DYNAMIC GAMES

Lecture 5

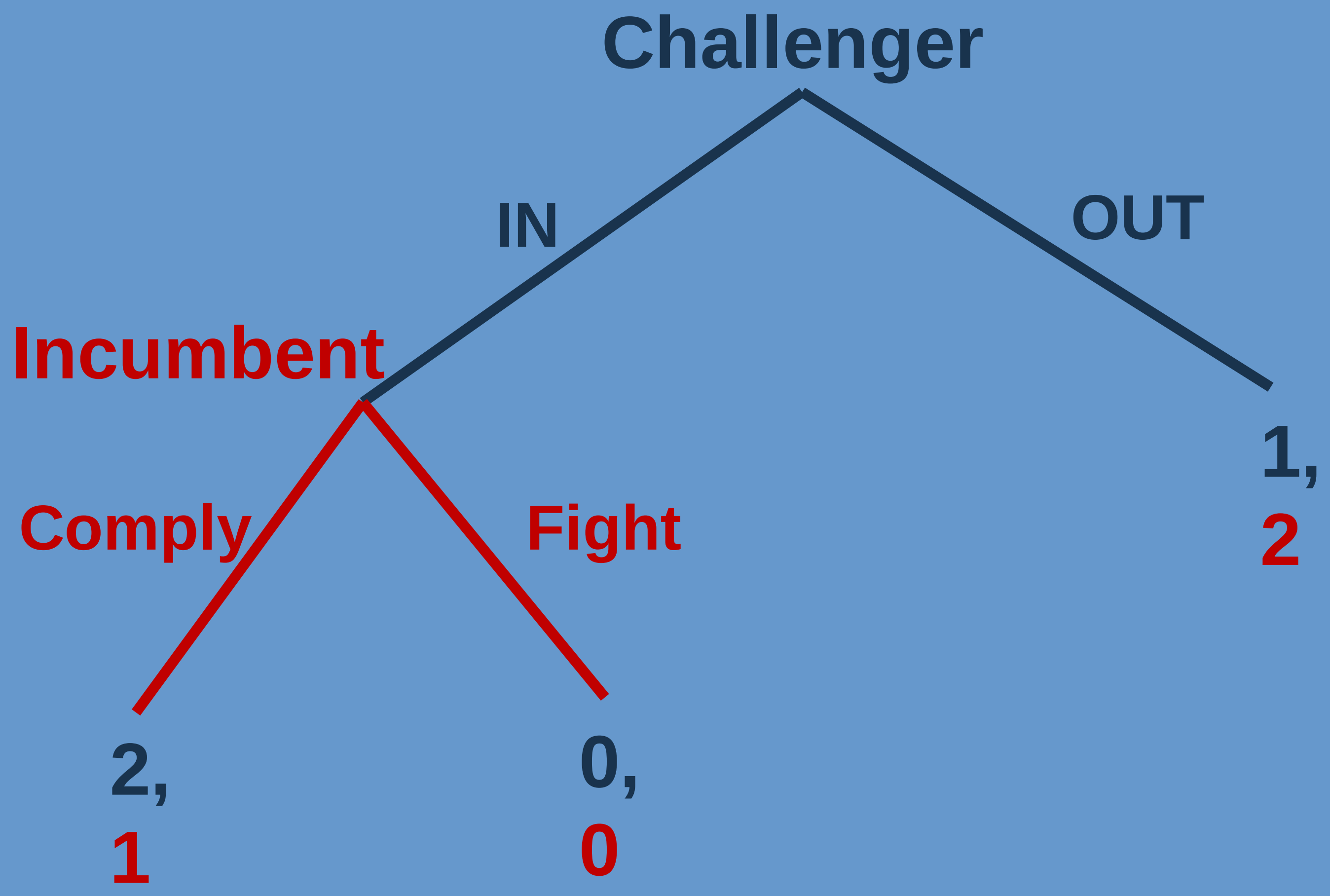
Revision

- Illustrations of NE, MSNE
  - several applications of game theory in real situations
  - will not be part of the midterm exam
- Symmetric games and equilibrium
  - **GAME: if the players' sets** of actions are the same and the players' preferences are represented by the expected values of payoff functions u_1 *and* u_2 *for which* $u_1(a_1, a_2) = u_2(a_2, a_1)$ *for* every action pair (*a1, a2)*
  - A profile α^* of mixed strategies in a strategic game with vNM preferences in which each player has the same set of actions is a **symmetric mixed strategy Nash equilibrium if it is a mixed** strategy Nash equilibrium and α^*_i is the same for every player i

Dynamic games

- In simple dynamic games players choose the actions sequentially one after each other (contrary to the static games where we modeled the decision of players as static – simultaneous)
- EXAMPLE: (Entry game) An incumbent faces the possibility of entry by a challenger. The challenger may enter or not. If it enters, the incumbent may either comply or fight. This game is illustrated in a following diagram.

Dynamic games – extensive form



Dynamic (Extensive) games

- Set of players:
  - Challenger and Incumbent
- Terminal histories:
  - All possible sequences of actions in the game
  - All possible ways how we can get at the ending node in the tree diagram
  - (IN,Comply); (IN, Fight); (OUT)
- Player function
- Preferences for the players

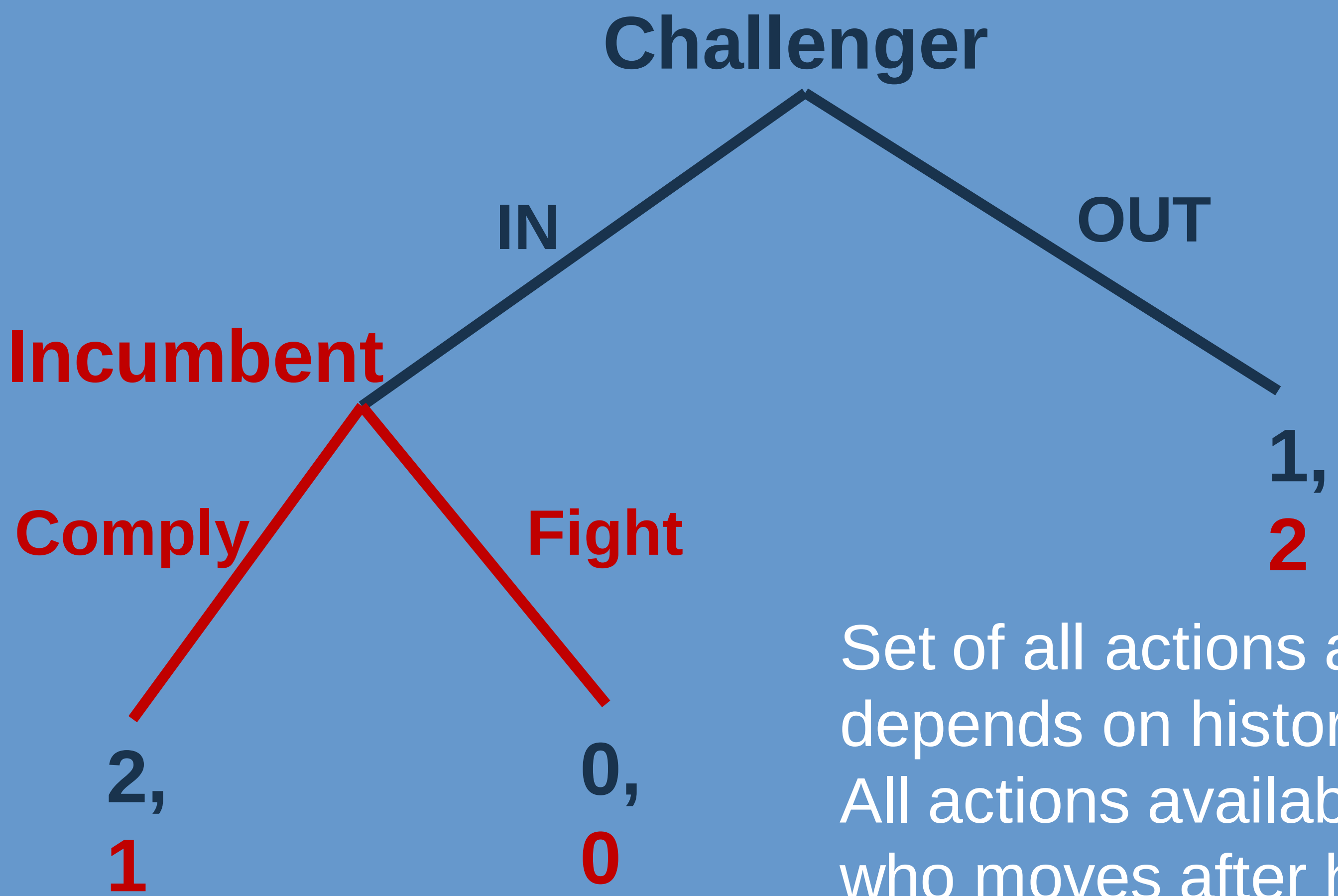
Dynamic (Extensive) games

- Set of players
- Terminal histories
  - proper subhistory (or simply history) of terminal history $(a_1,a_2,\ldots.a_k)$:
  - any sequence $(a_1,a_2,\ldots,a_m)$ such that $m<k$
  - Ø, IN in the case of challenger-incumbent game
- Player function:
  - set a player who takes an action after subhistory h
  - function that assigns a player to every proper subhistory
  - P(Ø) = Challenger ; P(IN) = Incumbent
- Preferences for the players

Dynamic (Extensive) games

- Set of players
- Terminal histories
- Player function
- Preferences for the players:
  - Preferences over terminal histories
  - Preferences over outcomes of terminal histories
  - Again represented by utility (payoff) function
  - challenger: u_1 for which u_1(In, Comply) = 2, u_1(Out) = 1, and u_1(In, Fight) = 0
  - Incumbent: u_2 for which u_2(Out) = 2, u_2(In, Comply) = 1, and u_2(In, Fight) = 0

Dynamic games – extensive form



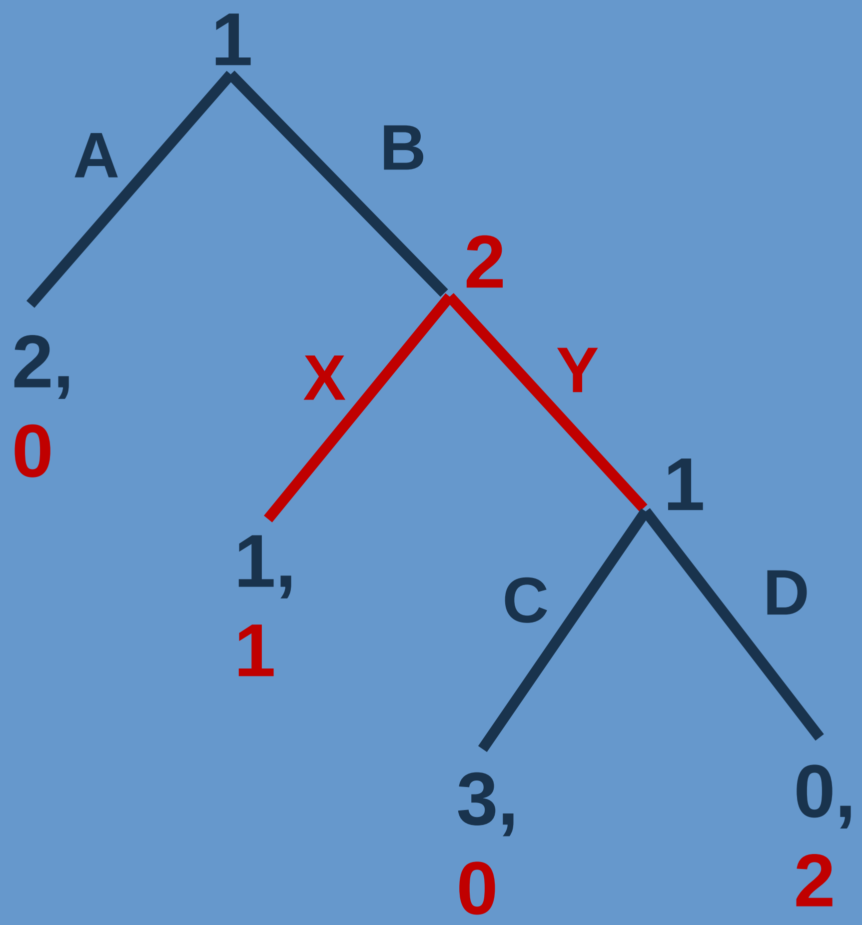
Set of all actions available: depends on history h

All actions available to the player who moves after h:

A(h) = {a: (h, a) is a history}

Dynamic games – extensive form

Set of players: 1 and 2
Terminal histories:
A, BX , BYC, BYD
Player function:
P(Ø)=1, P(B)=2,
P(BY)=1
Preferences
for the players:
1: BYC>A>BX>BYD
2: BYD>BX>A=BYC



Dynamic games – Example

Represent in extensive form diagram the two-player extensive game with perfect information in which the terminal histories are:

(C, E), (C, F), (D, G), and (D, H)

the player function is given by

$P(\emptyset) = 1$ and $P(C) = P(D) = 2$,

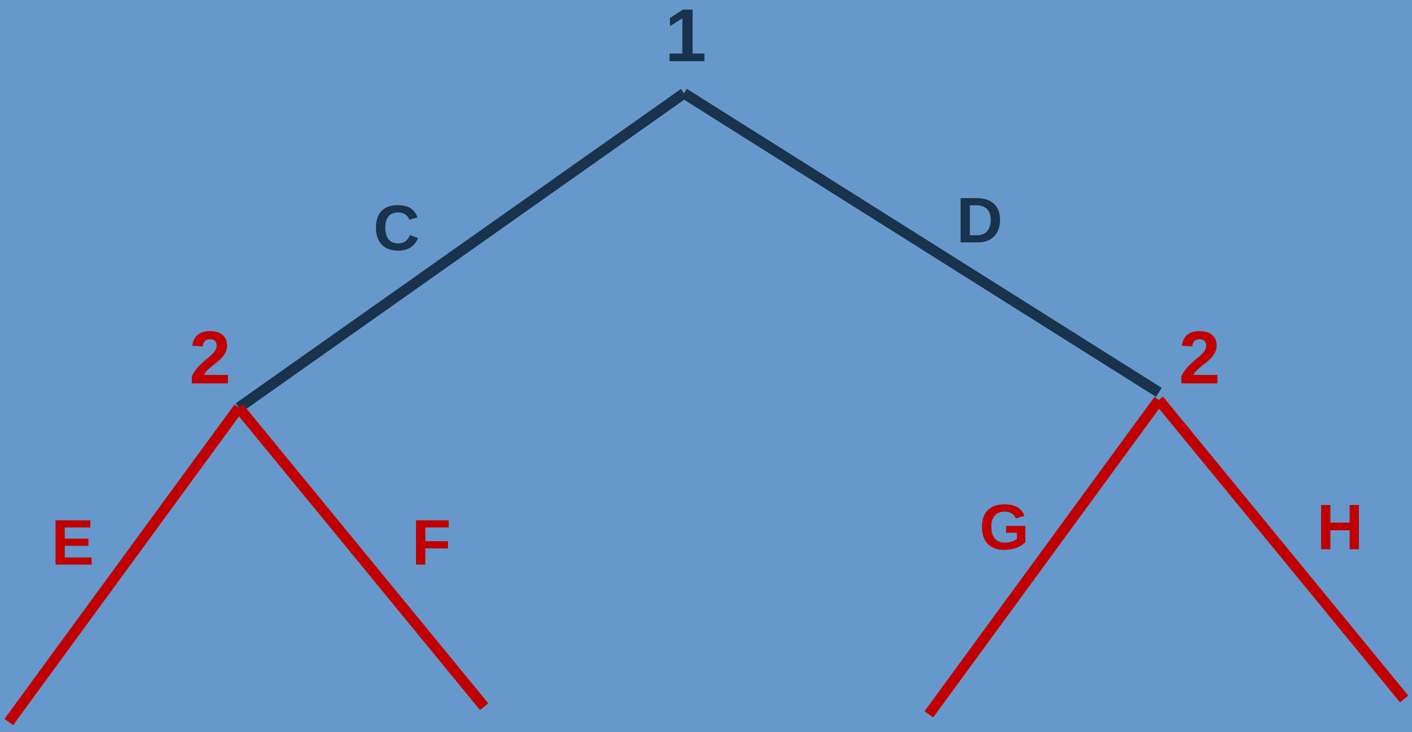
player 1 prefers (C, F) to (D, G) to (C, E) to (D, H)

player 2 prefers (D, G) to (C, F) to (D, H) to (C, E)

Dynamic games – Example

(C, E), (C, F), (D, G), and (D, H)

