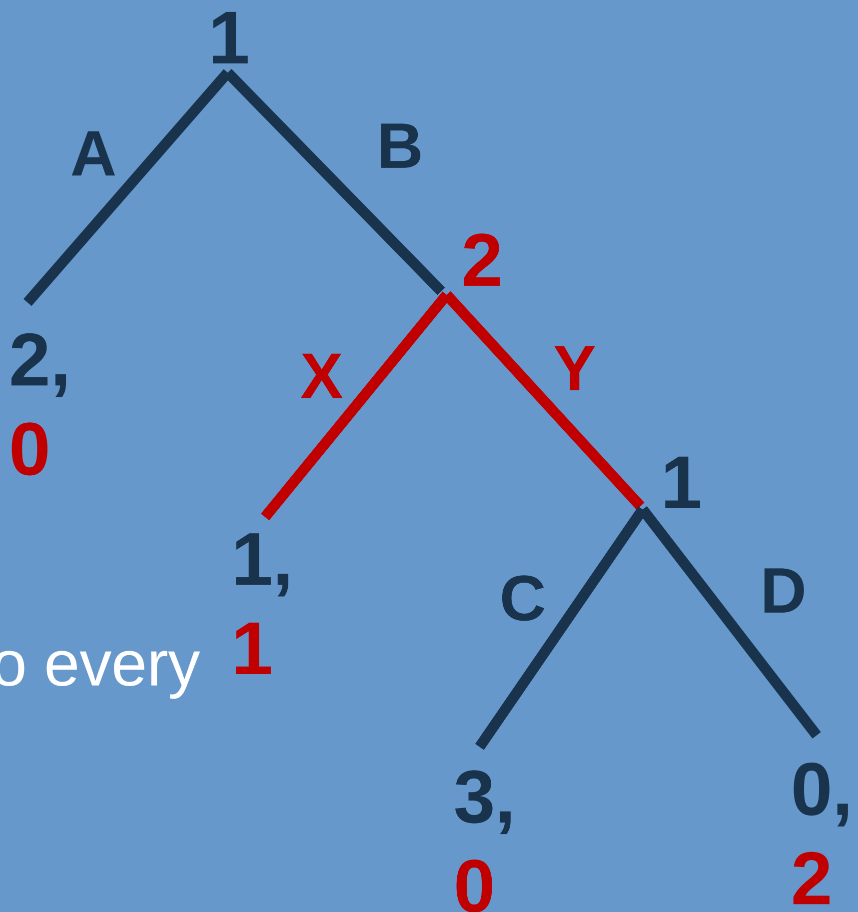
DYNAMIC GAMES

Lecture 6

Revision

Dynamic game:

- Set of players:
- Terminal histories:
  - all possible sequences of actions in the game
- Player function:
  - function that assigns a player to every proper subhistory
- Preferences for the players:
  - Preferences over terminal histories
  - represented by utility (payoff) function

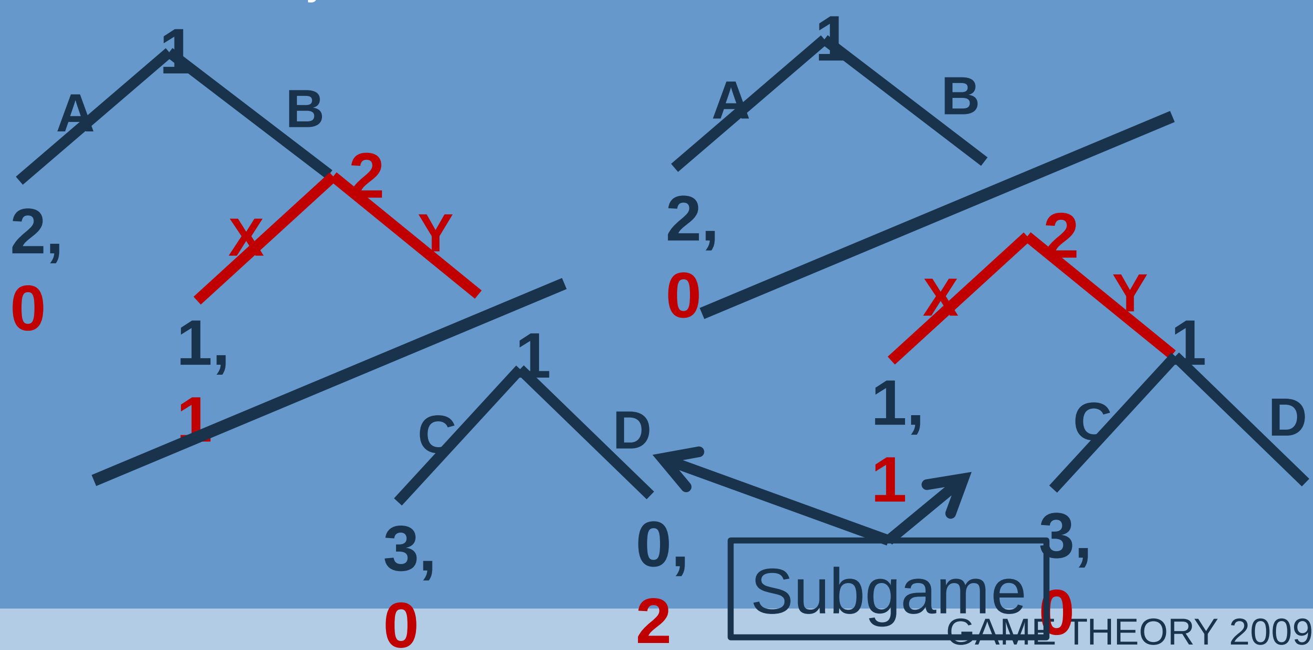


Revision

- Strategy:
  - specifies the action the player chooses for every history after which it is her turn to move
  - sufficient information to determine player's plan of action in every possible state in the game
- Outcome:
  - terminal history determined by strategy profile
  - particular strategies of all players in the game determines the terminal history that occurs

Revision

- Definition: The strategy profile s^* in an dynamic game with perfect information is a **Nash equilibrium** is such profile that none of the players have any incentive to deviate from equilibrium strategy s^*_i , given the other players adheres to s^*_{-j} .
- Subgame:



Revision

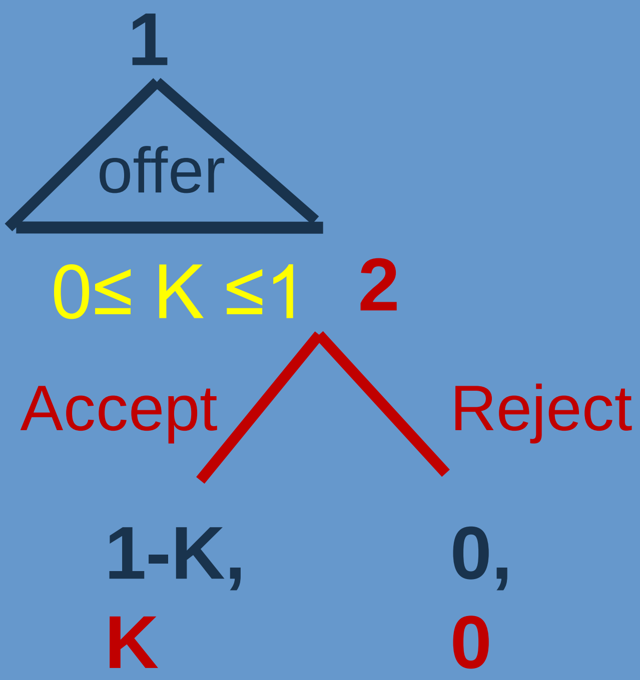
- Definition: A *subgame perfect equilibrium (SBNE) is a strategy profile s^* with the property* that in no subgame can any player *i do better by choosing a strategy* different from s^*_i *, given that every other player j adheres to* s^*_j *.*
- How to find SBNE:
  - Finite games -> Backward induction
  - Start with subgames of length 1, find all optimal actions
  - For each combination of these actions find optimal actions in subgames of length 2, continue …

Ultimatum game

Two people use the following procedure to split \$1. Person 1 offers person 2 an amount of money up to K=\$1. If 2 accepts this offer then 1 receives the remainder \$1-K. If 2 rejects the offer then neither person receives any payoff. Each person cares only about the amount of money she receives, and (naturally!) prefers to receive as much as possible.

Assume that the amount person 1 offers can be any number. Find all SBNE in the game.

