > W_1 + W_2$$

If the countries were identical, zero tariffs would be better for both of them, so there would an incentive for the governments to sign a treaty in which they commit to zero tariffs. In this example, the country with more attractive consumer market is better off when protecting the home producers with some tariff incurred on exports.

Summary

- Dynamic games with simultaneous moves
- Examples
- Gibbons 2.2; Osborne 7

NEXT WEEK no lecture, NEXT LECTURE:

Static games with incomplete information