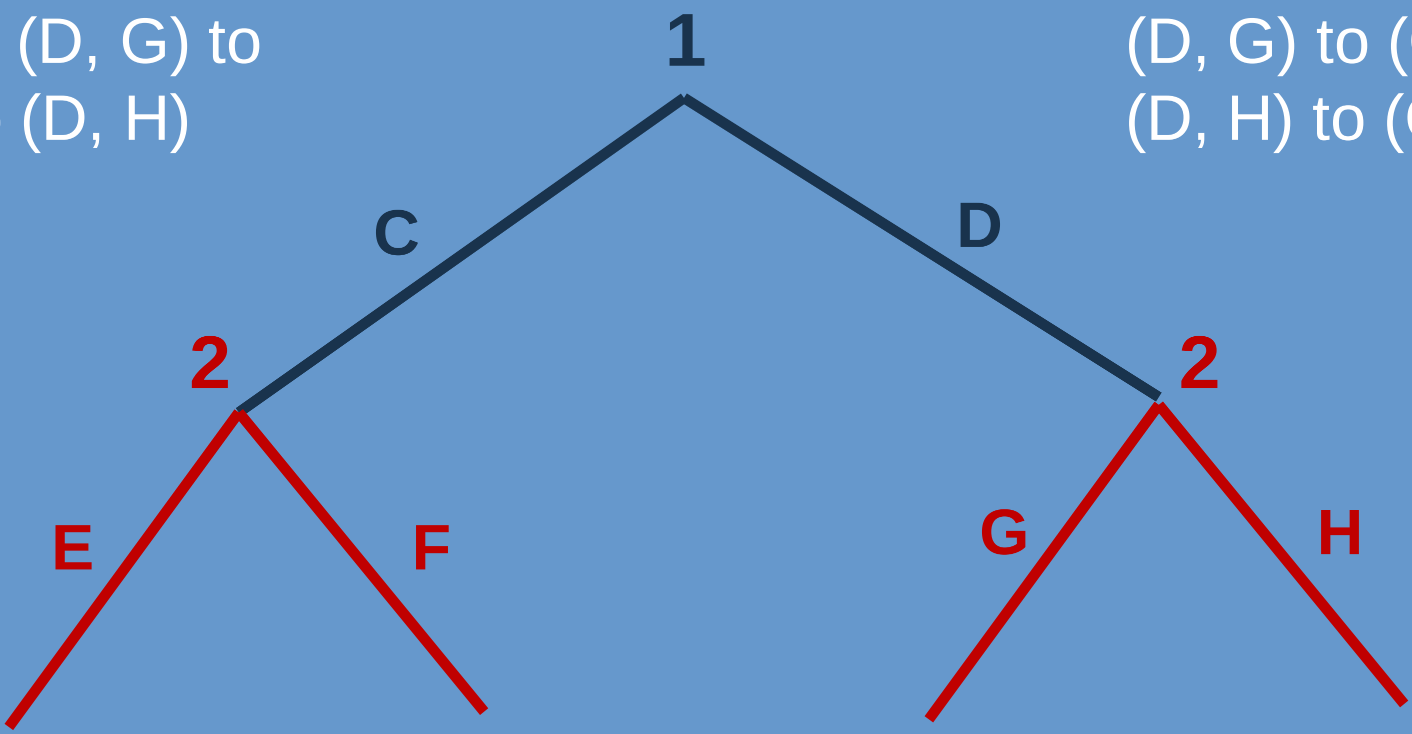
P(Ø) = 1 and P(C) = P(D) = 2



Dynamic games – Example

player 1:
(C, F) to (D, G) to
(C, E) to (D, H)

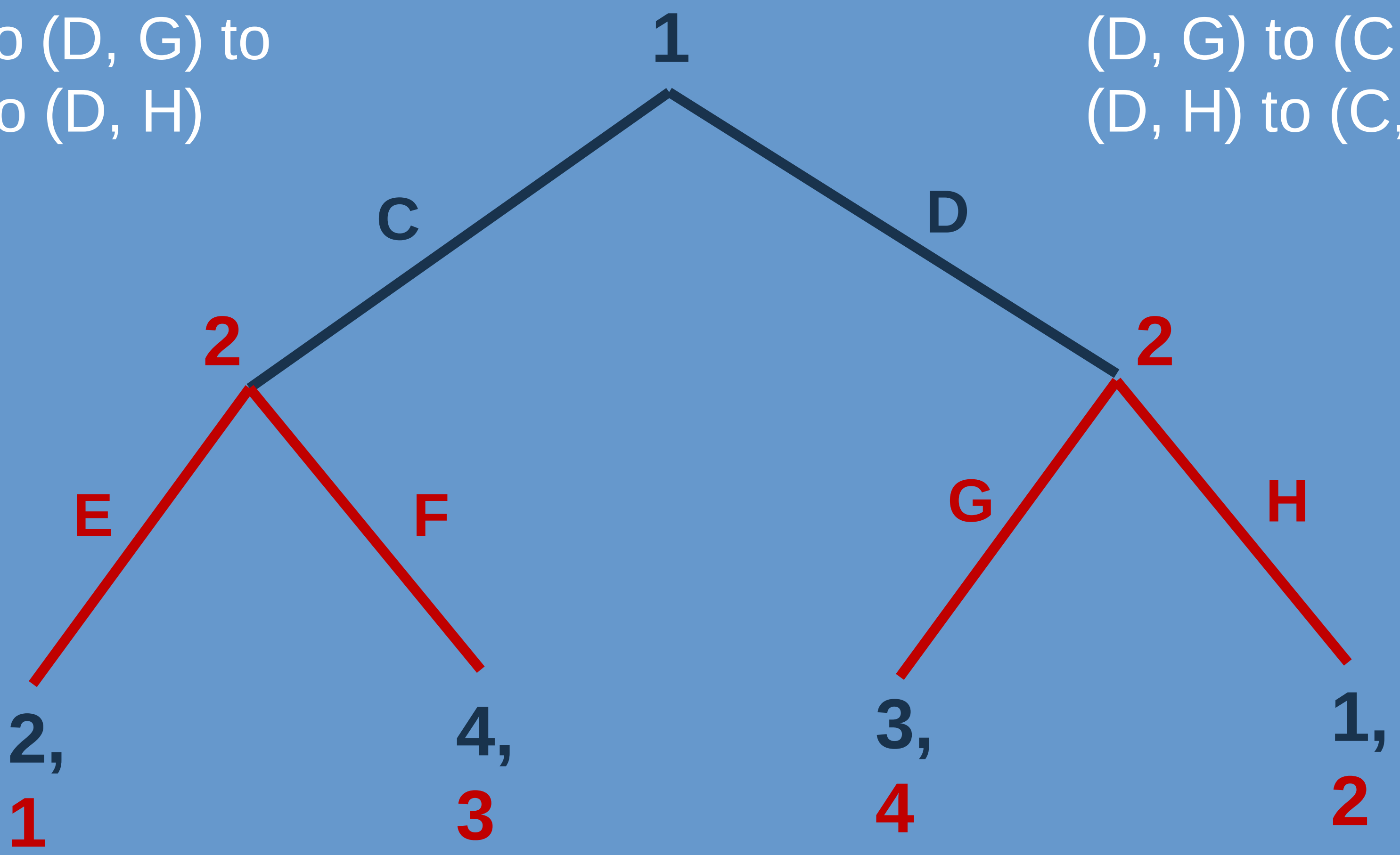
player 2
(D, G) to (C, F) to
(D, H) to (C, E)



Dynamic games – Example

player 1:
(C, F) to (D, G) to
(C, E) to (D, H)

player 2
(D, G) to (C, F) to
(D, H) to (C, E)

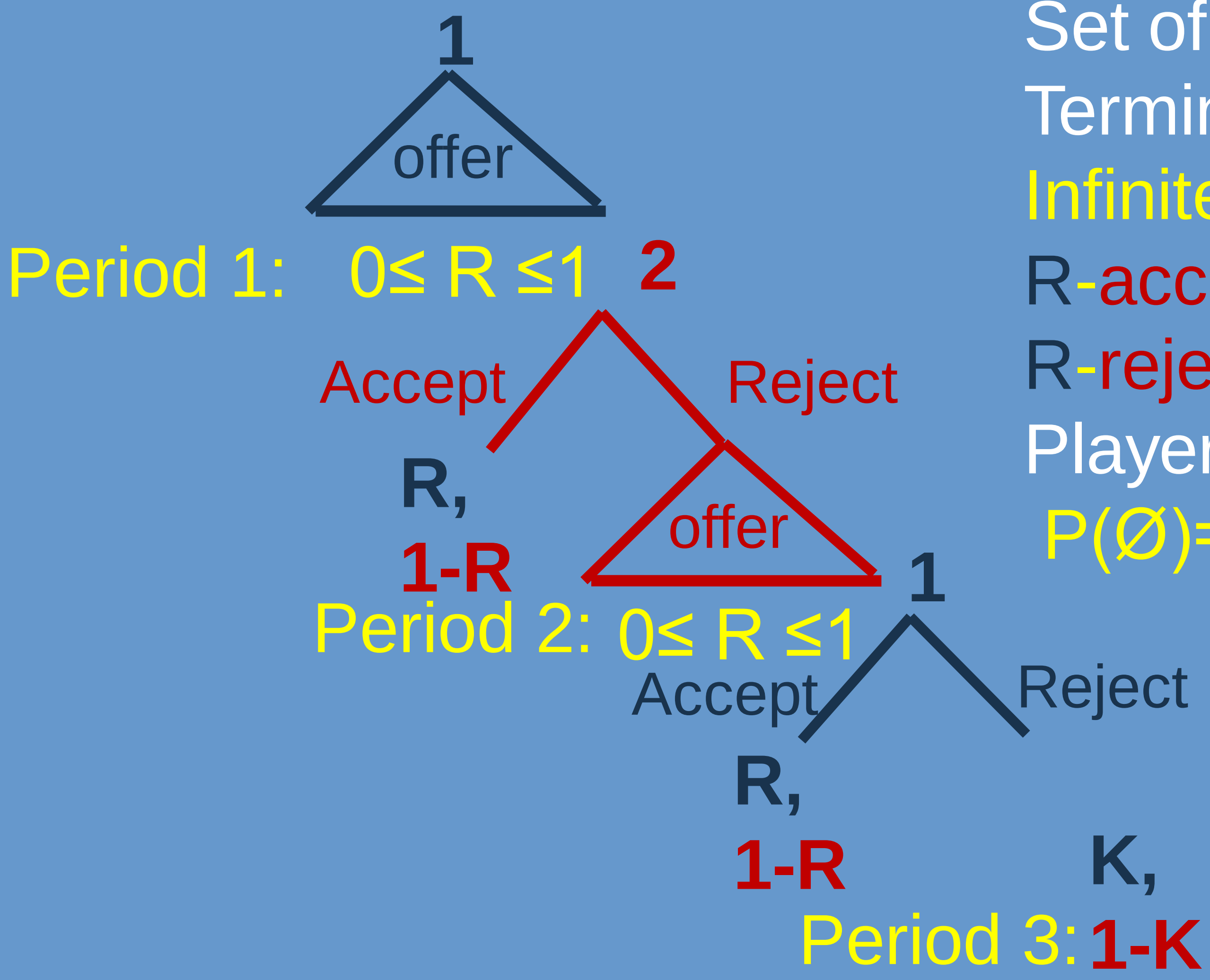


Dynamic games - Example

Players 1 and 2 are bargaining over one dollar over 3 periods. They alternate in making offers: first player 1 makes a proposal that player 2 can accept or reject; if 2 rejects then in second period 2 makes a proposal that 1 can accept or reject; if player 1 rejects then he receives K in third period and player 2 receives 1-K.

Once an offer has been rejected, it ceases to be binding and is irrelevant to the subsequent play of the game. Each offer takes one period and players are impatient: they discount payoffs received in later periods by the factor δ per period, where $0< \delta<1$.

Dynamic games - Example



Set of players: 1 and 2

Terminal histories:

Infinitely many

R-accept,

R-reject-R – accept, etc.

Player function:

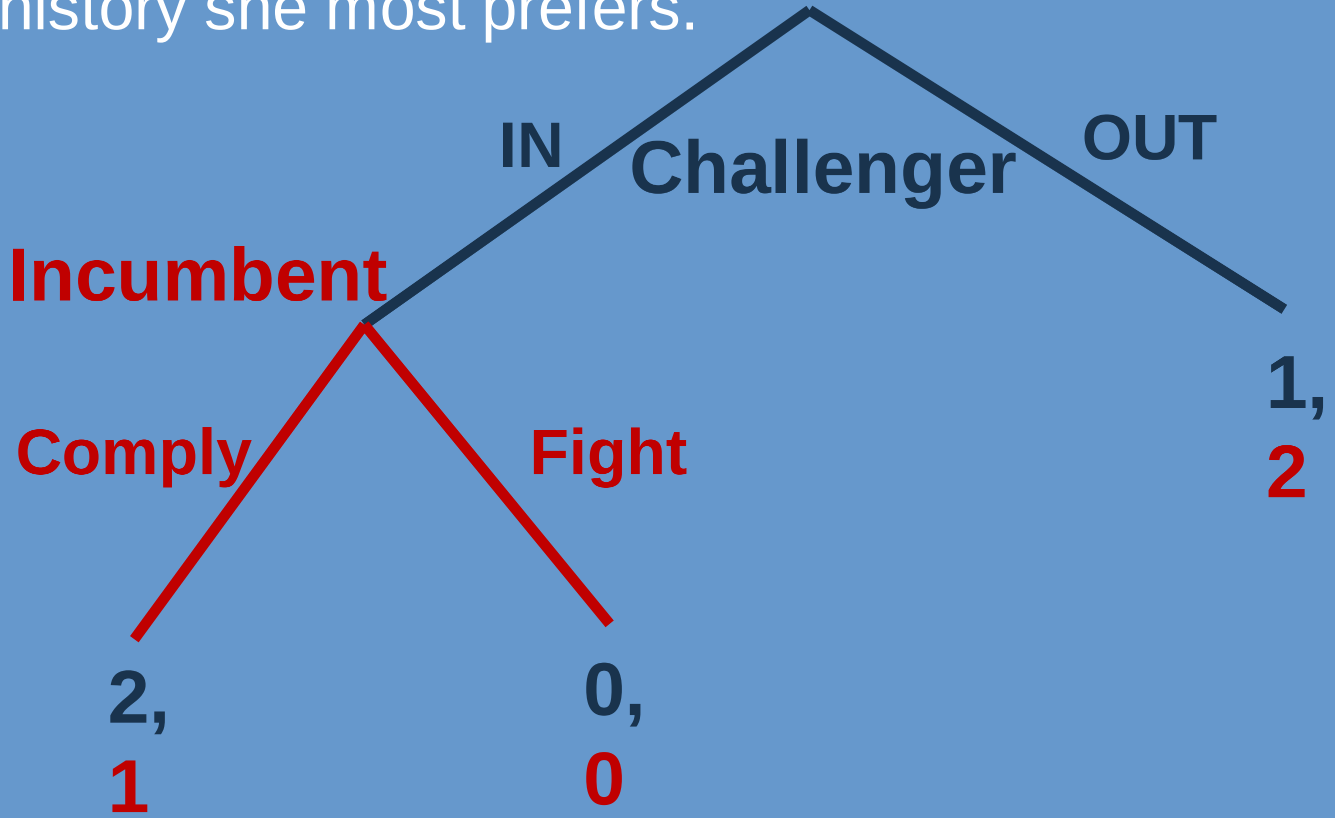
P(Ø)=1, P(R)=2,

Backward induction

- Common knowledge – all players are rational
- Players know that all the players are rational and therefore they may anticipate the moves of the other players as they know that they are rational
- Whenever a player has to move, she deduces, for each of her possible actions, the actions that the players (including herself) will subsequently rationally take, and chooses the action that yields the terminal history she most prefers.

Backward induction

Whenever a player has to move, she deduces, for each of her possible actions, the actions that the players (including herself) will subsequently rationally take, and chooses the action that yields the terminal history she most prefers.