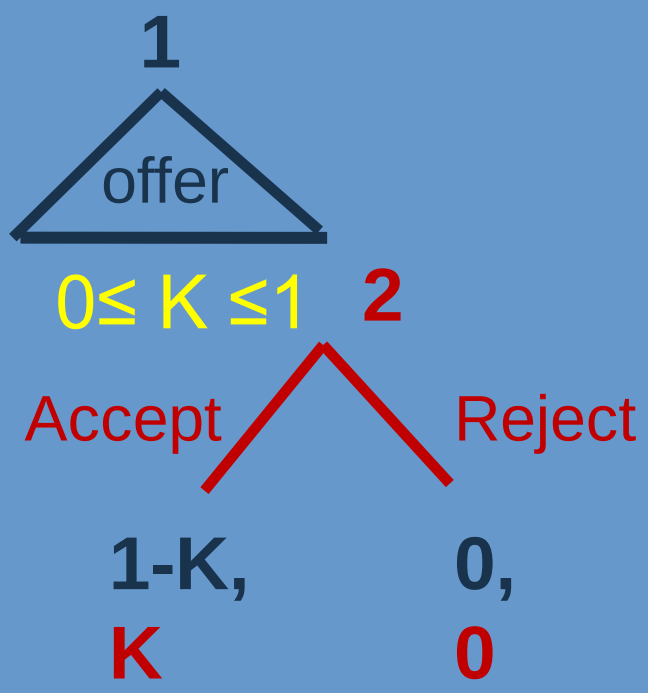
Ultimatum game



example of NE:

Player 1 offers K (gets 1-K)
Player 2 ,for every offer X of player 1, accepts K or more and rejects anything else

Ultimatum game



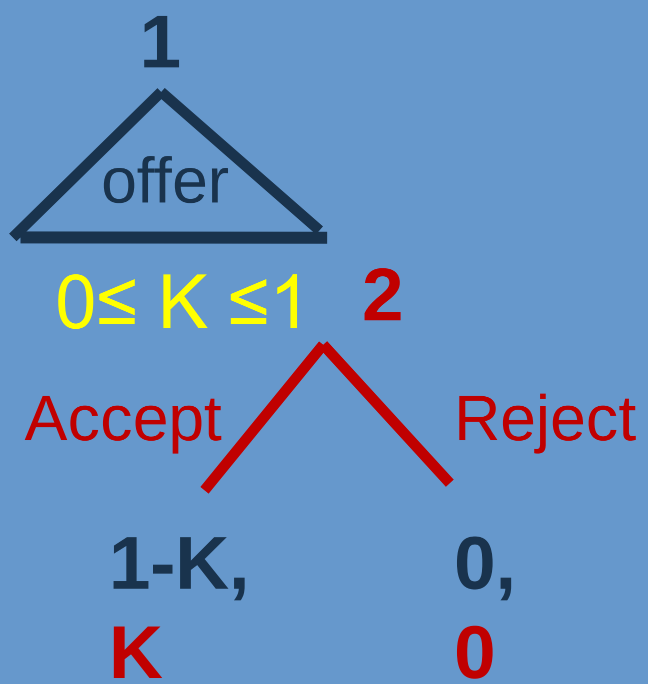
SBNE - Backward induction:

Optimal actions for player 2:
(subgame s of length 1)

first optimal strategy:
- If K>0 → accept
- If K=0 → reject

second optimal strategy:
- If K>0 → accept
- If K=0 → accept

Ultimatum game



Strategy for player 2:

If K>0 → accept

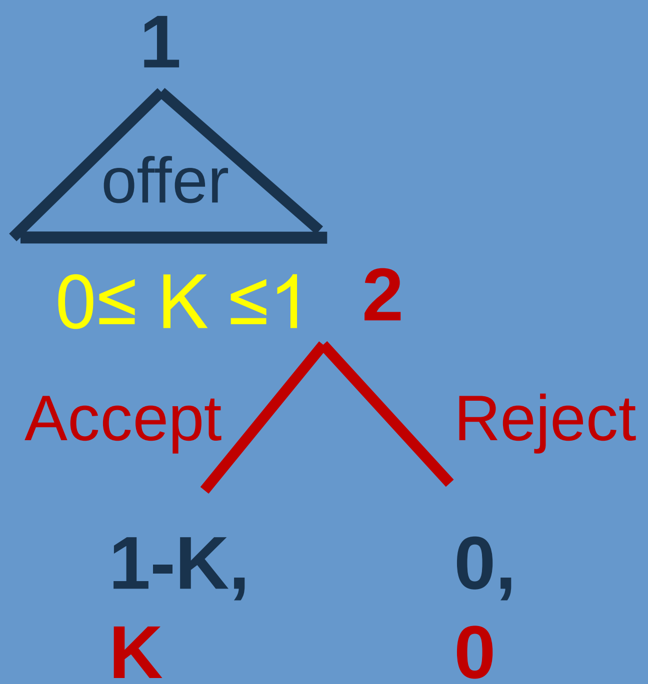
If K=0 → reject

Optimal action of player 1: (subgame of length 2)

If player 2 has strategy to accept the offer only if K>0 then no offer K is optimal action for player 1 as then L=K/2 will be better for him

→ NO SBNE such that K>0

Ultimatum game



Strategy for player 2:
 If K>0 → accept
 If K=0 → accept

Optimal action of player 1:
(subgame of length 2)
If player 2 has strategy to accept
then the only optimal action for player 1 is K=0

SBNE: offer K=0, always accept

Ultimatum game

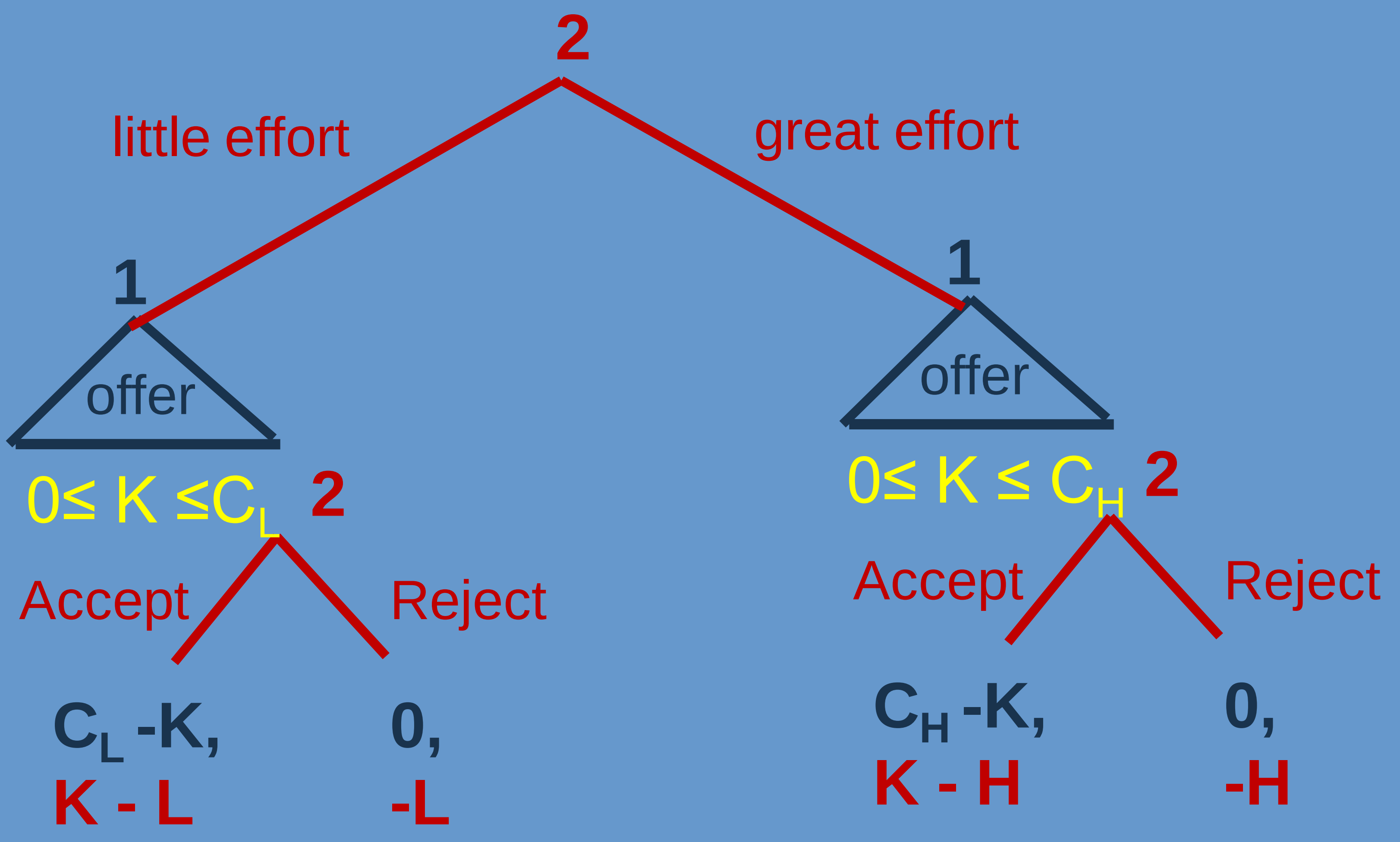
In the experiments in the late 1970s at the University of Cologne the average demand by people playing the role of player 1 was 0.65c in first series of experiments, and in the second series it was 0.69c, much less than the amount c or c − 0.01 predicted by the notion of subgame perfect equilibrium (0.01DM was the smallest monetary unit). Almost 20% of offers were rejected over the two experiments, including one of 3DM (out of a pie of 7DM) and five of around 1DM (out of pies of between 4DM and 6DM). Many other experiments, including one in which the amount of money to be divided was much larger (Hoffman, McCabe, and Smith 1996), have produced similar results. In brief, the results do not accord well with the predictions of subgame perfect equilibrium. In other words people are also equity-conscious and do not typically experience one shot games.