Backward induction

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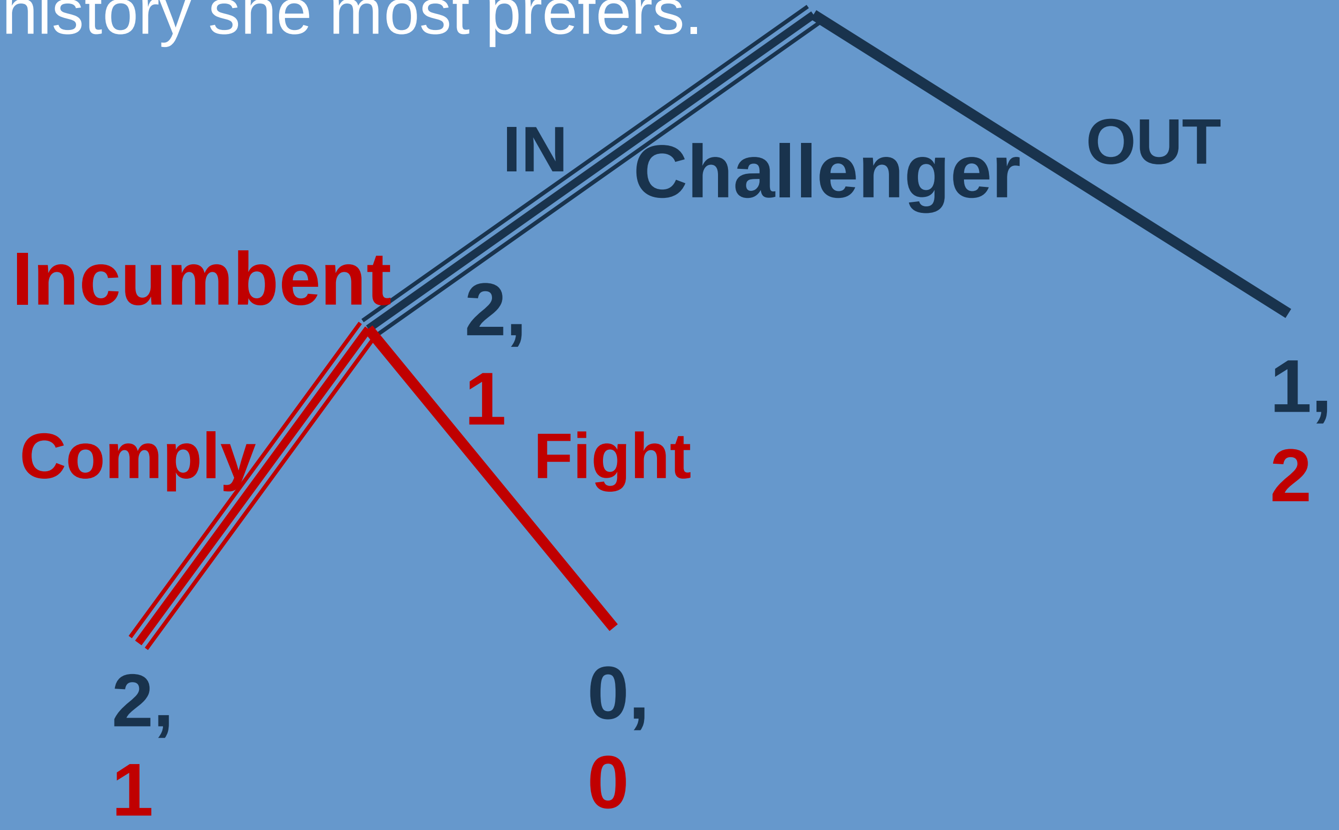
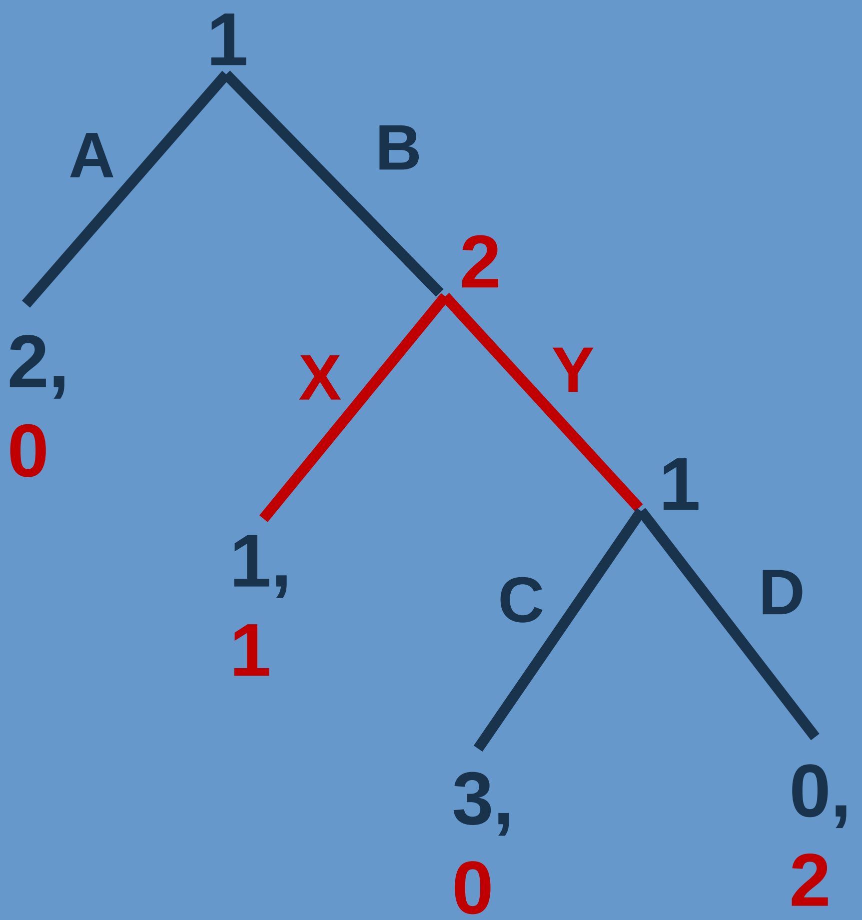


Backward induction

We are starting solving game from the latest node assuming that the last player is rational



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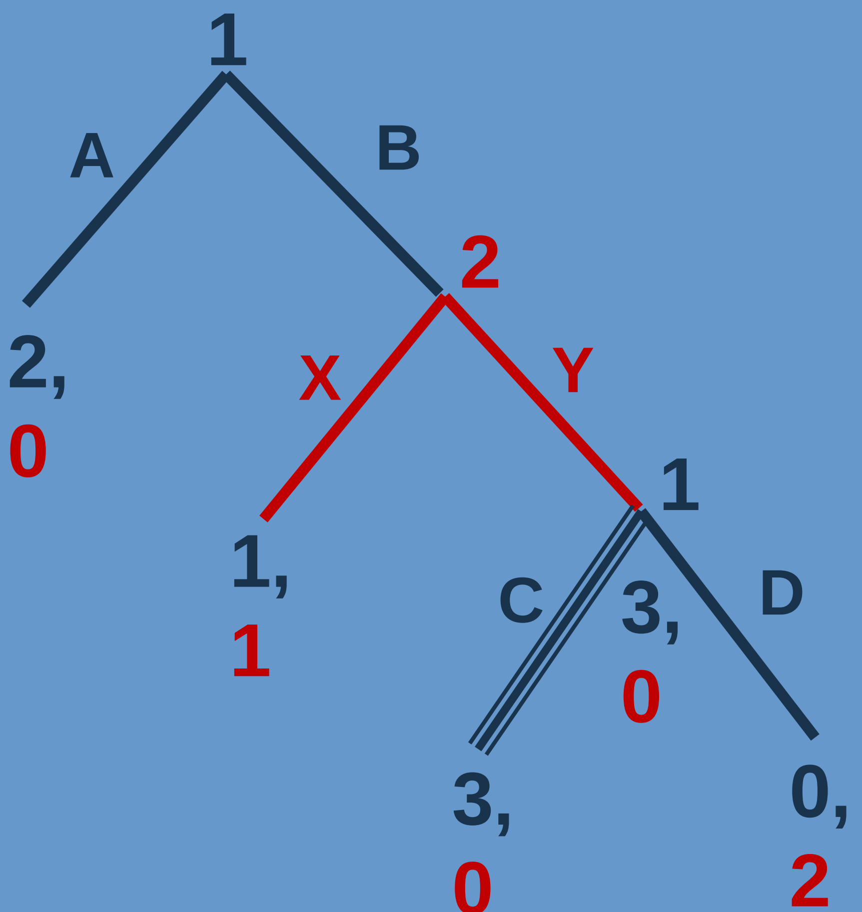
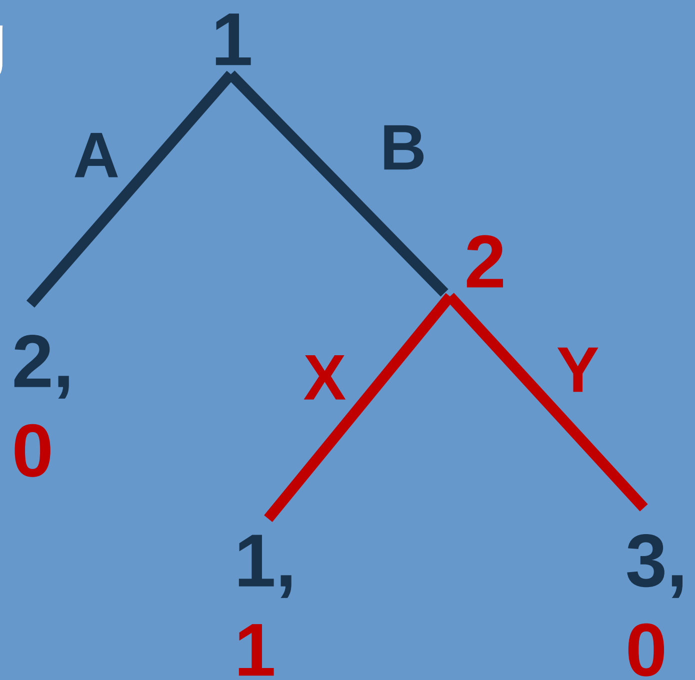


Backward induction

We are starting solving game from the latest node and continue backwards



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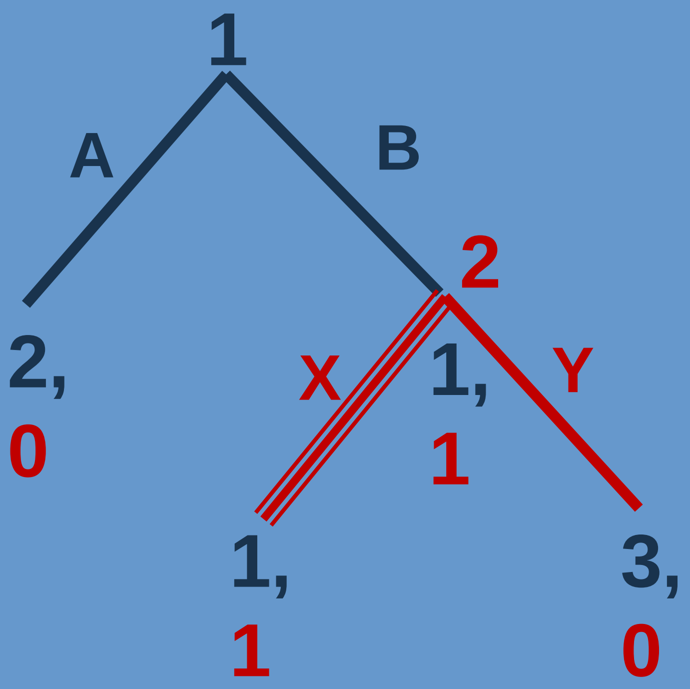

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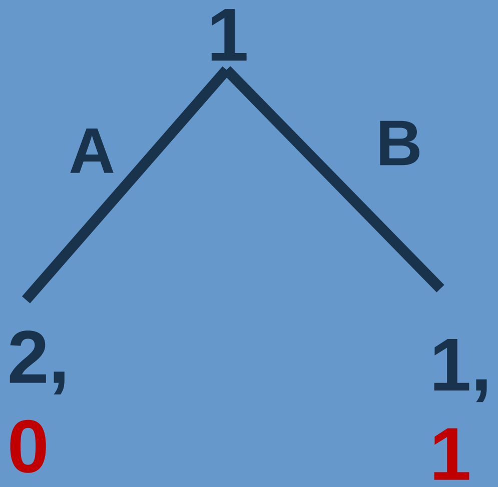

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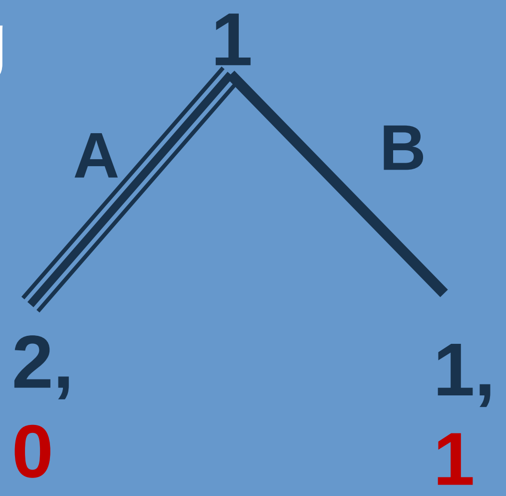
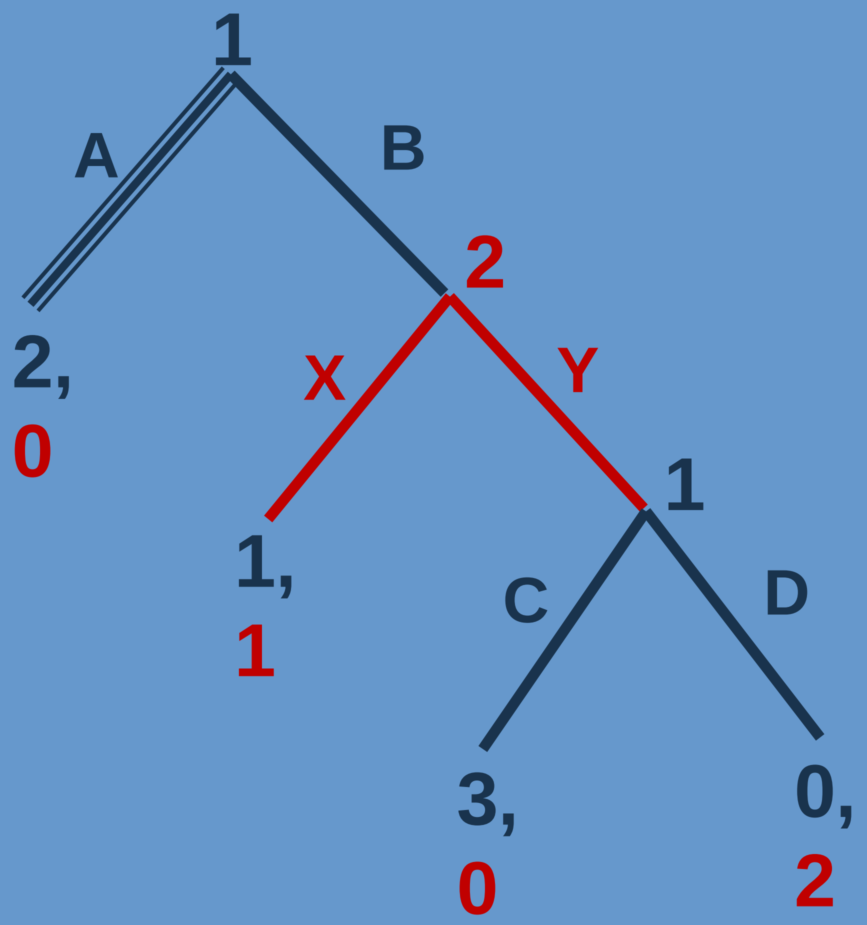
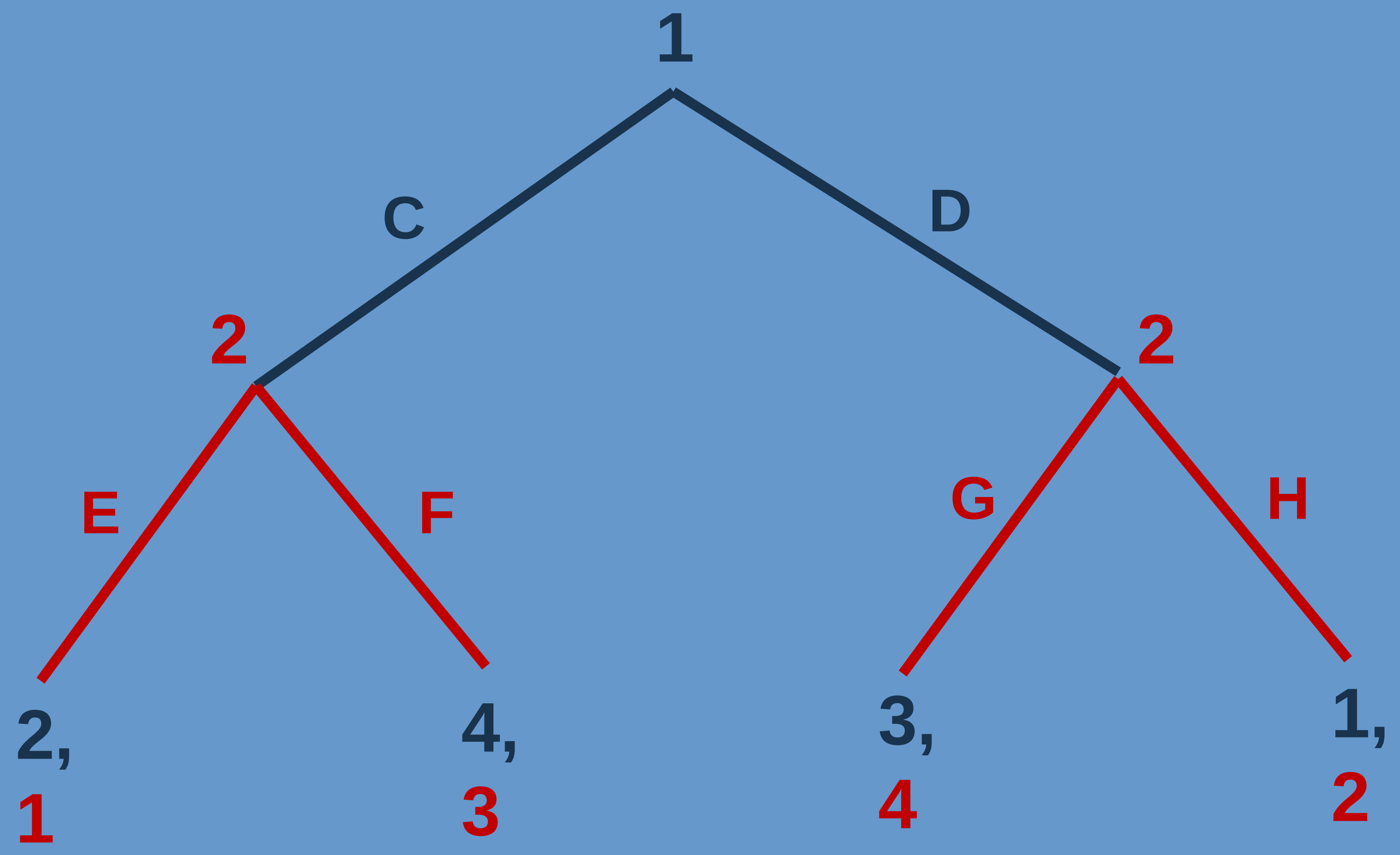


Backward induction

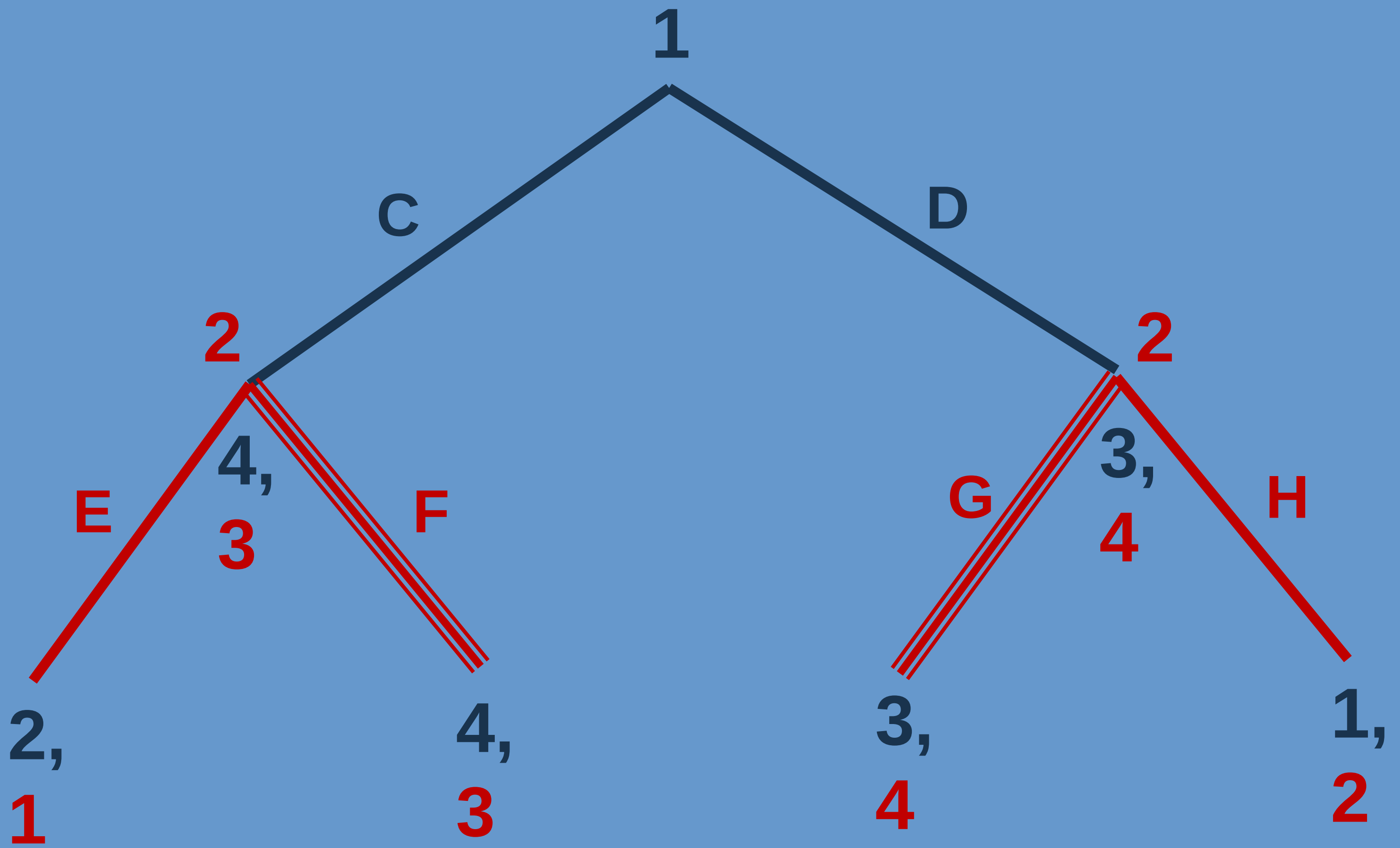
backward-induction outcome is player 1 choosing A and ending the game in the first stage



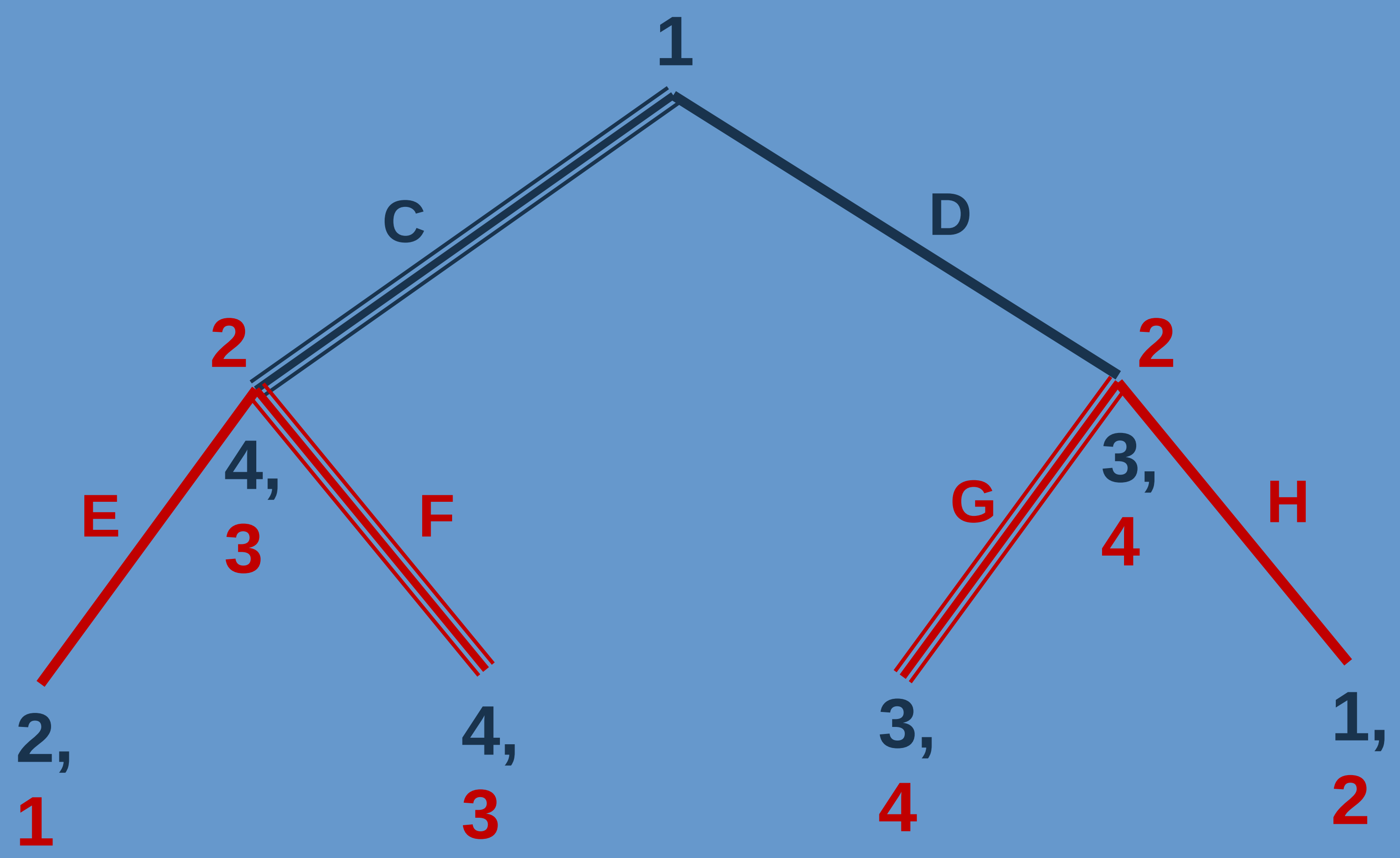
Backward induction



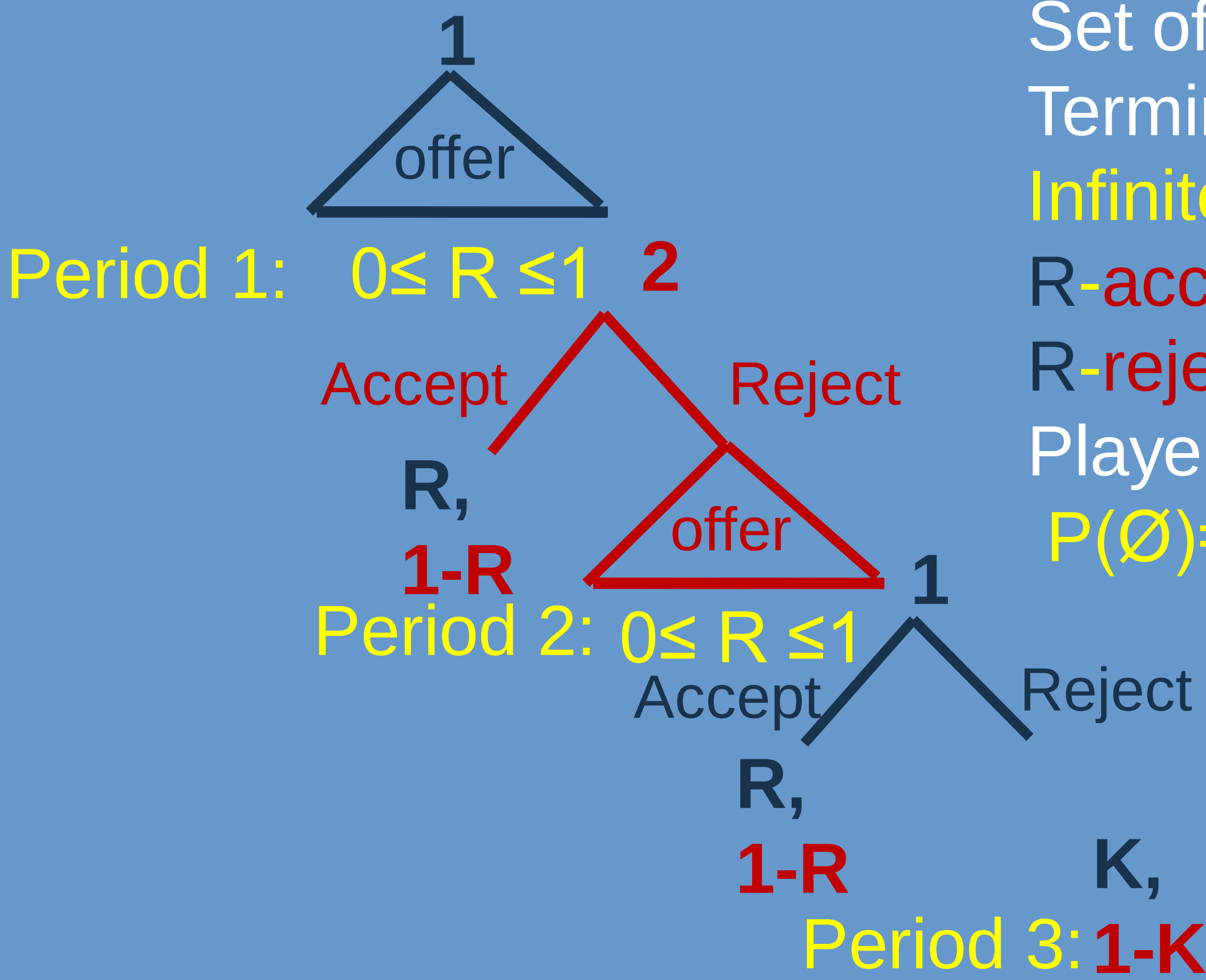
Backward induction



Backward induction



Backward induction



Set of players: 1 and 2

Terminal histories:

Infinitely many

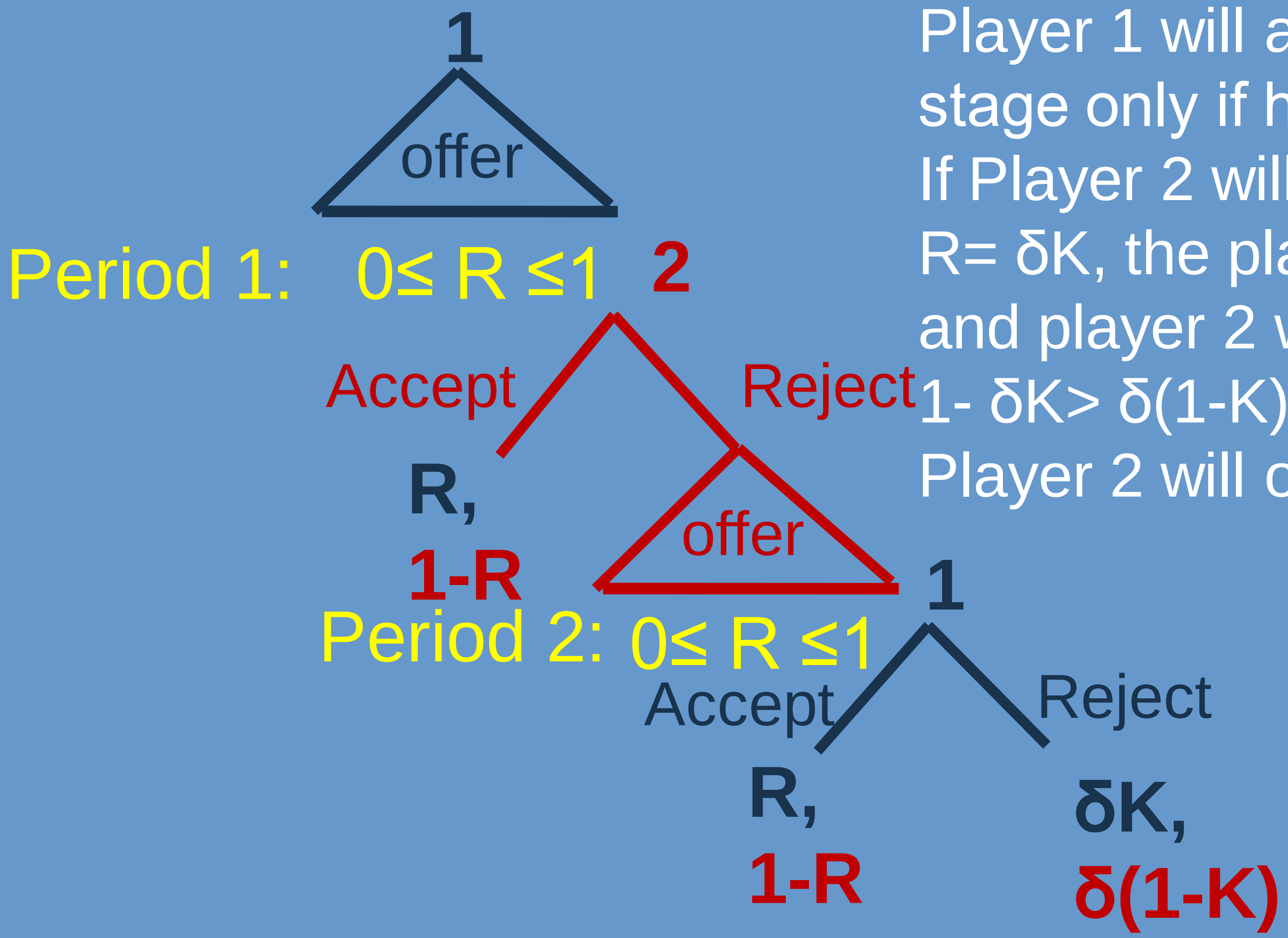
R-accept,

R-reject-R – accept, etc.

Player function:

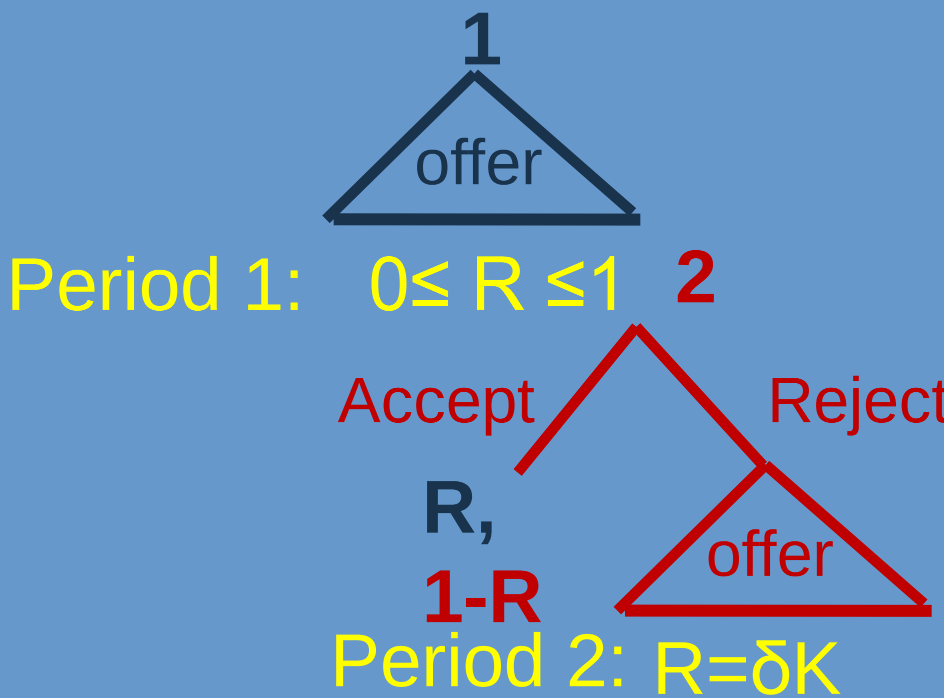
P(Ø)=1, P(R)=2,

Backward induction



Player 1 will accept the last stage only if he receives R≥δK If Player 2 will offer player 1 R= δK, the player 1 will accept and player 2 will receive 1- δK> δ(1-K) if player 1 reject (if Player 2 will offer less then δK)