Holdup game

Before engaging in an ultimatum game in which she may accept or reject an offer of person 1, person 2 takes an action that affects the size of \$c to be divided. She may exert little effort, resulting in a small amount of \$$c_L$, or great effort, resulting in a large amount of size \$$c_H$. She dislikes exerting effort. Specifically, assume that her payoff is $x - E$ if her share is \$x, where $E = L$ if she exerts little effort and

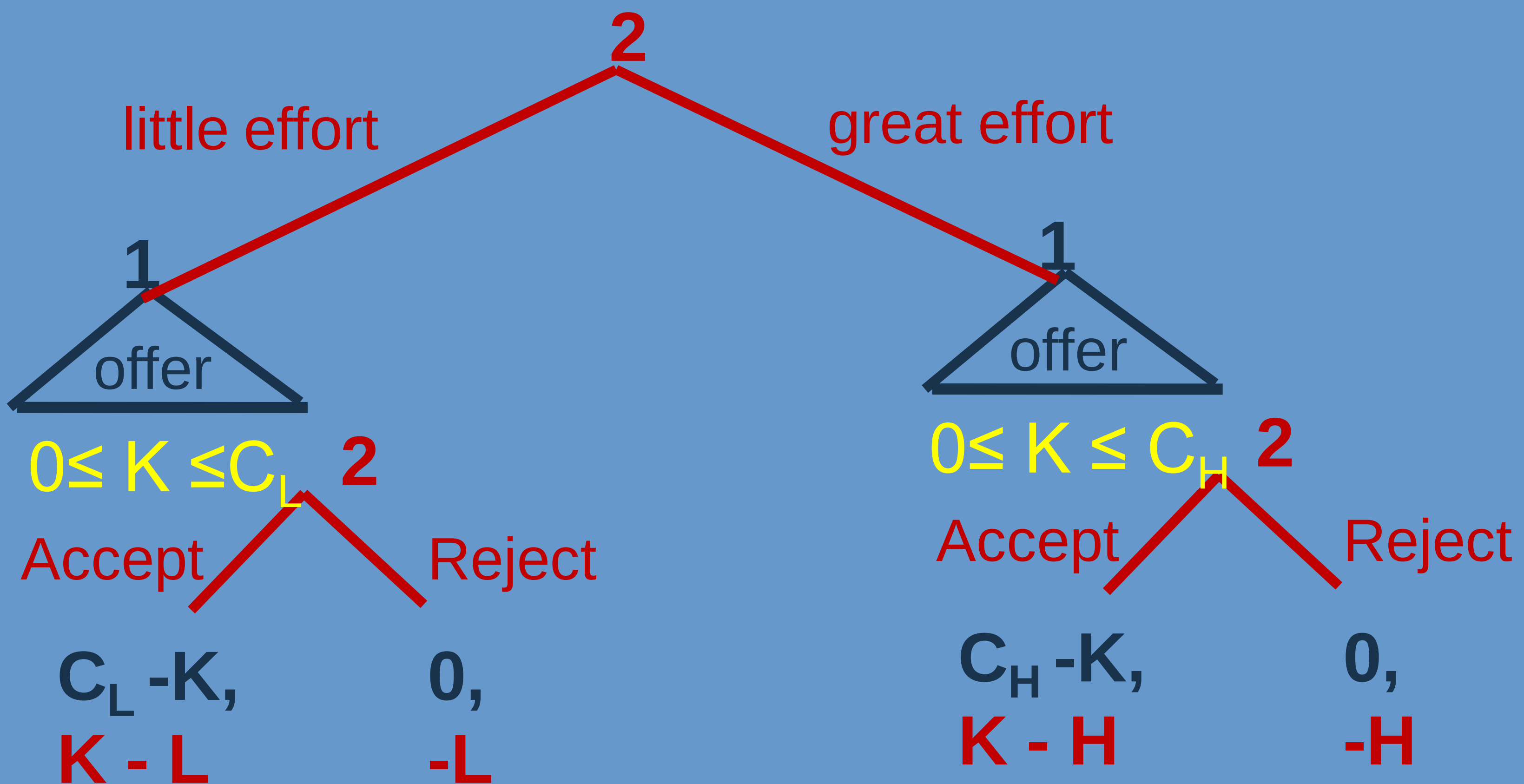
$E = H > L$ if she exerts great effort.

Holdup game


2
little effort
great effort
1
offer
0≤ K ≤C_L
2
Accept
Reject
C_L -K,
K - L
0,
-L
1
offer
0≤ K ≤ C_H
2
Accept
Reject
C_H -K,
K - H
0,
-H

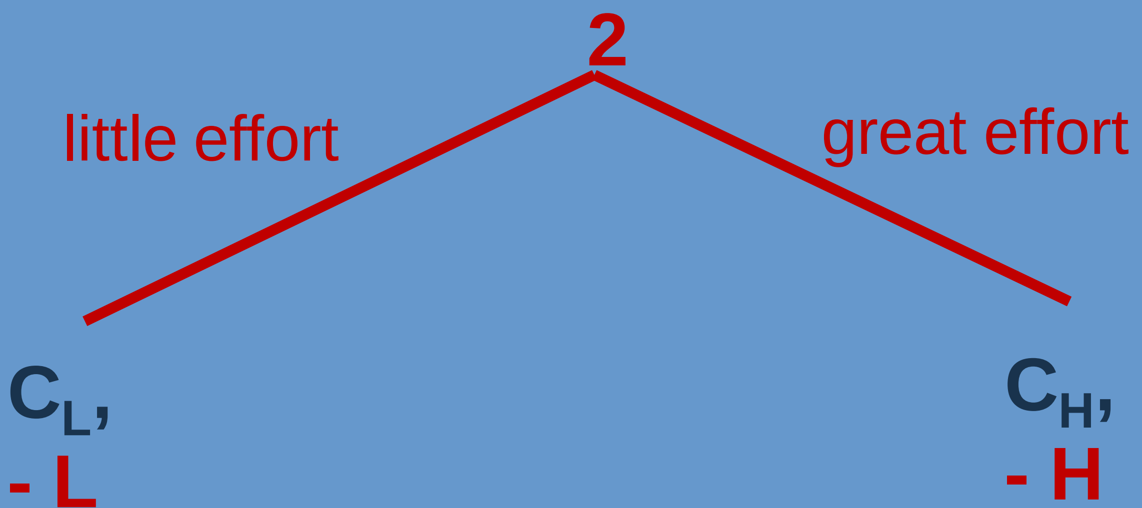
Holdup game

As we already know the optimal actions (strategies of subgames of length 2 we can start the analysis here and find the optimal actions of subgame of length 3



Holdup game

As we already know the optimal actions (strategies) of subgames of length 2 we can start the analysis here and find the optimal actions of subgame of length 3