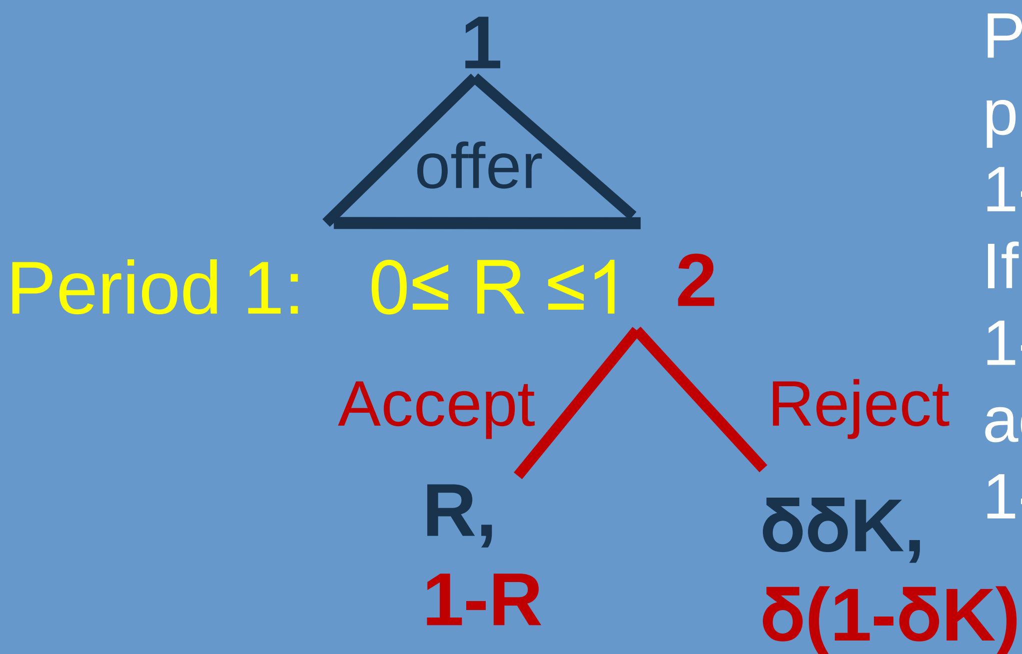
Backward induction



Player 1 will accept the last stage only if he receives R≥δK
If Player 2 will offer player one R= δK, the player 1 will accept and player 2 will receive 1- δK> δ(1-K) if player 1 reject (if Player 2 will offer less then δK)

δK,
1-δK

Backward induction



Player 2 will accept the offer of player 1 only if he receives $1-R \geq \delta\,(1-\delta K)$

If Player 1 will offer player 2 $1-R = \delta\,(1-\delta K)$, the player 2 will accept and player 1 will receive $1-\delta\,(1-\delta K) = 1-\delta+\delta\,\delta K > \delta\,\delta\,K$

if player 2 reject (if player 1 will offer less than $\delta\,(1-\delta K)$

Backward induction

Backward-induction outcome:
player 1 will offer 1-R= δ (1-δK),
player 2 will accept

Period 1: R=1-δ (1-δK)

2

Accept

1-δ (1-δK),
δ (1-δK)

Dynamic games - Example

- Backward induction does not tell us what the player will do in the case he is indifferent between several choices, and thus leaves open the question of which action the player should choose
- Games with infinitely long histories present another difficulty for backward induction: they have no end from which to start the induction

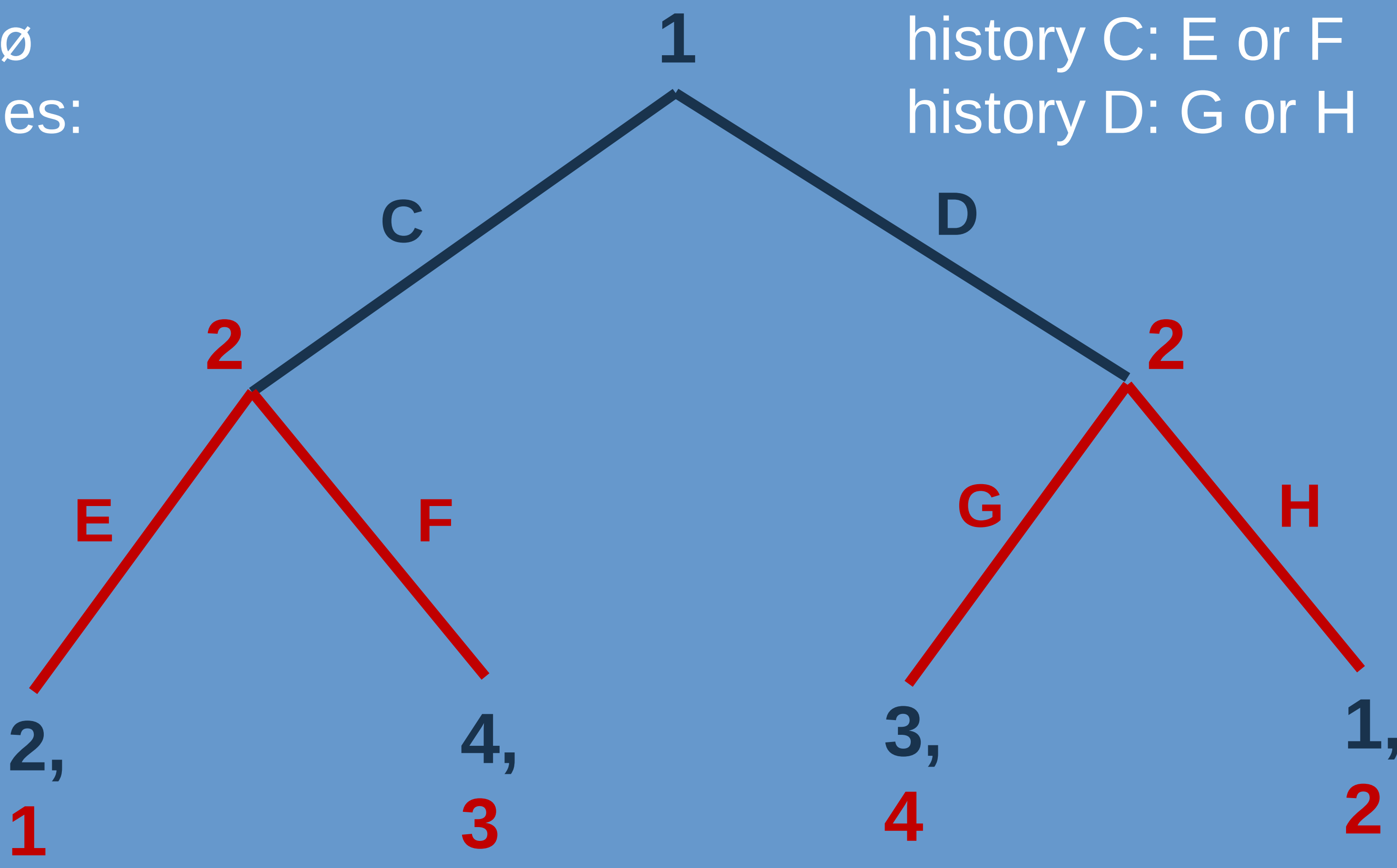
Strategies

- key concept in the study of extensive games
- strategy specifies the action the player chooses for every history after which it is her turn to move
- Definition: A **strategy of player i in an dynamic game with** perfect information is a function that assigns to each history h after which it is player i's turn to move (i.e. P(h) = i, where P is the player function) an action in A(h) (the set of actions available after h)

Strategies

player 1:moves only after history ø
strategies:
C, D

player 2: possible moves
history C: E or F
history D: G or H



Strategies

player 1:moves only after history ø
strategies:
C, D

player 2: possible moves
history C: E or F
history D: D or H

Player 2 strategies	Player 1 play C	Player 1 play D
Strategy 1	**E**	**G**
Strategy 2	**E**	**H**
Strategy 3	**F**	**G**
Strategy 4	**F**	**H**

We can describe the strategies of player 1 as C,D and player two as EG,EH,FG,FH where the first letter assign action for first history (C) and second for second history (D)

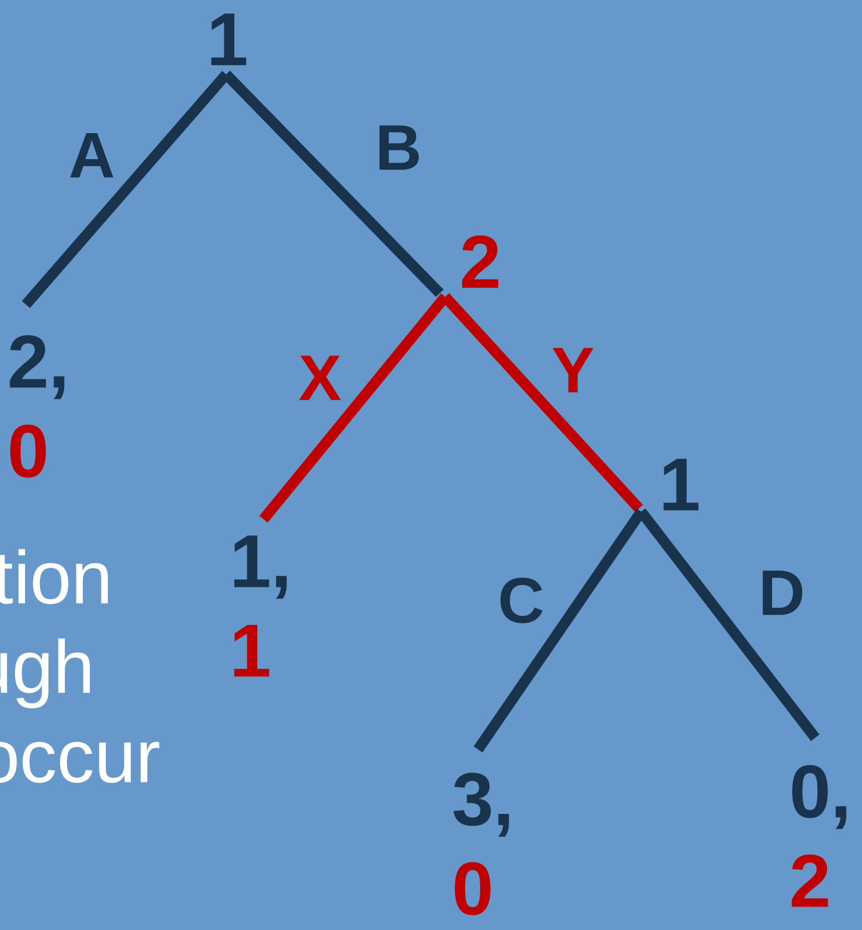
Strategies

- strategies may be interpreted as a plan of action
- strategy provides sufficient information to determine player's plan of action
- if a player appoints an agent to play the game for her, and tells the agent her strategy, then the agent has enough information to play the game according to her wishes, whatever actions the other players
- Definition requires that a strategy of any player i specify an action for every history after which it is player i's turn to move, even for histories that, if the strategy is followed, do not occur

Strategies

Player 1 have in both nodes where he plays two actions.
His strategies:
AC, AD, BC, BD

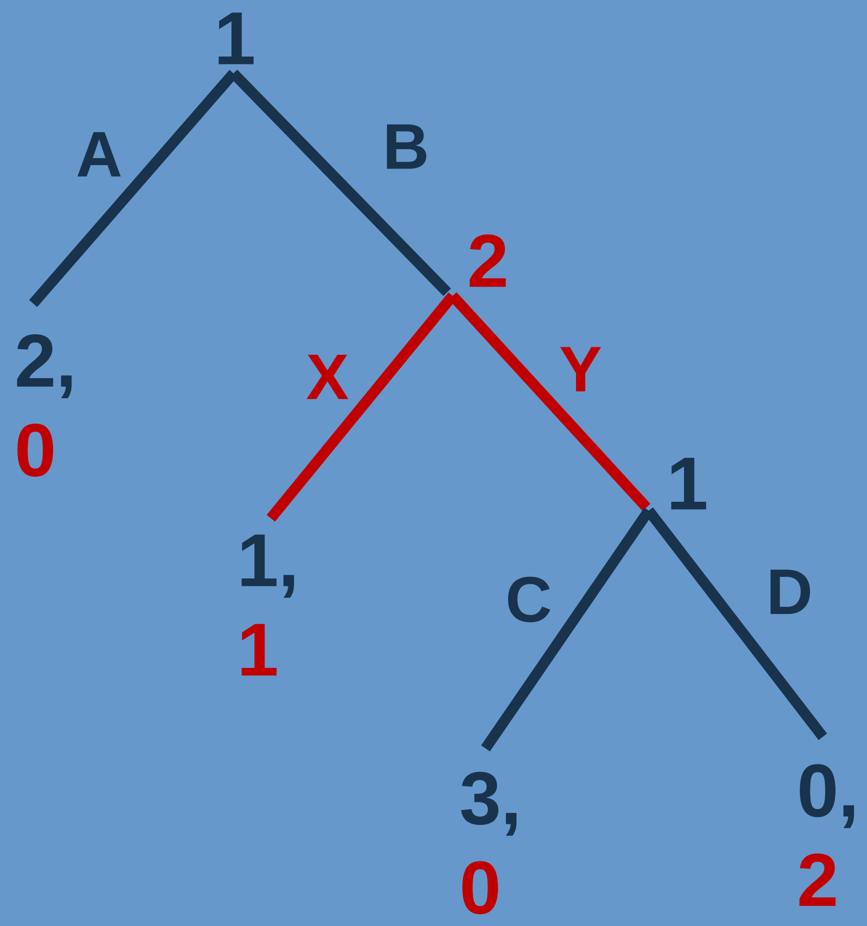
he has to define his action in history BY even though the history BY will not occur in strategies AC, AD



Outcomes

A strategy profile – particular strategies of all players in the game – determines the terminal history that occurs

terminal history as the outcome of s is denoted as O(s)



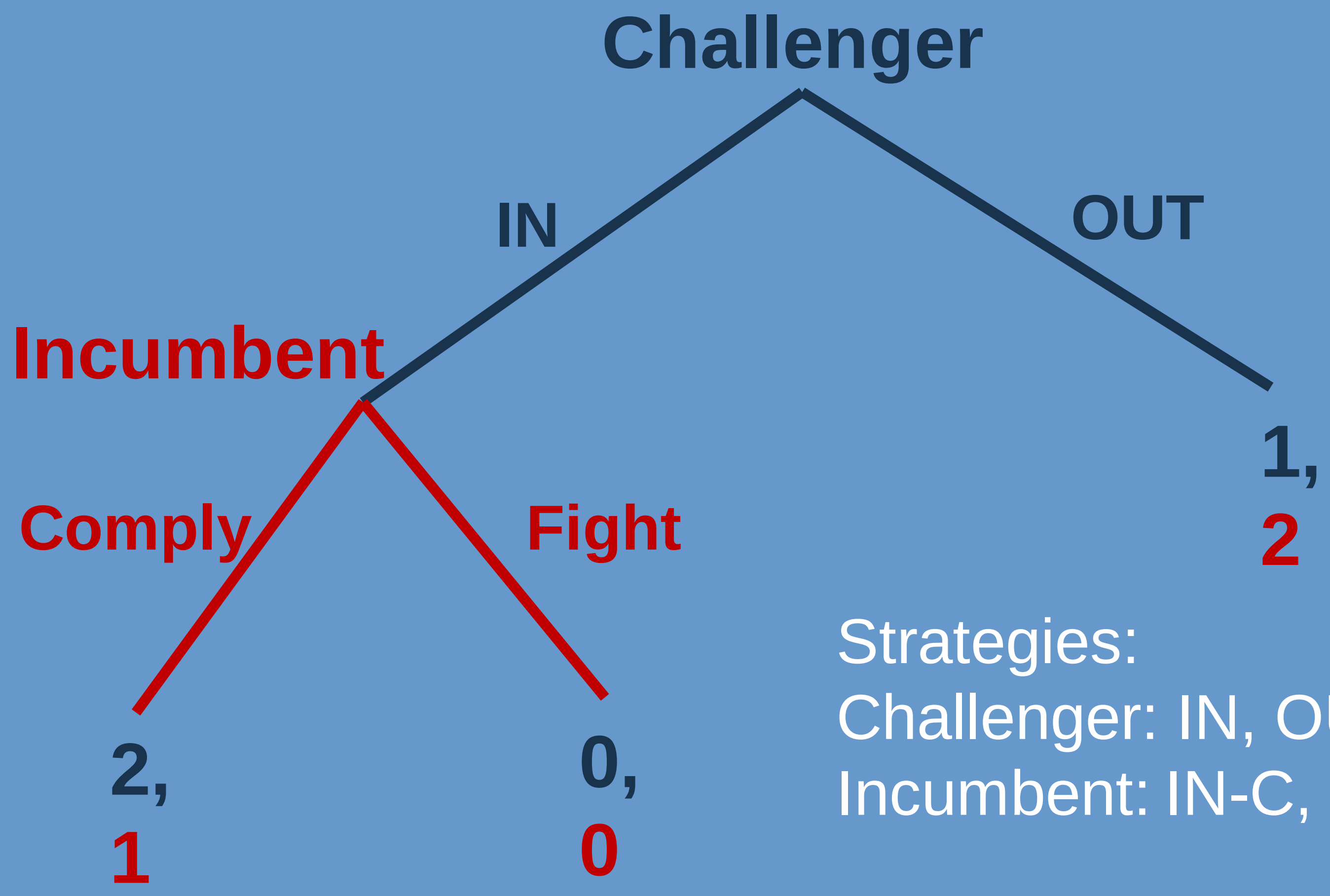
Nash Equilibrium

- Definition: The strategy profile s^* in an dynamic game with perfect information is a **Nash equilibrium if, for every player i and every strategy r_i of player i, the terminal history** $O(s^*)$ generated by s^* is at least as good according to player i's preferences as the terminal history $O(ri, s^*_{-i})$ generated by the strategy profile (r_i, s^*_{-i}) in which player i chooses r_i while every other player j chooses s^*_j . Equivalently, for each player i, $u_i(O(s*)) \geq u_i(O(ri, s^*_{-i}))$ for every strategy ri of player I,where ui is a payoff function that represents player i's preferences and O is the outcome function of the game.

How to find NE

- Analyze the normal form (static game) of dynamic game
- Set of players: same as in dynamic game
- Set of actions: strategies of all players
- Set of preferences: payoff to action profiles is actually payoffs to terminal histories

How to find NE



Strategies:
Challenger: IN, OUT
Incumbent: IN-C, IN-F

How to find NE

Incumbent

Challenger

	Comply	Fight
IN	**2**, **1**	**0**, **0**
OUT	**1**, **2**	**1**, **2**

How to find NE

Incumbent

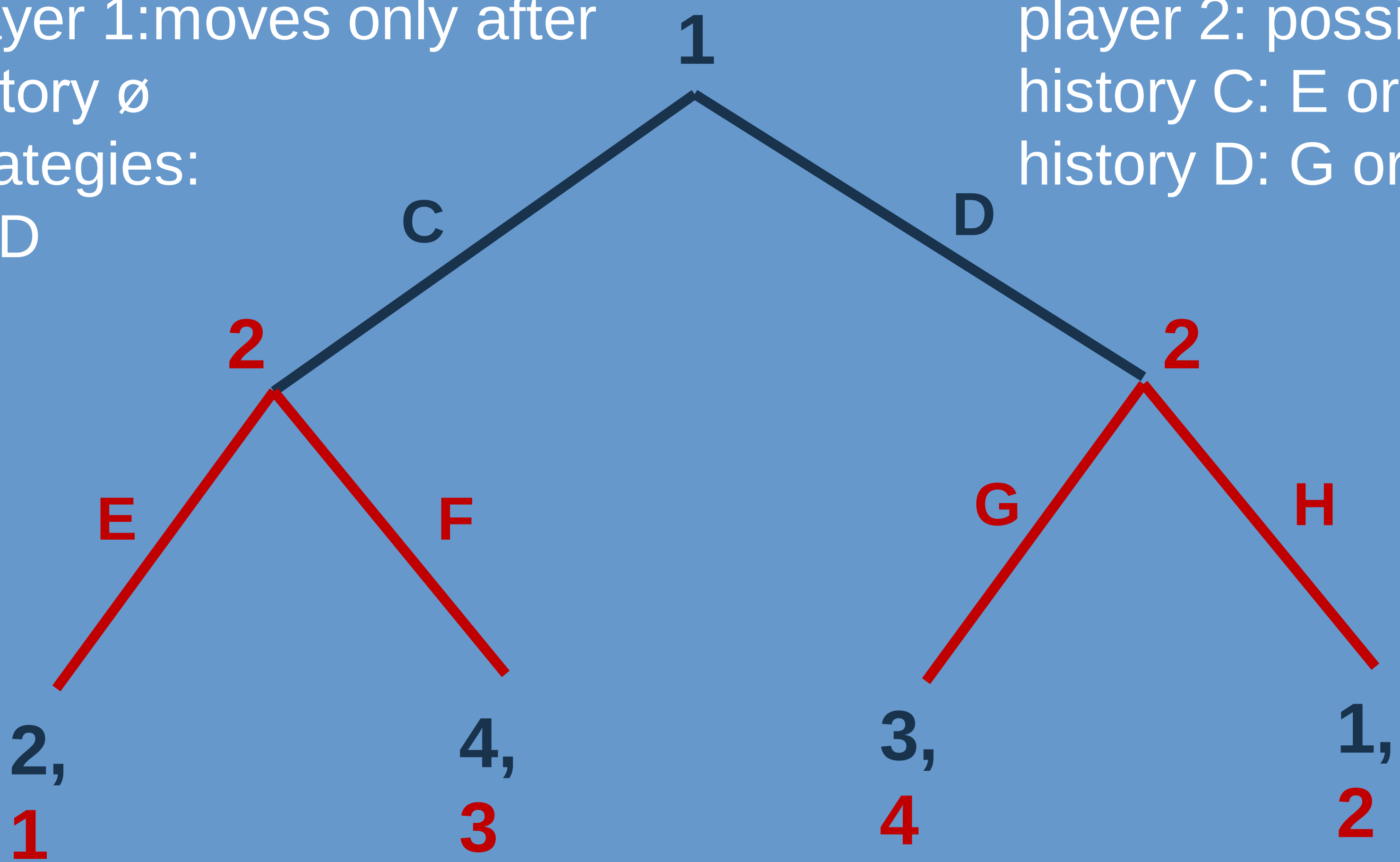
Challenger

	Comply	Fight
IN	**2**, **1**	**0**, **0**
OUT	**1**, **2**	**1**, **2**

Strategies

player 1:moves only after
history ø
strategies:
C, D

player 2: possible moves
history C: E or F
history D: G or H



player 1: C,D
player 2: EG,EH,FG,FH

How to find NE

Player 2

Player 1

	EG	EH	FG	FH
C	**2, 1**	**2, 1**	**4, 3**	**4, 3**
D	**3, 4**	**1, 2**	**3, 4**	**1, 2**

How to find NE

Player 2

Player 1

	EG	EH	FG	FH
C	**2**, **1**	**2**, **1**	**4**, **3**	**4**, **3**
D	**3**, **4**	**1**, **2**	**3**, **4**	**1**, **2**

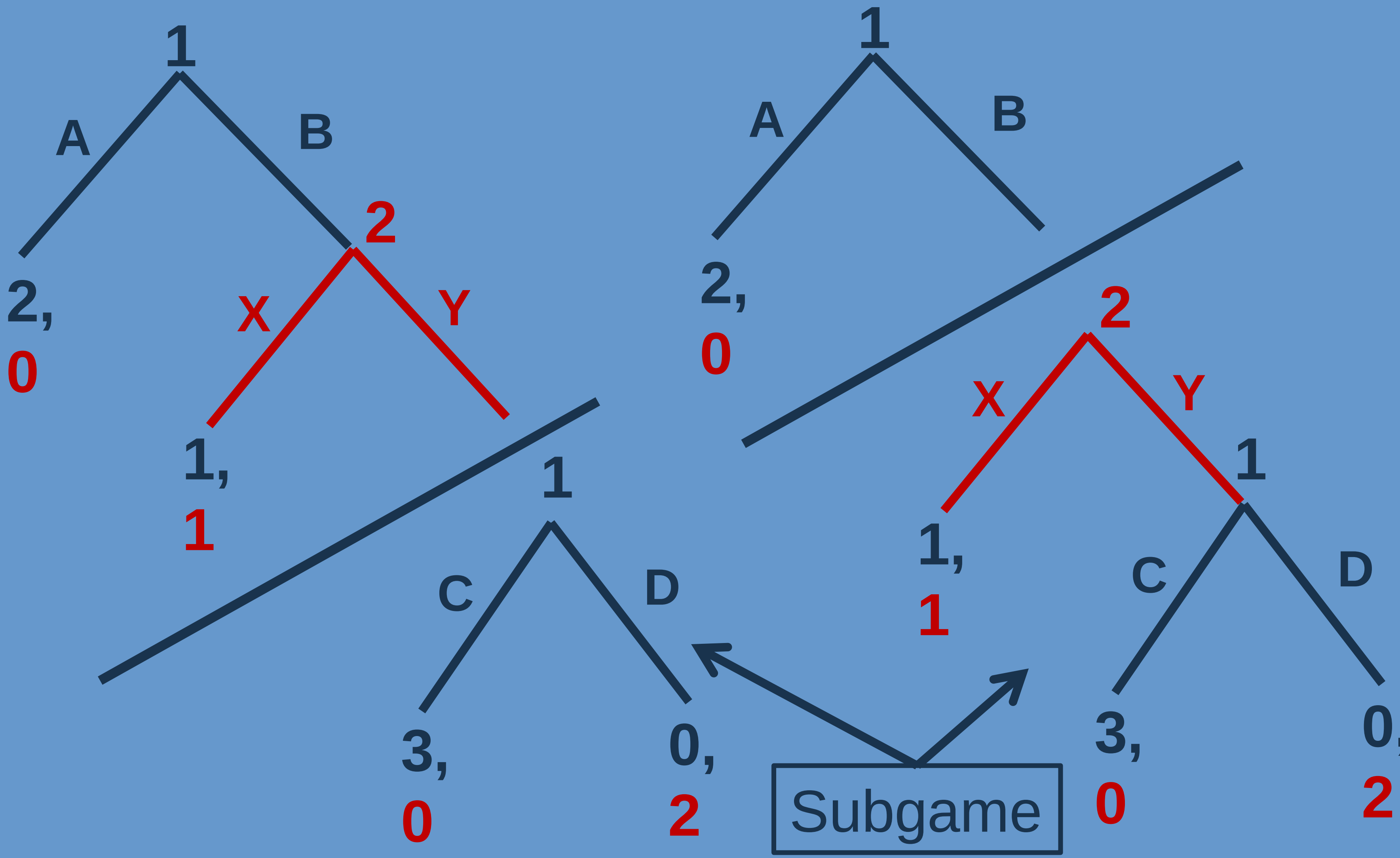
Subgame Perfect Equilibrium

- Definition: **SUBGAME**

  Let Γ be an dynamic game with perfect information, with player function P. For any nonterminal history h of Γ, the subgame Γ(h) following the history h is the following extensive game:

  - Set of players: players in Γ
  - Terminal histories: set of all sequences h' of actions such that (h, h') is a terminal history of Γ
  - Player function: The player P(h, h') is assigned to each proper subhistory h' of a terminal history
  - Preferences for the players: Each player prefers h' to h'' if and only if she prefers (h, h') to (h, h'') in Γ

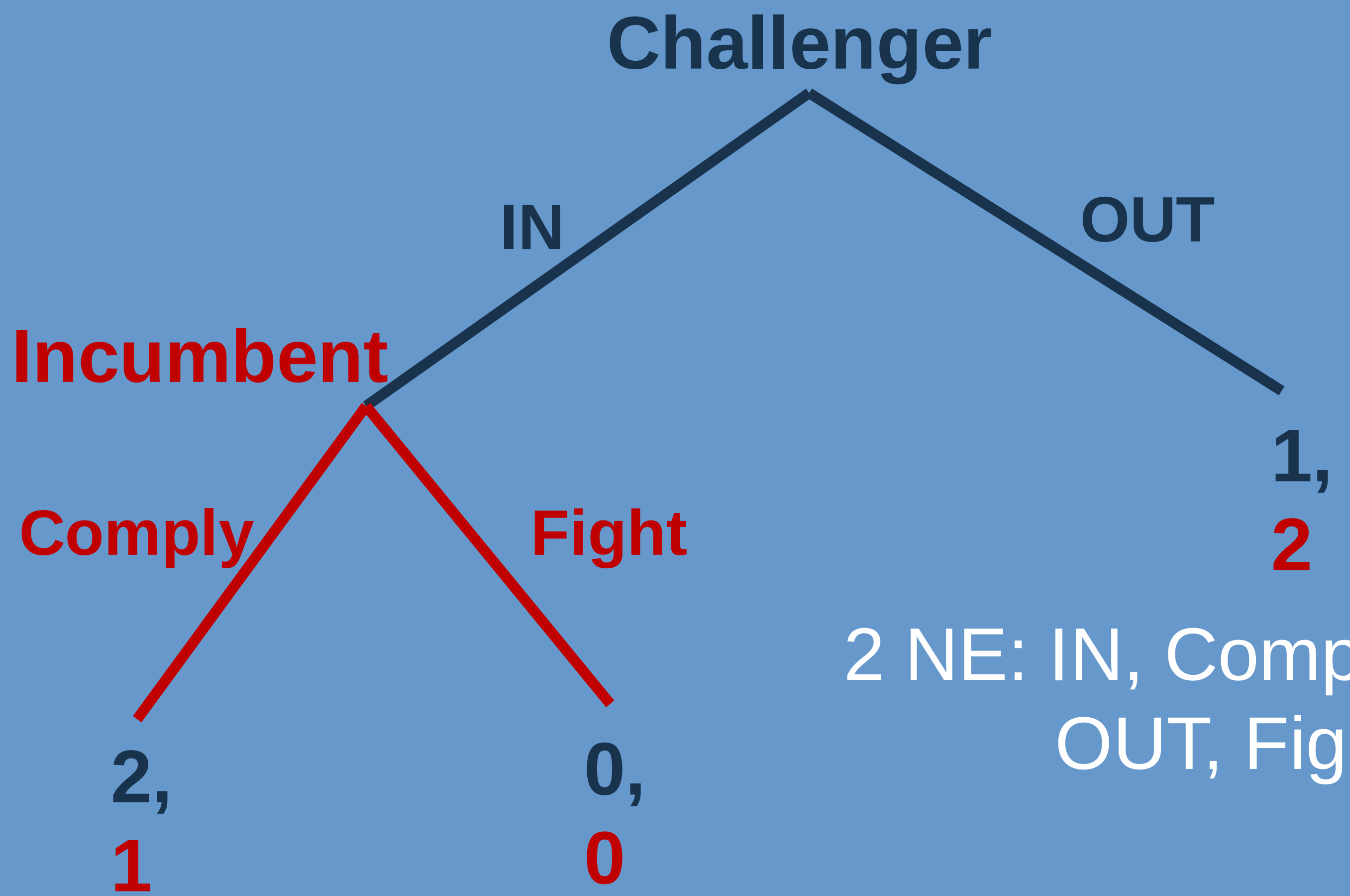
Subgame – example



Subgame Perfect Equilibrium

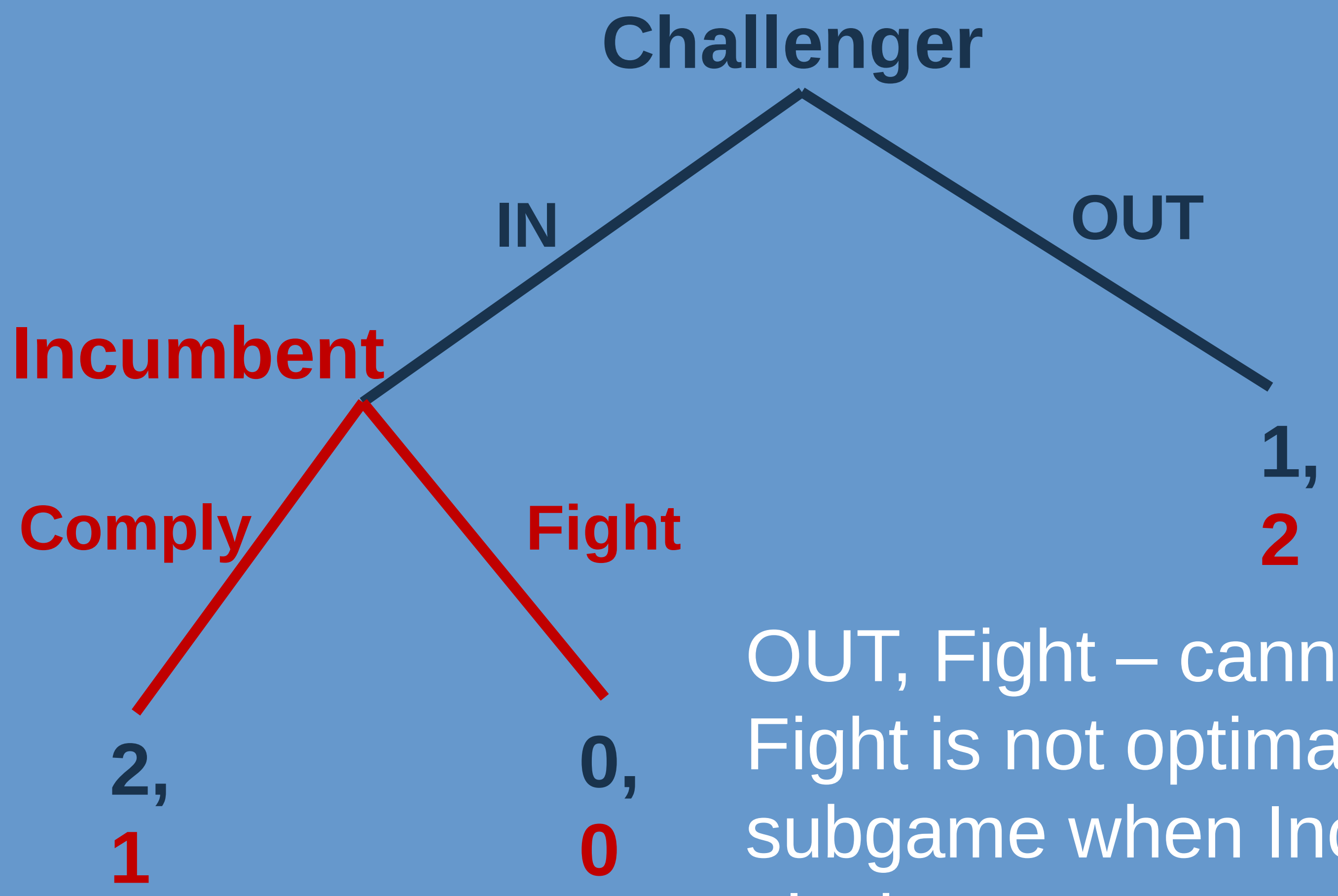
- Definition: A *subgame perfect equilibrium (SBNE) is a strategy profile s^* with the property* that in no subgame can any player *i do better by choosing a strategy* different from s^*_i *, given that every other player j adheres to* s^*_j .