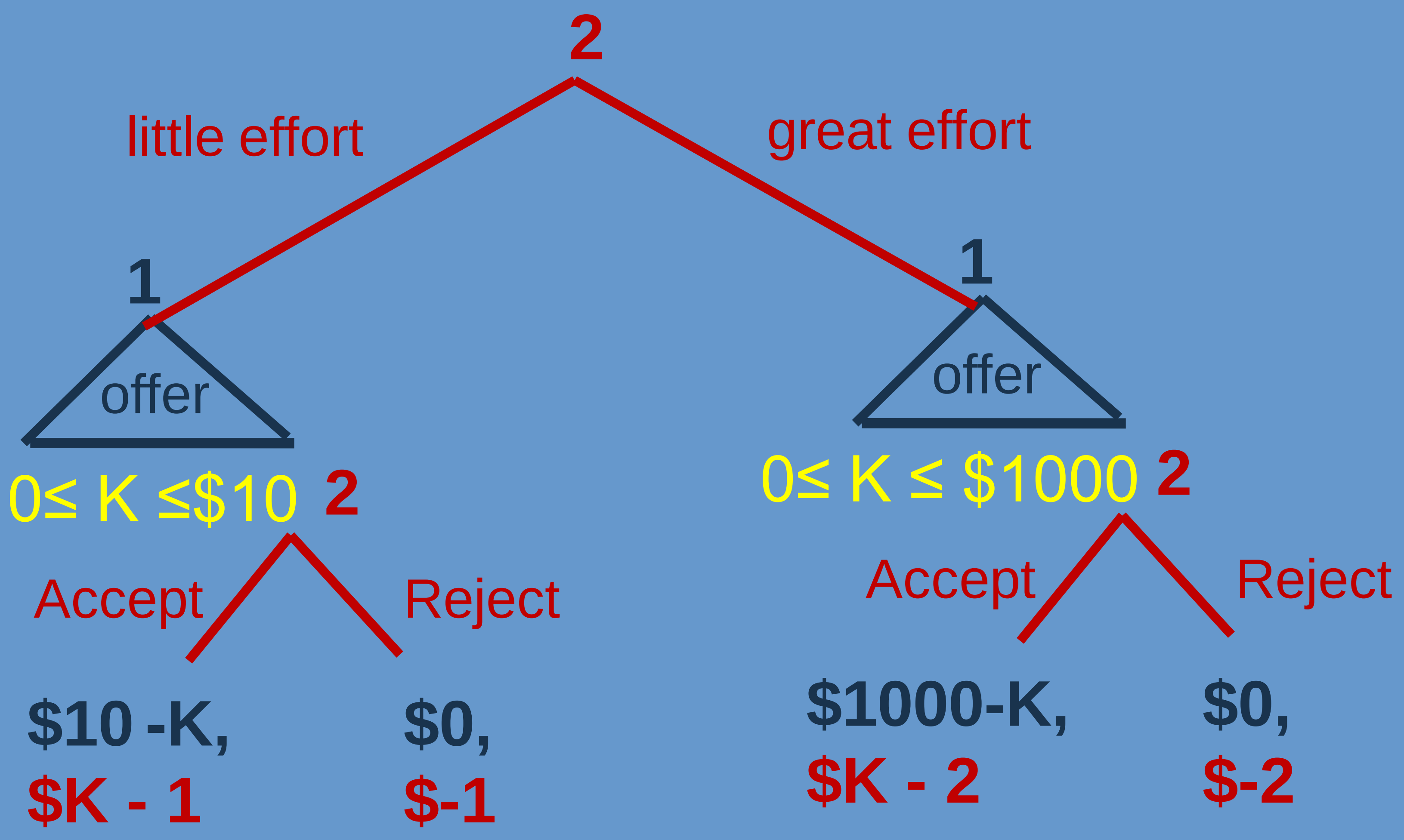
SBNE: offer K=0, accept

SBNE: offer K=0, accept

The optimal action for player 2 given the SBNE of the subgames is to exert just little effort:

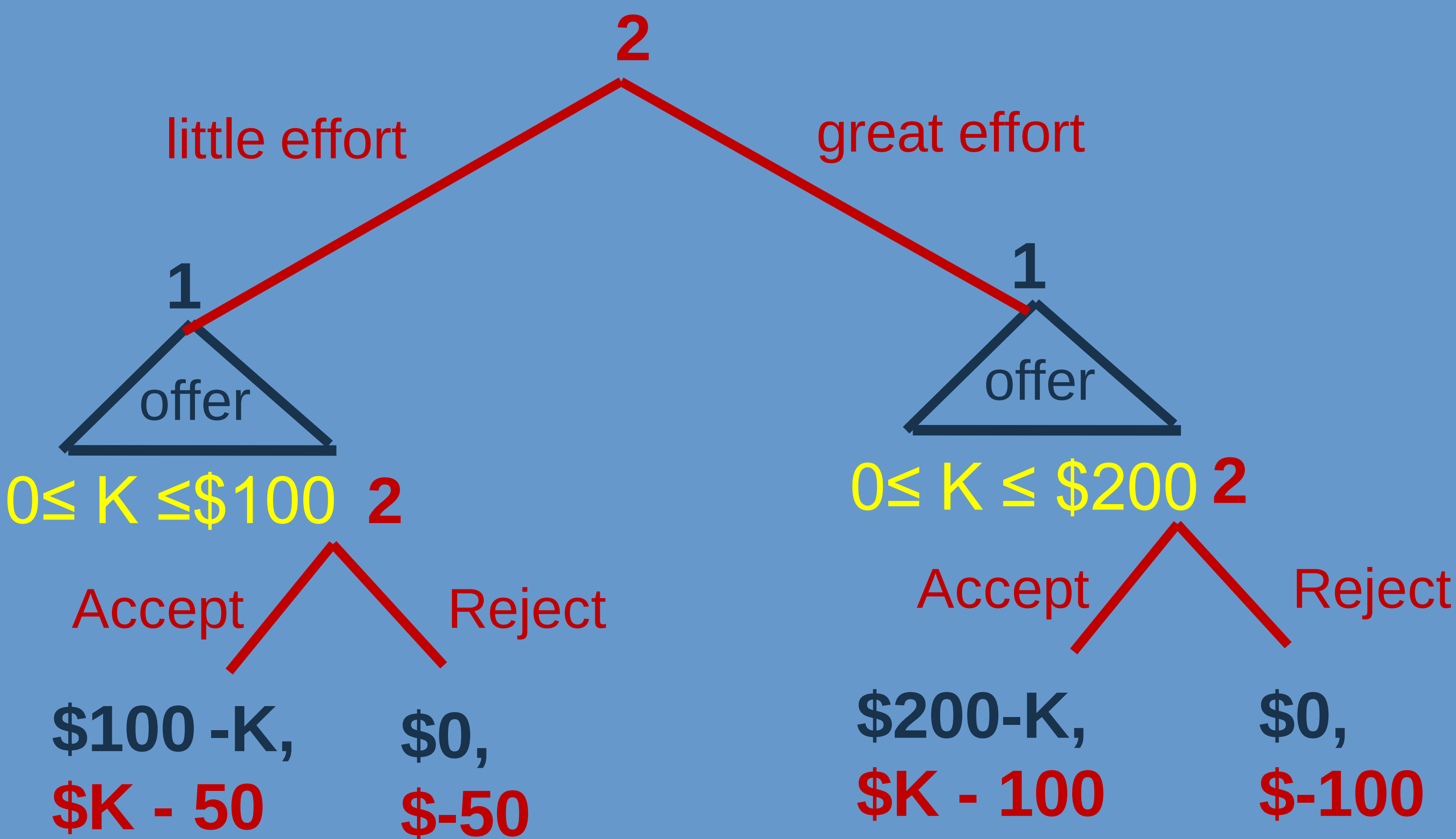
SBNE: little effort, allways offer K=0 (C_L), always accept (-L)

Holdup game



2
little effort
great effort
1
offer
1
offer
0≤ K ≤$10
2
0≤ K ≤ $1000
2
Accept
Reject
Accept
Reject
$10 -K,
$K - 1
$0,
$-1
$1000-K,
$K - 2
$0,
$-2

Holdup game


2
little effort
great effort
1
1
offer
offer
0≤ K ≤$100
2
0≤ K ≤ $200
2
Accept
Reject
Accept
Reject
$100 -K,
$K - 50
$0,
$-50
$200-K,
$K - 100
$0,
$-100

Synergistic relationship

Consider a variant of the situation in, in which two individuals are involved in a synergistic relationship. If both individuals devote more effort to the relationship, they are both better off. For any given effort of individual *j, the return to individual i's* effort first increases, then decreases.

Suppose that the players choose their effort levels sequentially, rather than simultaneously. First individual 1 chooses her effort level a_1*, then individual 2 chooses* her effort level a_2*. An effort level is a nonnegative number, and individual i's preferences* (for $i = 1, 2$*) are represented by the utility (payoff) function*

$$u_i = a_i(c + a_j - a_i)$$

where j is the other individual and $c > 0$ *is a constant.*

Synergistic relationship

From 2nd lecture we know that the NE of the simultaneous game is (c,c)

- we were deriving best response function and then analyzing the situation when every player's action is best response to the other players' action.

NE: (c,c) $u_1 = c^2$ $u_2 = c^2$

However we have many NE in the current game when we have strategy for first player 1 – choosing effort a_1 and strategy for player 2 – for every possible effort of player 1 choose effort a_2.

Example of NE: $a_1 = c$; $a_2 = c$ if $a_1 = c$; $a_2 = c$ otherwise ;

or: $a_1 = c$; $a_2 = c$ if $a_1 = c$; $a_2 = 0$ otherwise ;

another examples: $a_1 = \frac{3}{4} c$; $a_2 = 7/8c$ if $a_1 = \frac{3}{4} c$;

$a_2 = 0$ otherwise ;

Synergistic relationship

NOW – SBNE

we start with subgame of length 1 and analyze the optimal actions

assume that the first player chose effort a_1:

player 2 is choosing a_2 in such way to maximize his utility:

max $u_2 = a_2(c + a_1 - a_2) = -a_2^2 + a_2(c + a_1)$

$\rightarrow a_2 = \frac{1}{2}(c + a_1)$

Synergistic relationship

we know that given history a_1

player 2 is choosing $a_2 = ½ (c + a_1)$

so in the subgame of length 2 (whole game) player 1 is choosing such strategy a_1 to maximize his utility, given he is aware that the player 2 will play afterwards $a_2 = ½ (c + a_1)$

$$\max u_1 = a_1(c + a_2 - a_1) = a_1(c + ½ (c + a_1) - a_1) =$$

$$\max u_1 = - ½ a_1^2 + a_1 . 3/2c$$

SPNE → $a_1 = 3/2\ c$; $a_2 = ½ (c + 3/2\ c) = 5/4\ c$

$u_1 = 9/8\ c^2$ $u_2 = 25/16\ c^2$

Synergistic relationship

Similar to comparison of Cournot model of duopoly and Stackelberg model of duopoly

However the leader in Stackelberg (SBNE) when playing first produce more and get higher profit than in Cournot (NE) and the second firm produce less and get less profit than in NE.