Subgame Perfect Equilibrium



2 NE: IN, Comply
OUT, Fight

Subgame Perfect Equilibrium



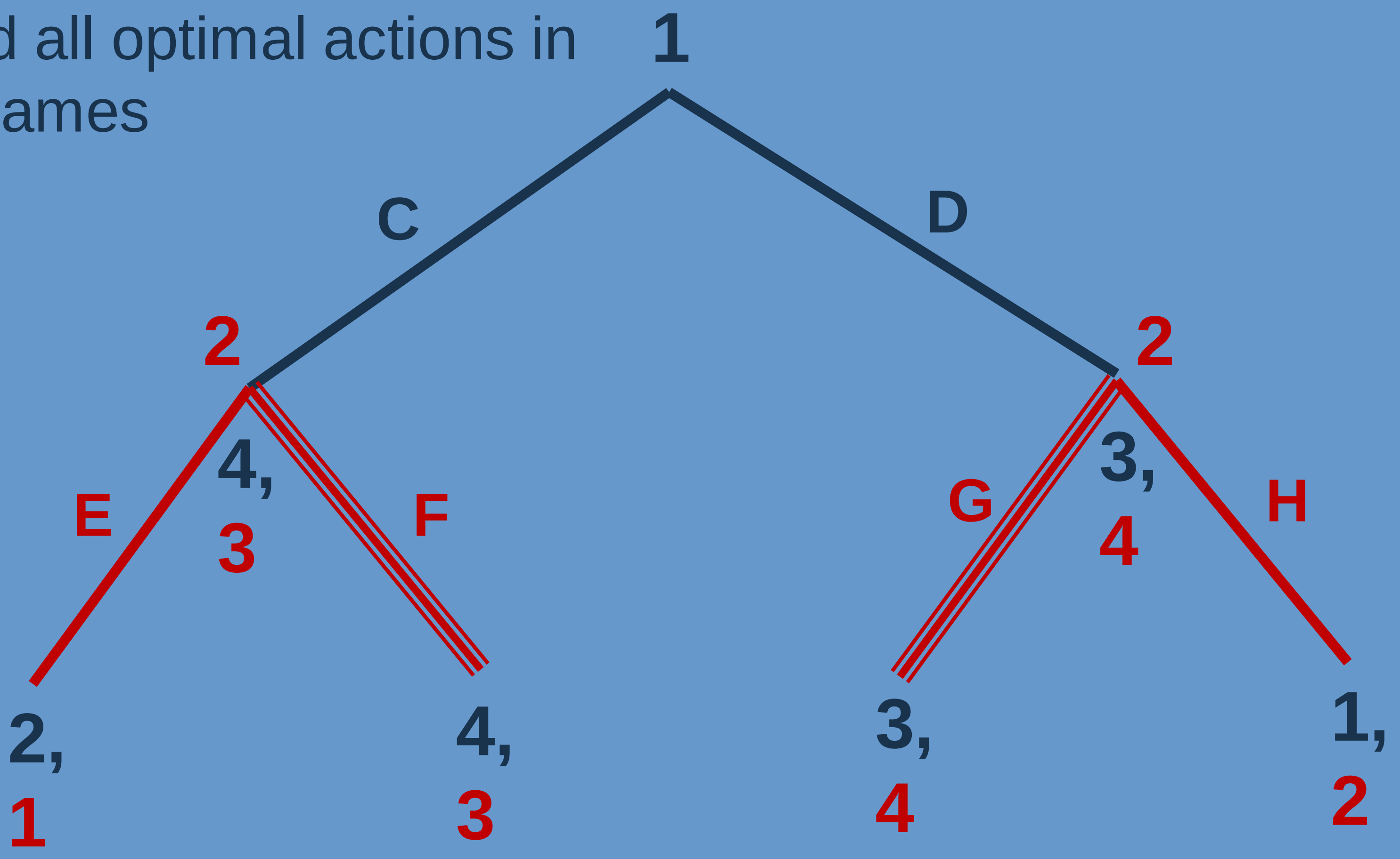
OUT, Fight – cannot be SPNE
Fight is not optimal for the subgame when Incumbent is playing

Subgame Perfect Equilibrium

- Definition: The strategy profile s^* *in an dynamic* game with perfect information is a **subgame perfect equilibrium if, for every** player *i, every history h after which it is player i's turn to move (i.e. P(h) = i), and* every strategy r_i *of player i, the terminal history* $O_h(s^*)$ *generated by* s^* *after the* history *h is at least as good according to player i's preferences as the terminal history* $O_h(r_i , s^*_{-i})$ *generated by the strategy profile* (r_i , s^*_{-i}) *in which player i chooses* r_i *while every other player j chooses* s^*_j *. Equivalently, for every player i and every* history *h after which it is player i's turn to move,* $u_i(O_h(s^*)) \geq u_i(O_h(r_i , s^*_{-i}))$ *for every strategy* r_i *of player i,* where u_i *is a payoff function that represents player i's preferences and* $O_h(s)$ *is the* terminal history consisting of *h followed by the sequence of actions generated by s after h.*

Subgame Perfect Equilibrium

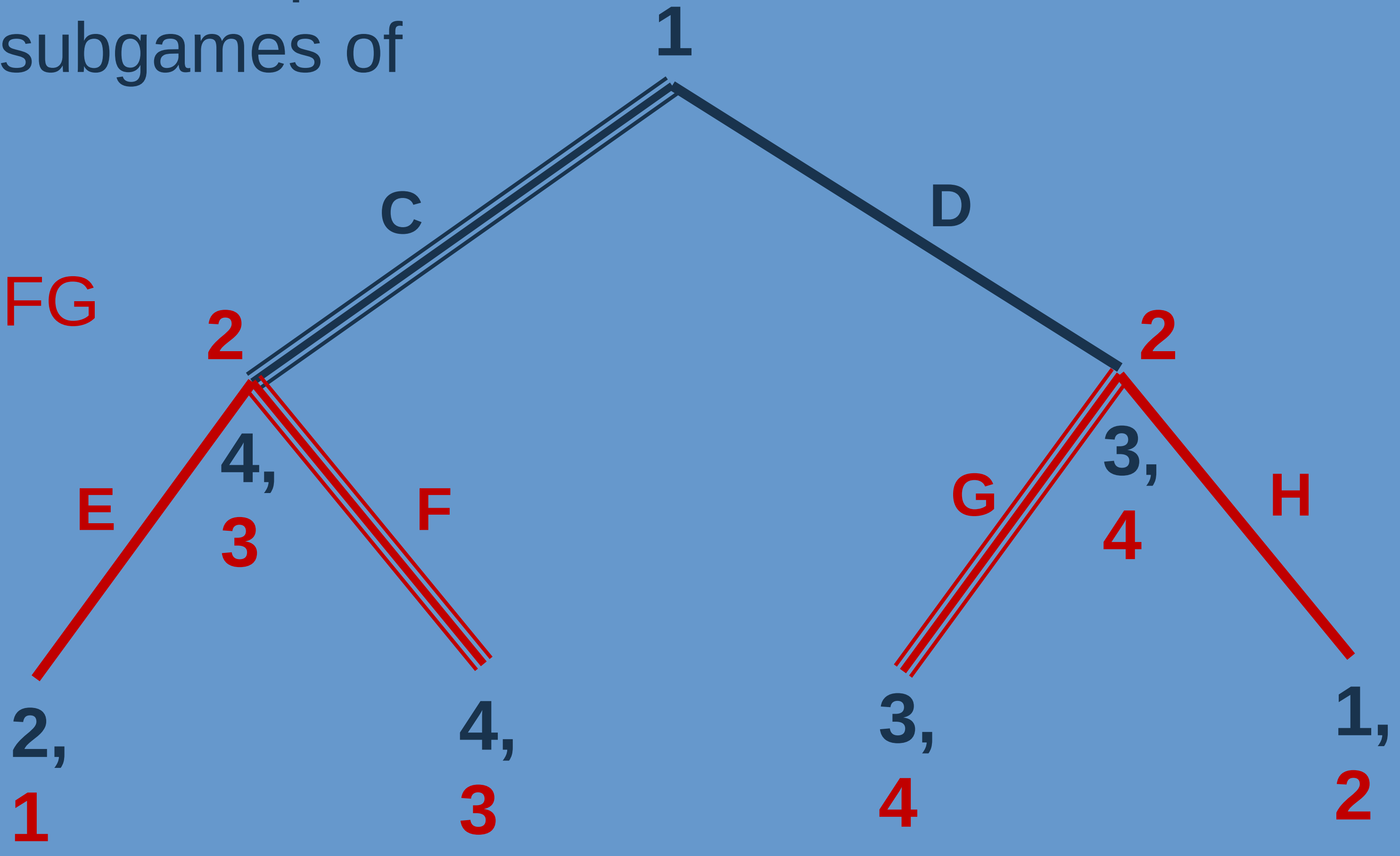
Start with subgames of length 1 and find all optimal actions in these games



Subgame Perfect Equilibrium

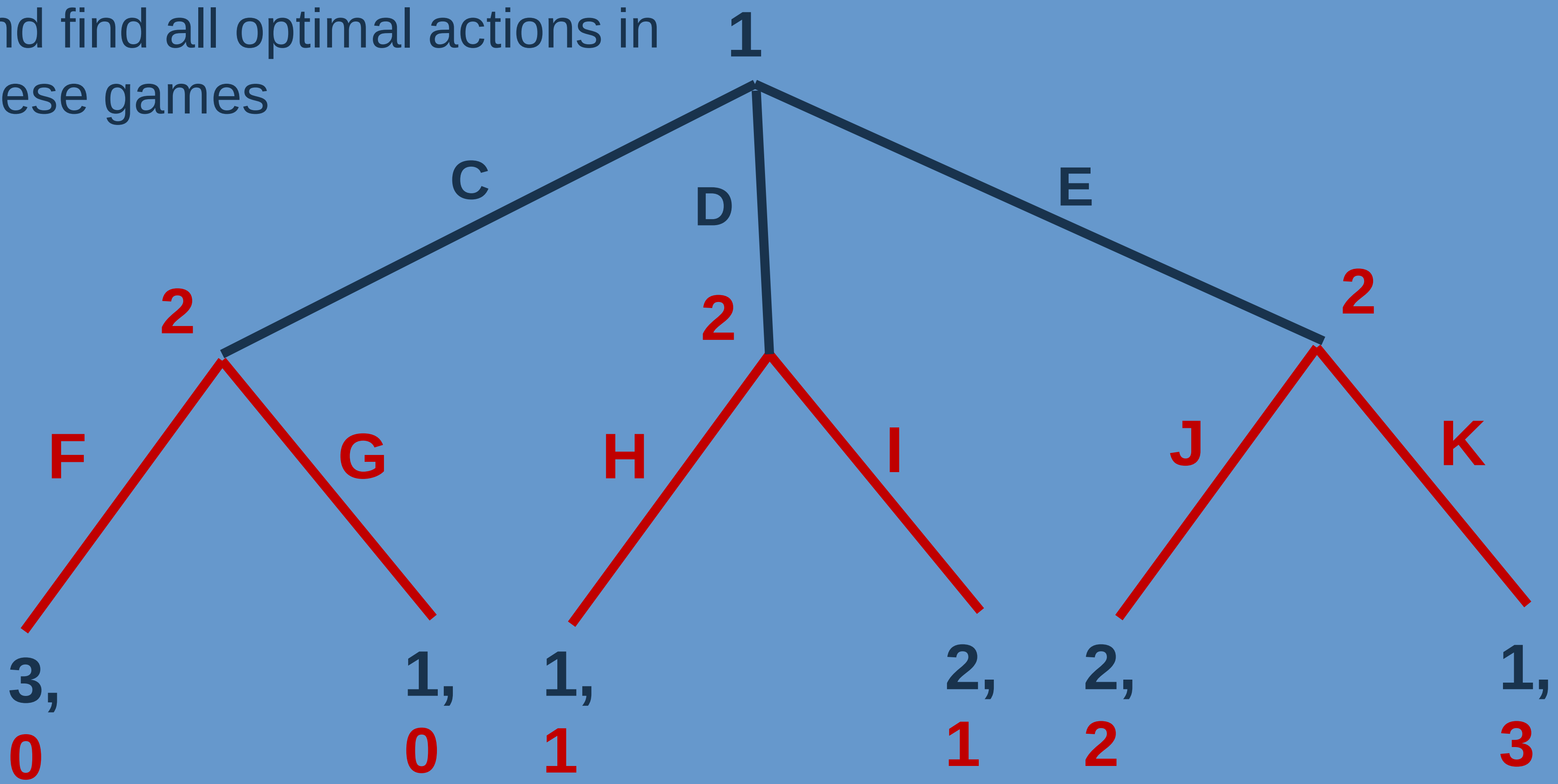
For each combination of these actions find optimal actions in subgames of length 2

SPNE: C, FG



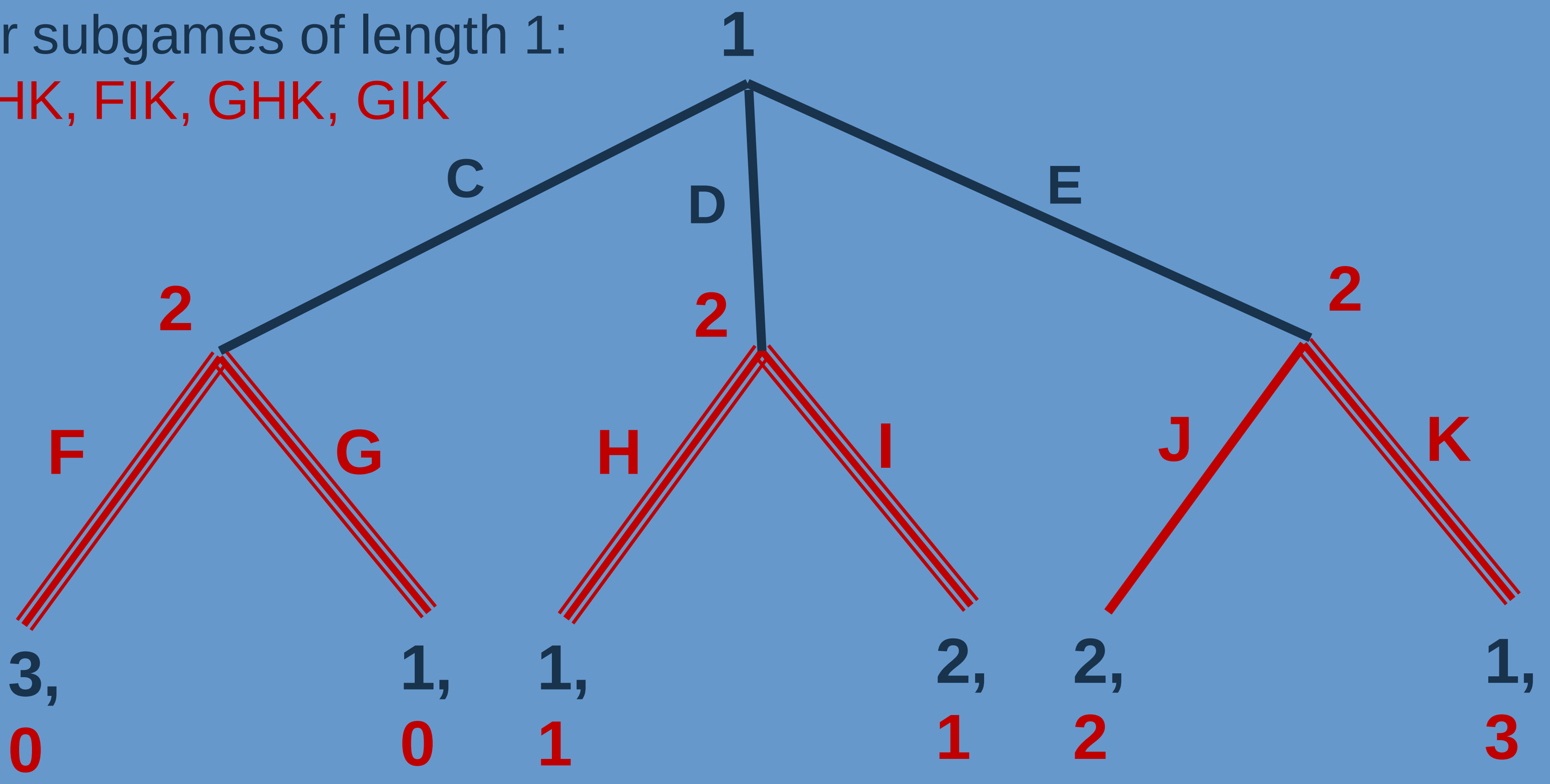
Subgame Perfect Equilibrium

Start with subgames of length 1 and find all optimal actions in these games



Subgame Perfect Equilibrium

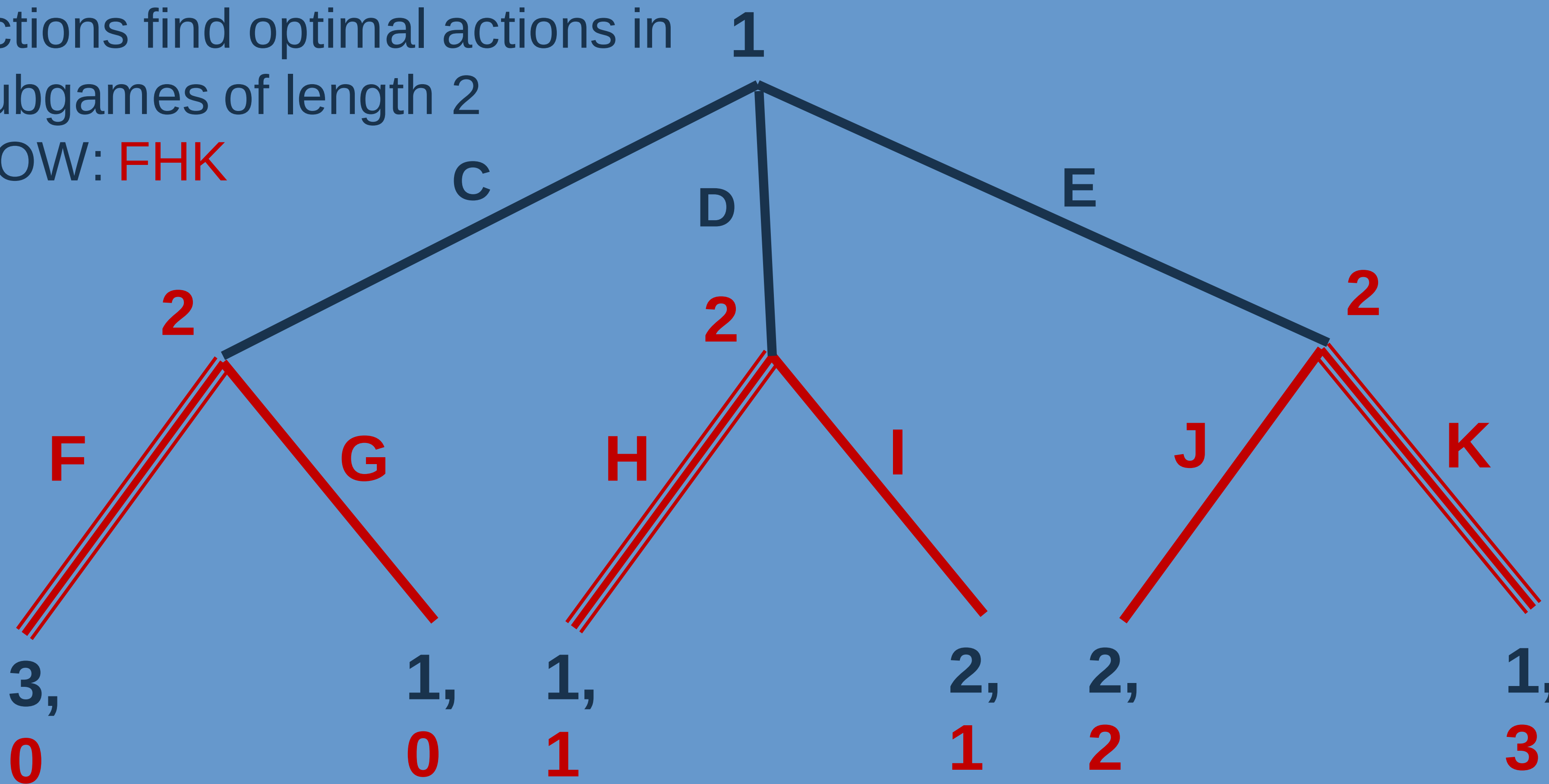
Optimal combinations of actions (strategies)
for subgames of length 1:
FHK, FIK, GHK, GIK