(if we have classic downward sloping reaction curves)

Here – the leader has to exert higher effort and get lower profit than the second player, however, both of them are bettor of compared to simultaneous decision

NE: (c,c) $u_1 = c^2$ $u_2 = c^2$

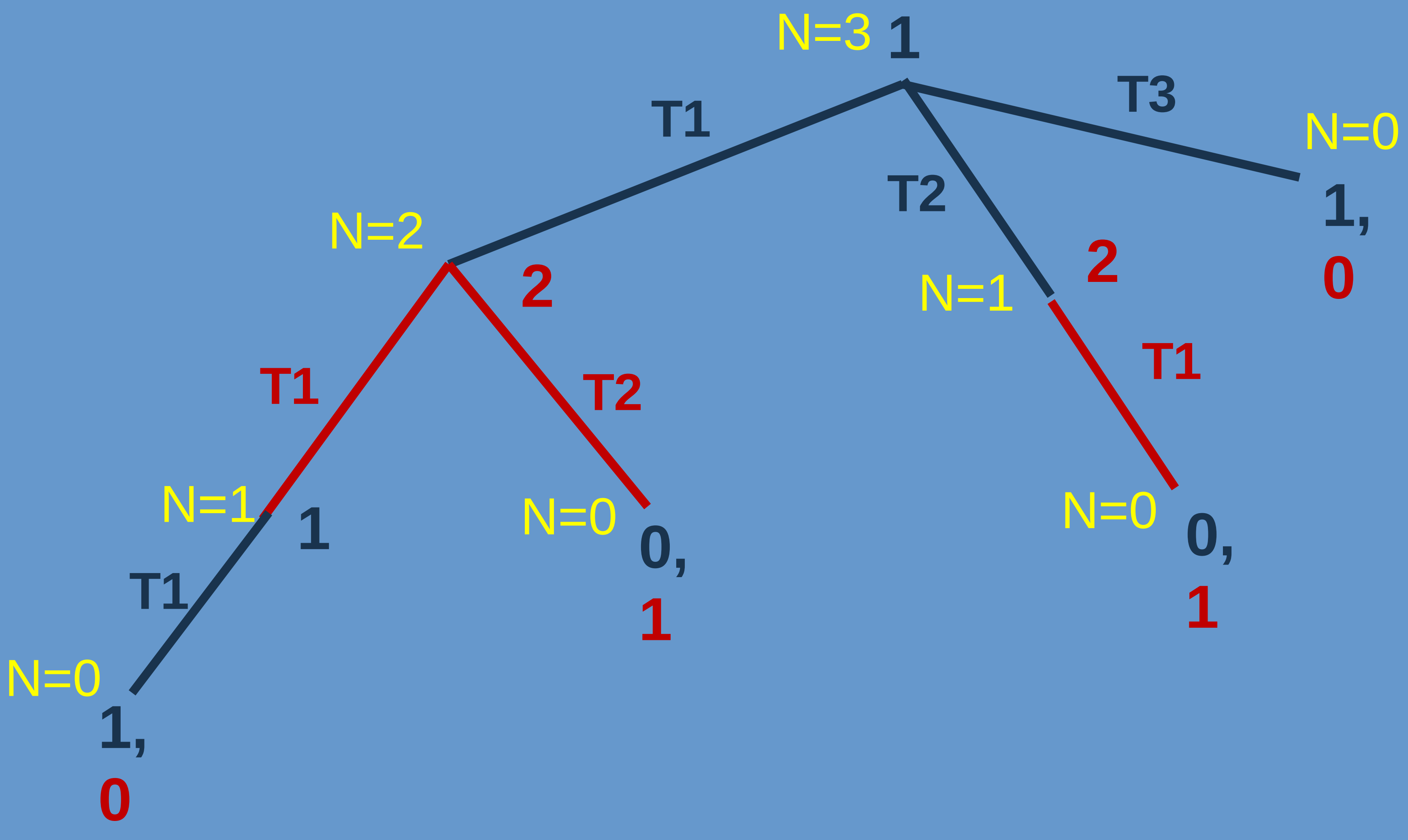
SPNE → $a_1 = 3/2\, c$; $a_2 = ½\,(c + 3/2\, c) = 5/4\, c$

$u_1 = 9/8\, c^2$ $u_2 = 25/16\, c^2$

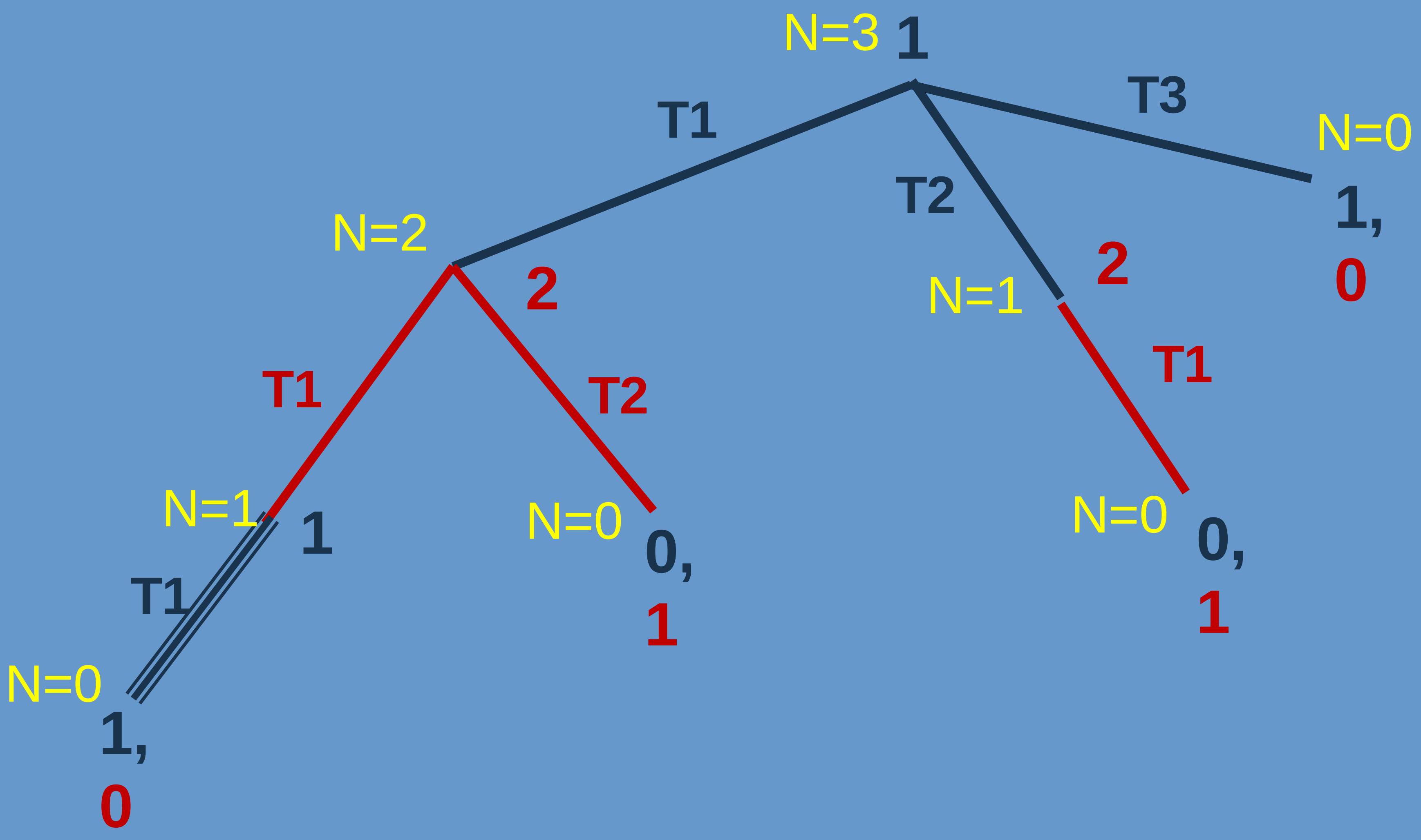
Removing stones

Two people take turns removing stones from a pile of n stones. Each person may, on each of her turns, remove either one, two or three stones. The person who takes the last stone is the winner; she gets $1 from her opponent. Find the subgame perfect equilibria of the games that model this situation for n = 1, 2, …. Find the winner in each subgame perfect of n= 1, 2, 3 and use the same technique to find the winner in each subgame perfect equilibrium for n = 4, and, if you can, for an arbitrary value of n.

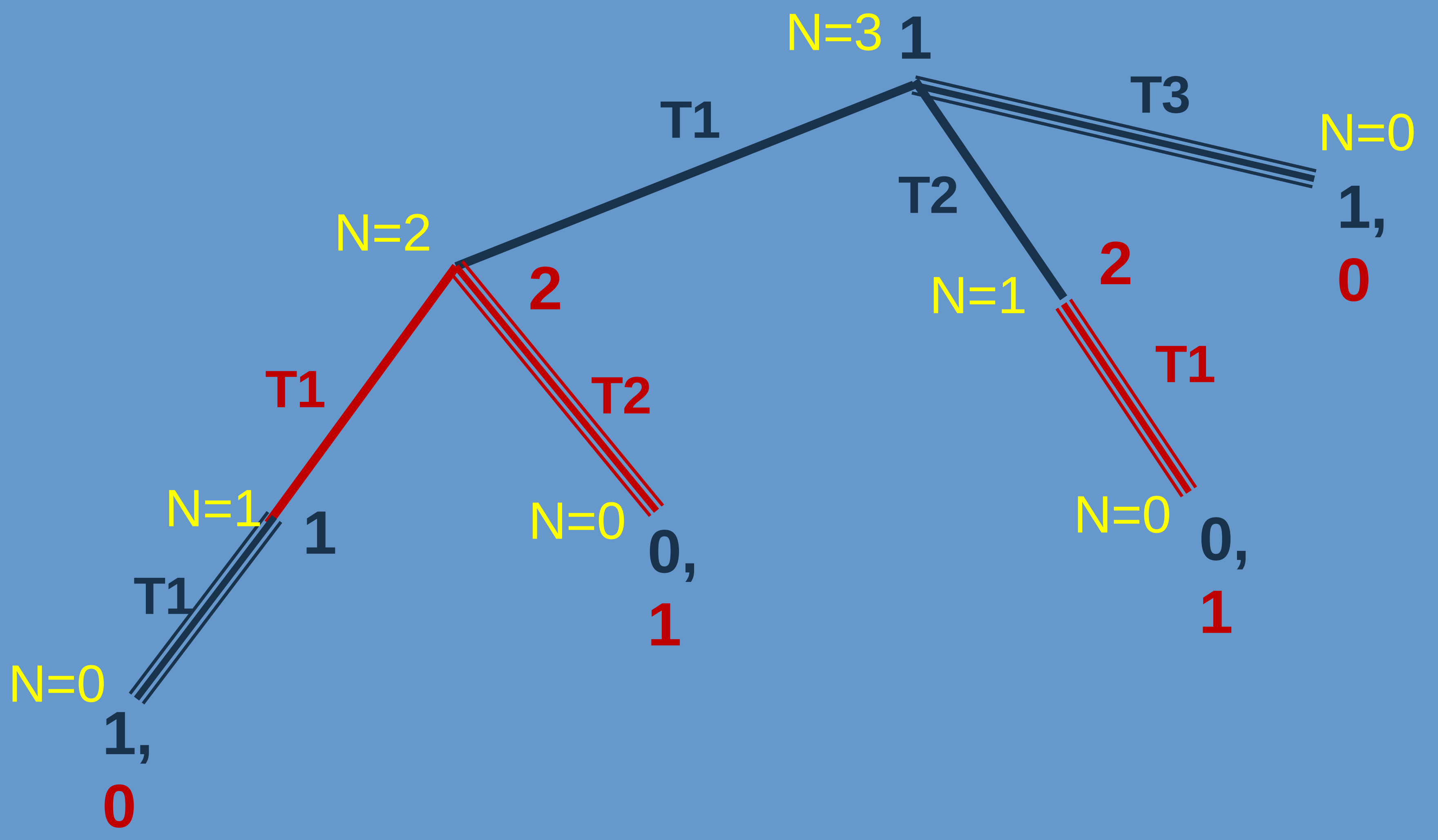
Removing stones



Removing stones

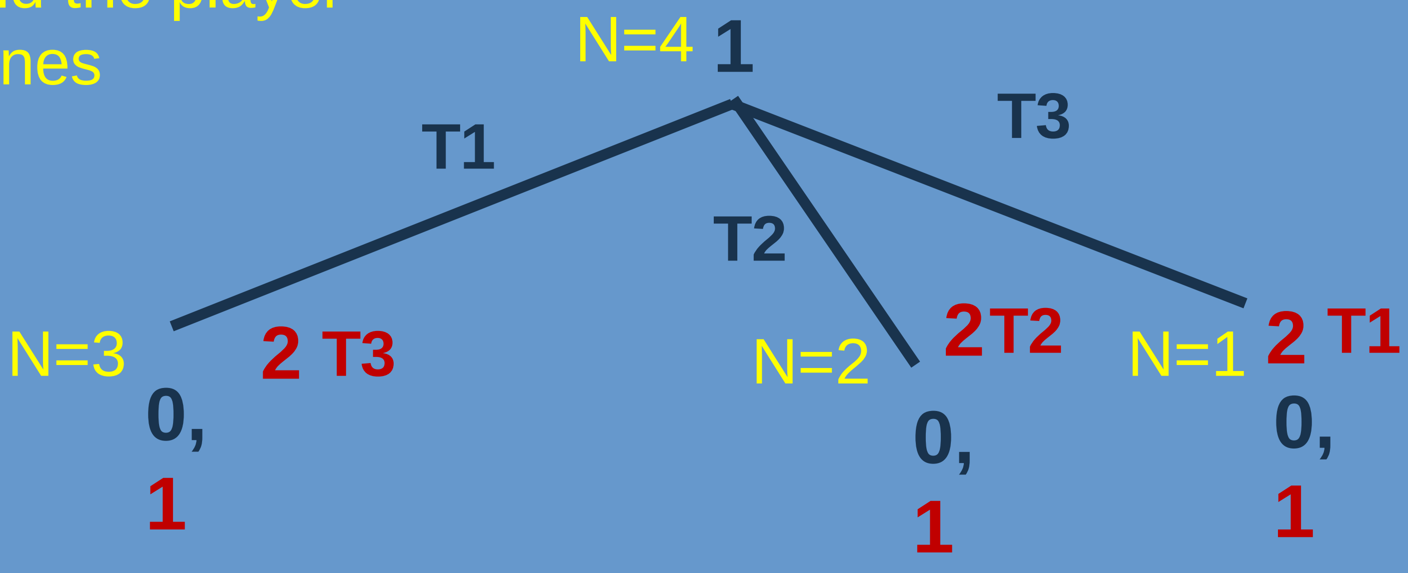


Removing stones



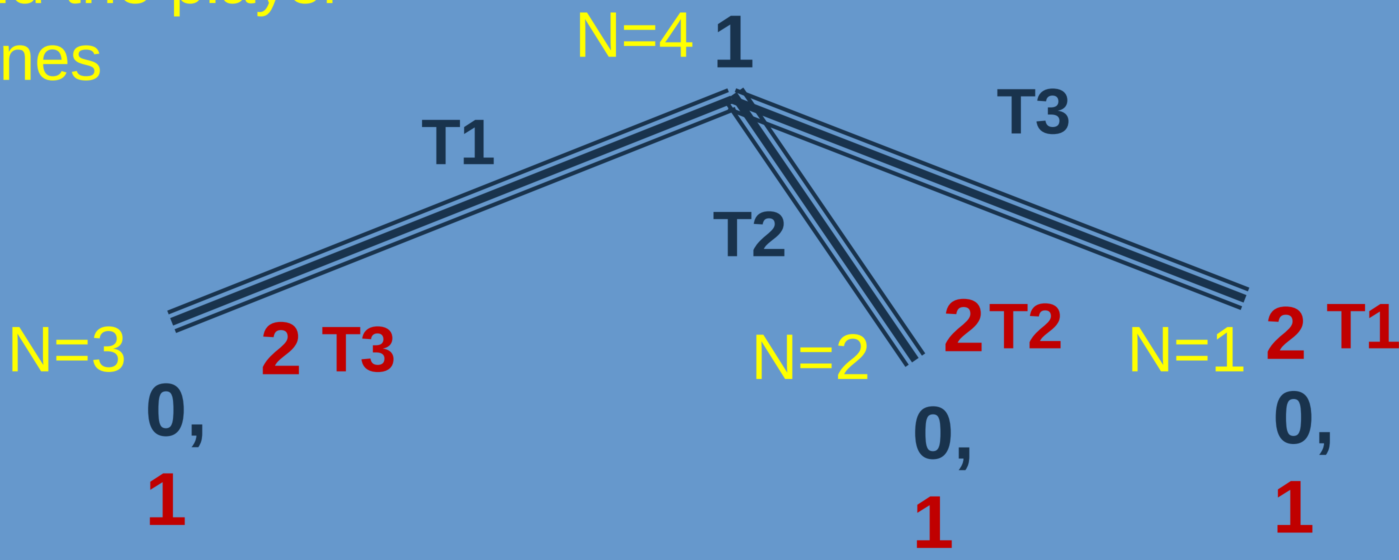
Removing stones

If N=1,2 or 3 and the player is removing stones he will win



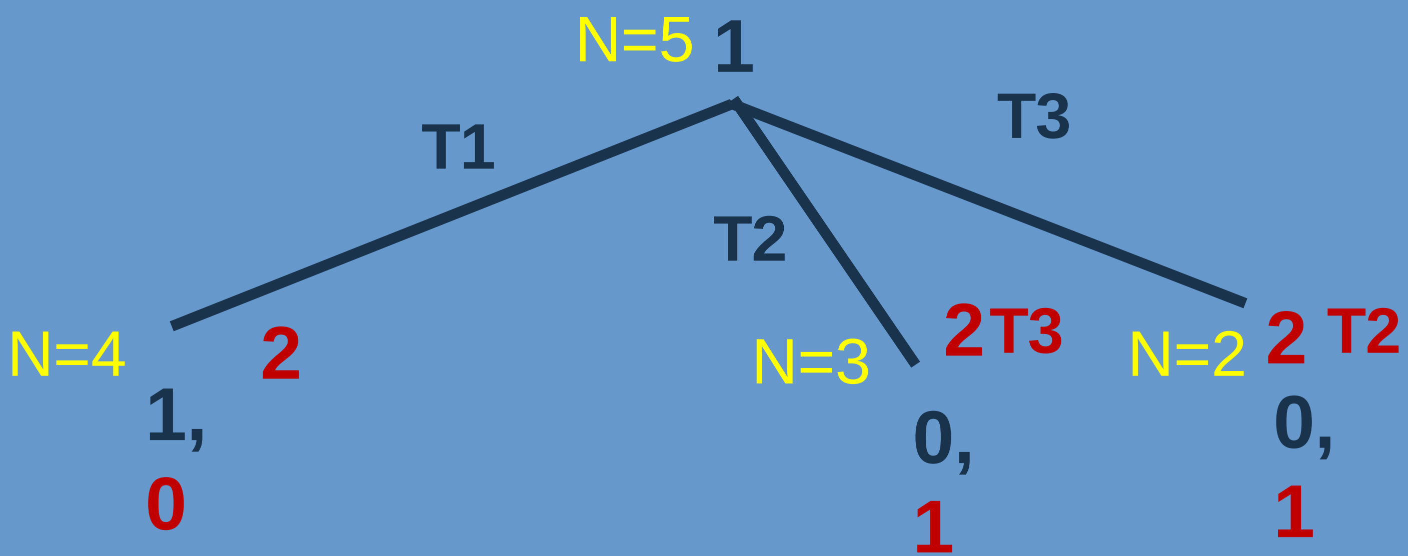
Removing stones

If N=1,2 or 3 and the player
is removing stones
he will win

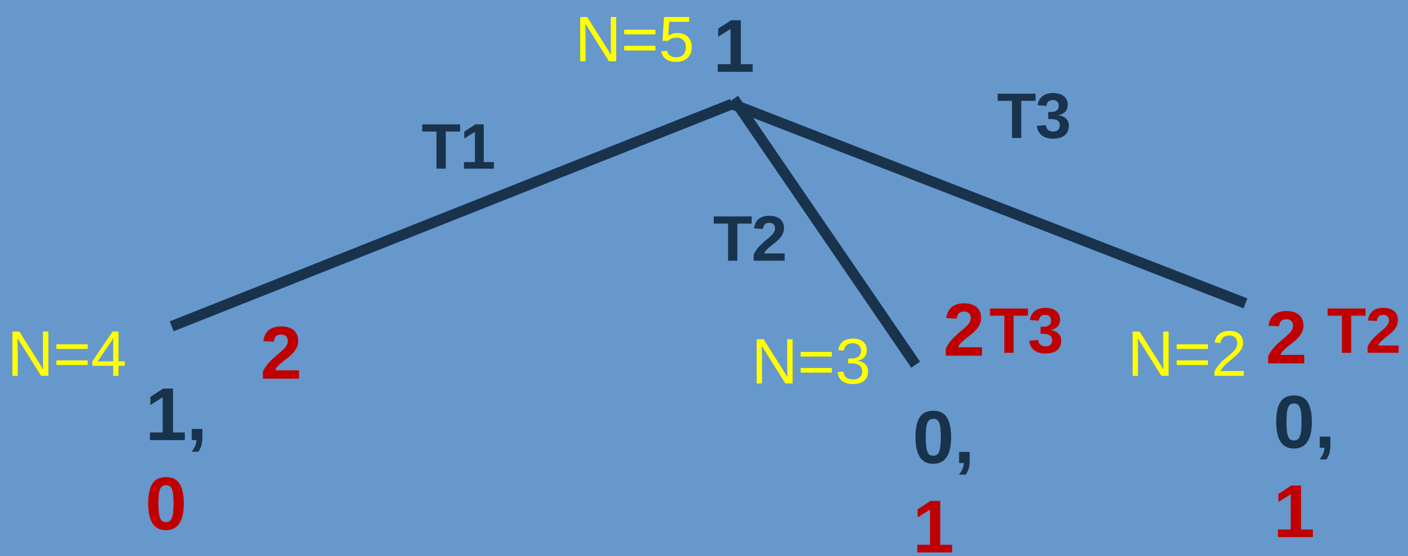


If N=4 and the player is removing stones he will lose

Removing stones



Removing stones



If N=5 and the player is removing stones he will lose

→ N=4 lose
N=5,6,7 win
N=8 lose

N=9,10,11 win
N=12 lose
N=13,14,15 win

Removing stones

If we will continue we will see that if the player 1 is on the move and N=4k+C, C=1,2,3 he will win

In the first move he will take C and every other move he will take such move that P2+P1=4 where, P2 represent the number of stones taken by player 2, P1 by player 1. Therefore, he will force player 2 to take action when N=4 and thus player 2 will lose.

Otherwise if N=4k, he will lose

Player 2 will force player 1 to take action when N=4 and thus player 1 will lose. In every move player 2 will take such number of stones P2 that P2+P1=4 where, P1 represent the number of stones taken by player 1.

Race games

In situations that can be represented as similar games firms compete with each other to develop new technologies; authors compete with each other to write books and film scripts about momentous current events; scientists compete with each other to make discoveries. In each case the winner enjoys a significant advantage over the losers, and each competitor can, at a cost, increase her pace of activity.

Race games

Simple example:

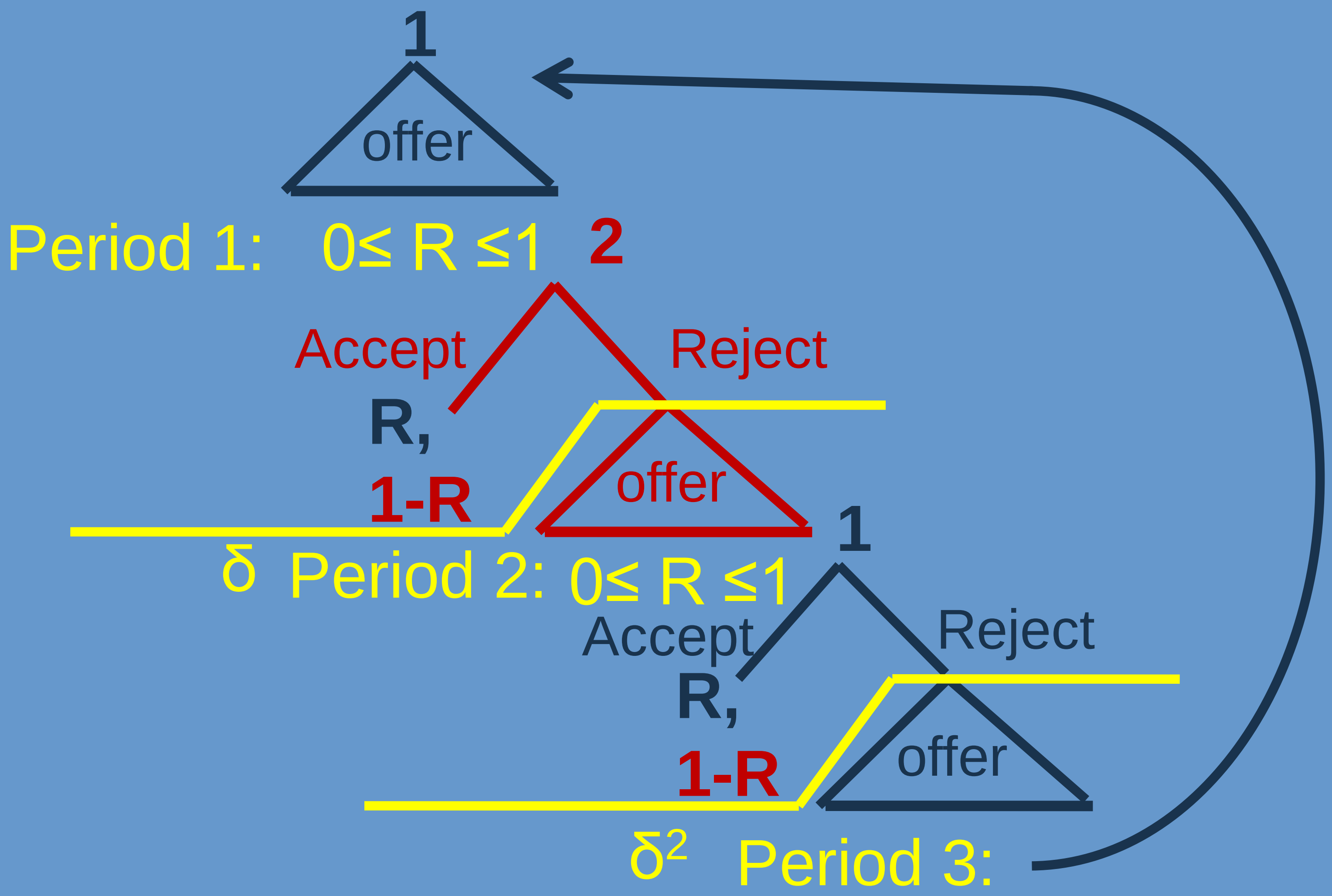
Player i is initially $k_i > 0$ steps from the finish line, for i = 1, 2. On each of her turns, a player can either not take any steps (at a cost of 0), or can take one step, at a cost of c(1), or two steps, at a cost of c(2). The first player to reach the finish line wins a prize, worth $v_i > 0$ to player i; the losing player's payoff is 0. To make the game finite, I assume that if, on successive turns, neither player takes any step, the game ends and neither player obtains the prize. I denote the game in which player i moves first by $G_i(k_1, k_2)$.

Sequential bargaining

Players 1 and 2 are bargaining over one dollar over infinite number of periods. They alternate in making offers: first player 1 makes a proposal that player 2 can accept or reject; if 2 rejects then in second period 2 makes a proposal that 1 can accept or reject; if player 1 rejects then he is again making offer and so on...

Once an offer has been rejected, it ceases to be binding and is irrelevant to the subsequent play of the game. Each offer takes one period and players are impatient: they discount payoffs received in later periods by the factor δ per period, where $0<\delta<1$.

Sequential bargaining



Sequential bargaining

1

Period 1: R=1-δ (1-δK)

2

Accept

1-δ (1-δK),
δ (1-δK)

From previous lecture we know that in 3 periods model with payoffs K and 1-K in the third period, the game has SBNE:

offer δ (1-δK) to player 2,
accept δ (1-δK) or more, reject less,
offer δK to player 1,
accept δK or more, reject less

Sequential bargaining

Period 1: **1 offer 2 decides**

↓

Period 2: δ **2 offer 1 decides**

↓

Period 3: δ^2 **1 offer 2 decides**

↓

Period 4: δ^3 **2 offer 1 decides**

↓

Period 5: δ^4 **1 offer 2 decides**

↓

Period 6: δ^5 **2 offer 1 decides**

↓

Strategies:

Player 1:

$(S_{11}, S_{12}, S_{13}, S_{14}, \ldots)$

S_{1n} = offers K if n odd

S_{1n} = A or R if n even

Player 2:

$(S_{11}, S_{12}, S_{13}, S_{14}, \ldots)$

S_{1n} = A or R if n odd

S_{1n} = offers L if n even

If any of the players accept in period T, it yields payoffs $(\delta^{T-1}K, \delta^{T-1}L)$

If they never agree, they get (0,0)

Sequential bargaining

Period 1: **1 offer 2 decides**

↓

Period 2: δ **2 offer 1 decides**

↓

Period 3: δ^2 **1 offer 2 decides**

↓

Period 4: δ^3 **2 offer 1 decides**

↓

Period 5: δ^4 **1 offer 2 decides**

↓

Period 6: δ^5 **2 offer 1 decides**

↓

Nash equilibria:

Player 1:

S_{1n} = always offers division (K,L) if n odd

S_{1n} = accepts $X \geq K$, rejects all other offers if n even

Player 2:

S_{1n} = accepts $Y \geq L$, rejects all other offers if n odd

S_{1n} = always offers L if n even

NOT SBNE:

In T=1 player 2 should accept $\delta L \leq Y \leq L$, in T=2 will get L…

Sequential bargaining

Period 1: **1 offer 2 decides**

Period 2: δ **2 offer 1 decides**

Period 3: δ^2 **1 offer 2 decides**

Period 4: δ^3 **2 offer 1 decides**

Period 5: δ^4 **1 offer 2 decides**

Period 6: δ^5 **2 offer 1 decides**

Subgame perfect NE:

Subgames T=1,3,…,2k+1 have same SBNE

Subgames T=2,4,…,2k have same SBNE

Sequential bargaining

Period 1: **1 offer 2 decides**

↓

Period 2: δ **2 offer 1 decides**

↓

Period 3: δ^2 **1 offer 2 decides**

↓

Period 4: δ^3 **2 offer 1 decides**

↓

Period 5: δ^4 **1 offer 2 decides**

↓

Period 6: δ^5 **2 offer 1 decides**

↓

Let

S={(K,L); (K,L) is the SBNE equilibrium payoff if T is odd }

further

$$M = \max_{(K,L) \text{ from } S} K \qquad m = \min_{(K,L) \text{ from } S} K$$

Sequential bargaining

Period 1: **1 offer 2 decides**

$1-\delta(1-\delta M)$, $\delta(1-\delta M)$

Maximum possible

Period 2: δ **2 offer 1 decides**

$1-\delta M$, δM

Minimum player 2 will get as he can offer δM

Period 3: δ^2 **1 offer 2 decides**

M Maximum possible payoff of P1 in T odd

even

But SBNE in T=3 and T=1 is same

$$1-\delta(1-\delta M) = M$$
$$1-\delta = M + \delta^2 M$$
$$1-\delta = (1+\delta^2) M$$
$$1-\delta = (1+\delta)(1-\delta) M$$
$$1/(1+\delta) = M$$

Sequential bargaining

Period 1: **1 offer 2 decides**

$1-\delta(1-\delta m)$, $\delta(1-\delta m)$

minimum possible

Period 2: δ **2 offer 1 decides**

$1-\delta m$, δm

Maximum player 2 will get as he has to offer at least δm

Period 3: δ^2 **1 offer 2 decides**

m ← minimum possible payoff of P1 in T odd

even

But SBNE in T=3 and T=1 is same

$$1-\delta(1-\delta m) = m$$
$$1-\delta = m + \delta^2 m$$
$$1-\delta = (1+\delta^2)\, m$$
$$1-\delta = (1+\delta)(1-\delta)\, m$$
$$1/(1+\delta) = m = M$$

Sequential bargaining

Period 1: **1 offer 2 decides**

$1/(1+\delta)$, $\delta/(1+\delta)$

Period 2: δ **2 offer 1 decides**

$1/(1+\delta)$, $\delta/(1+\delta)$

Period 3: δ^2 **1 offer 2 decides**

$1/(1+\delta)$

Only candidate for SPNE:

offer to the other player $\delta/(1+\delta)$ (yields $1/(1+\delta)$ to the offering player)

accept offers $X \geq \delta/(1+\delta)$

Checking - NE in all subgames:
If I accept $1/(1+\delta)$ in T=3, I should accept at least $\delta/(1+\delta)$ in T=2, the other player should offer at most $\delta/(1+\delta)$ and get $1/(1+\delta)$ in T=2 ….

Sequential bargaining

Period 1: **1 offer 2 decides**

$1/(1+\delta)$, $\delta/(1+\delta)$

Period 2: δ **2 offer 1 decides**

$1/(1+\delta)$, $\delta/(1+\delta)$

Period 3: δ^2 **1 offer 2 decides**

$1/(1+\delta)$

Only candidate for SPNE:

offer to the other player $\delta/(1+\delta)$ (yields $1/(1+\delta)$ to the offering player)

accept offers $X \geq \delta/(1+\delta)$

Predicts quite even division:

$\delta=.9 \rightarrow 1/(1+\delta) = 0.526$

$\delta/(1+\delta) = 0.474$

As $\delta \rightarrow 1$ then $1/(1+\delta) \rightarrow 0.5$

$\delta/(1+\delta) \rightarrow 0.5$

Summary

- Dynamic games
- Backward induction
- Nash equilibrium
- Subgame perfect equilibrium
- Gibbons 2-2.1.D; Osborne 5 and 6

NEXT WEEK:

MIDTERM

MIDTERM – 3.11.2009

!!!! Surnames starting A-N – 14:30 !!!!

!!!! Surnames starting O-Z – 15:15 !!!!

Topics: Static games: actions, action profiles, Iterative elimination of dominated strategies, Nash equilibrium, Mixed strategies, Dominated strategies in mixed strategies, mixed strategy NE, symmetric games and NE

Dynamic games: Backward induction, strategies, NE, SBNE, synergic relationship – NE in static, NE and SBNE in dynamic version, finite sequential bargaining

Will not be in midterm: electoral competition, war of attrition, reporting crime, expert diagnosis, sequential bargaining with infinite number of moves (time periods)