Subgame Perfect Equilibrium

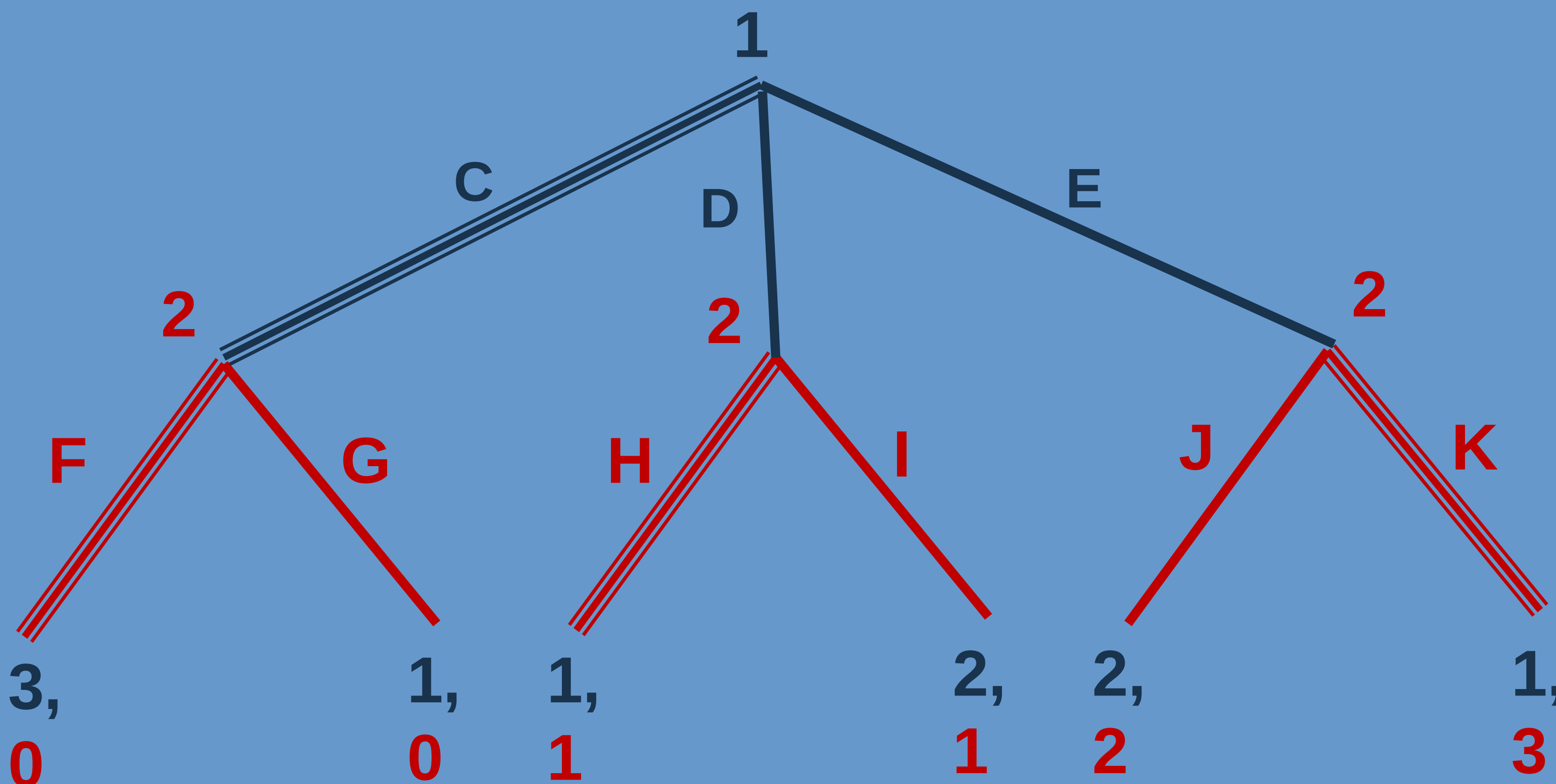
For each combination of these actions find optimal actions in subgames of length 2

NOW: FHK



Subgame Perfect Equilibrium

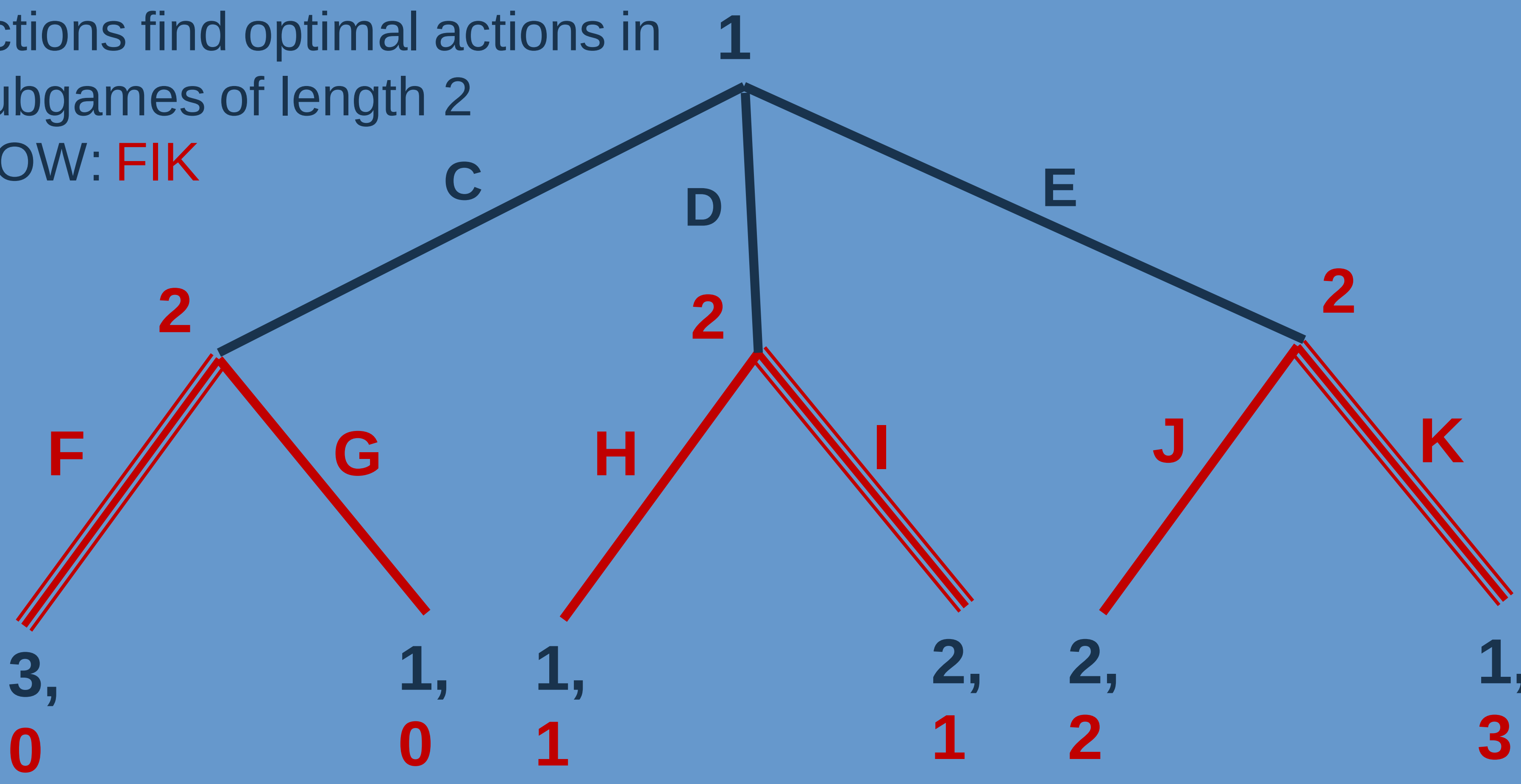
SPNE: C, FHK



Subgame Perfect Equilibrium

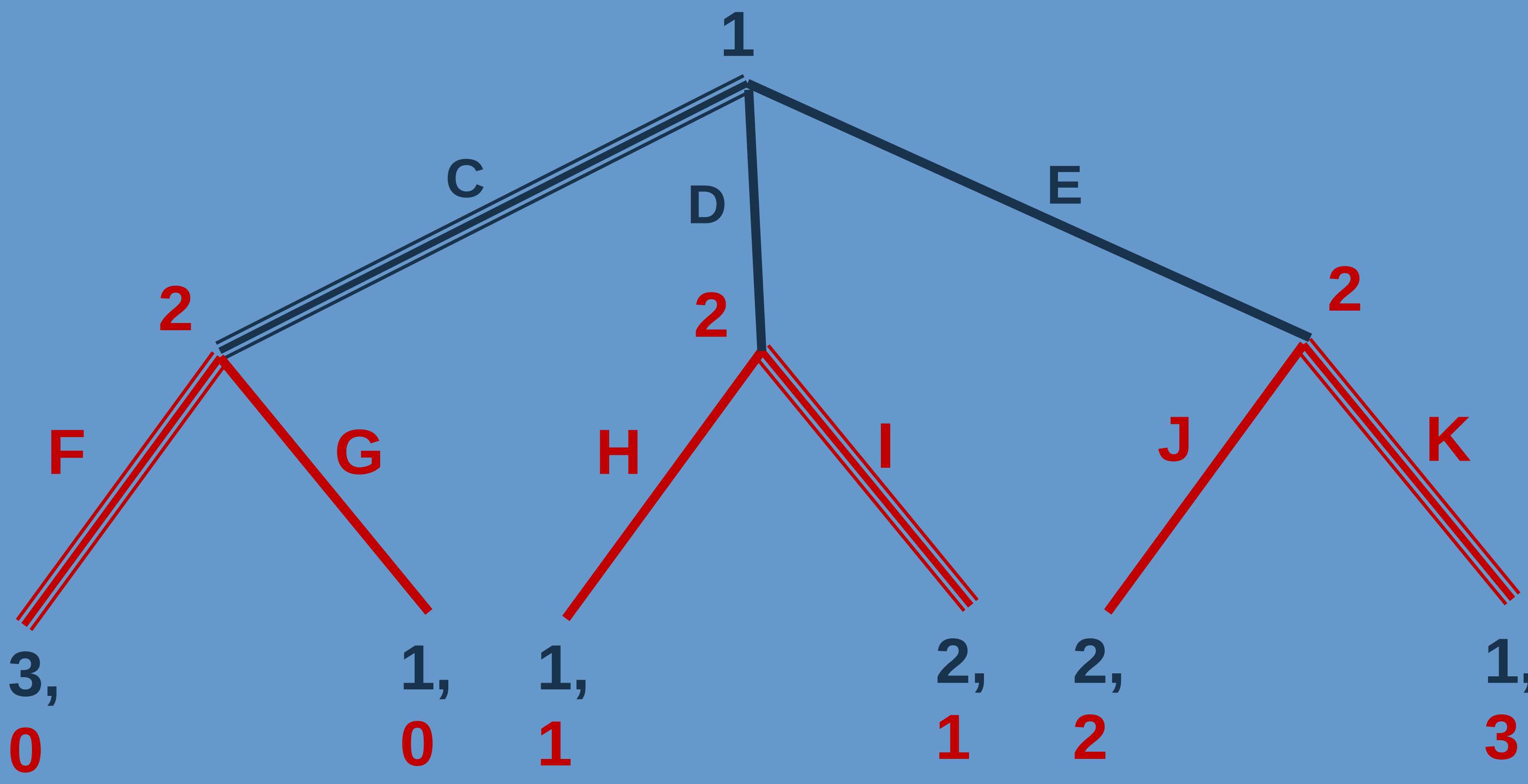
For each combination of these actions find optimal actions in subgames of length 2

NOW: FIK



Subgame Perfect Equilibrium

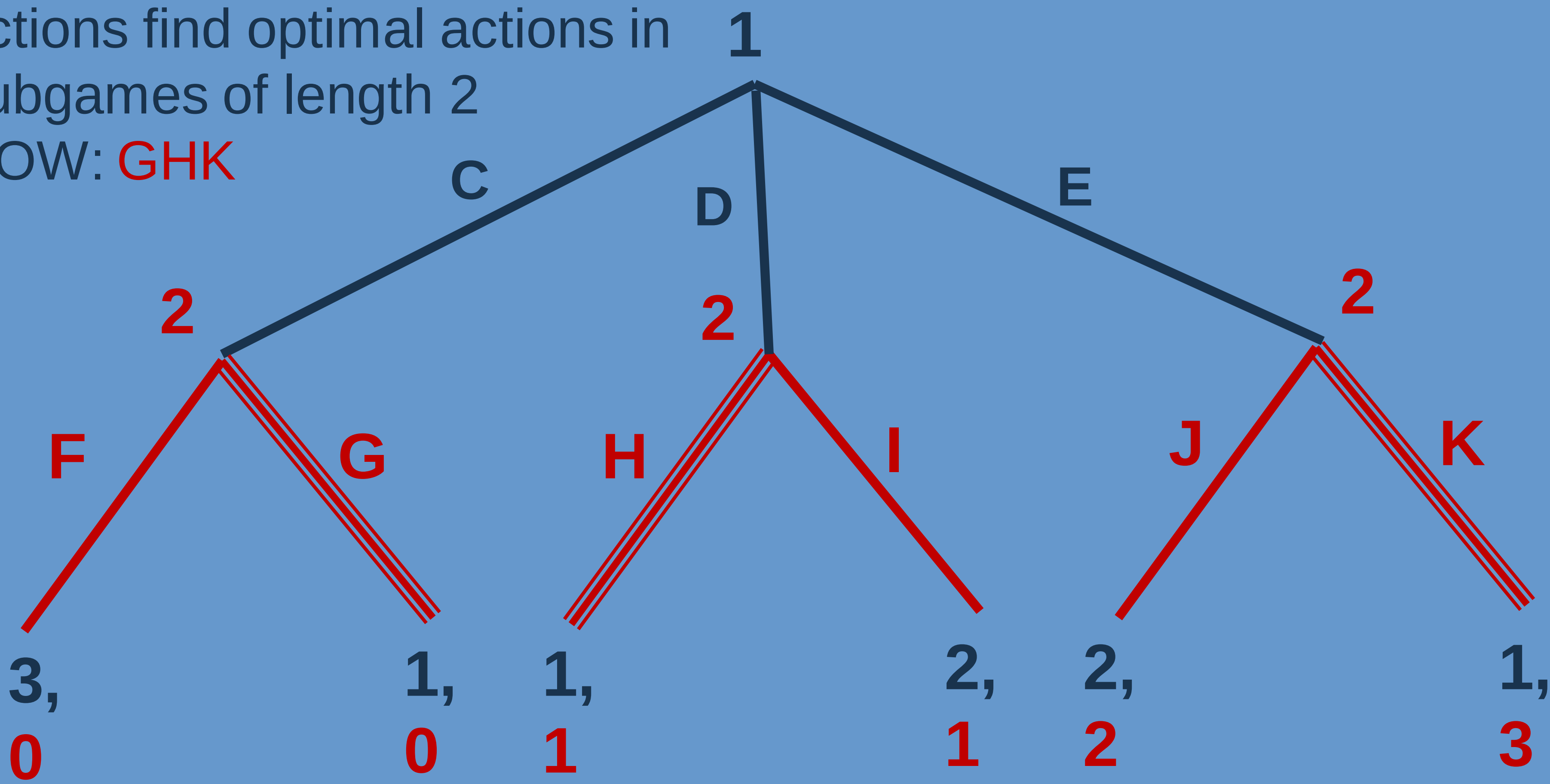
SPNE: C, FIK



Subgame Perfect Equilibrium

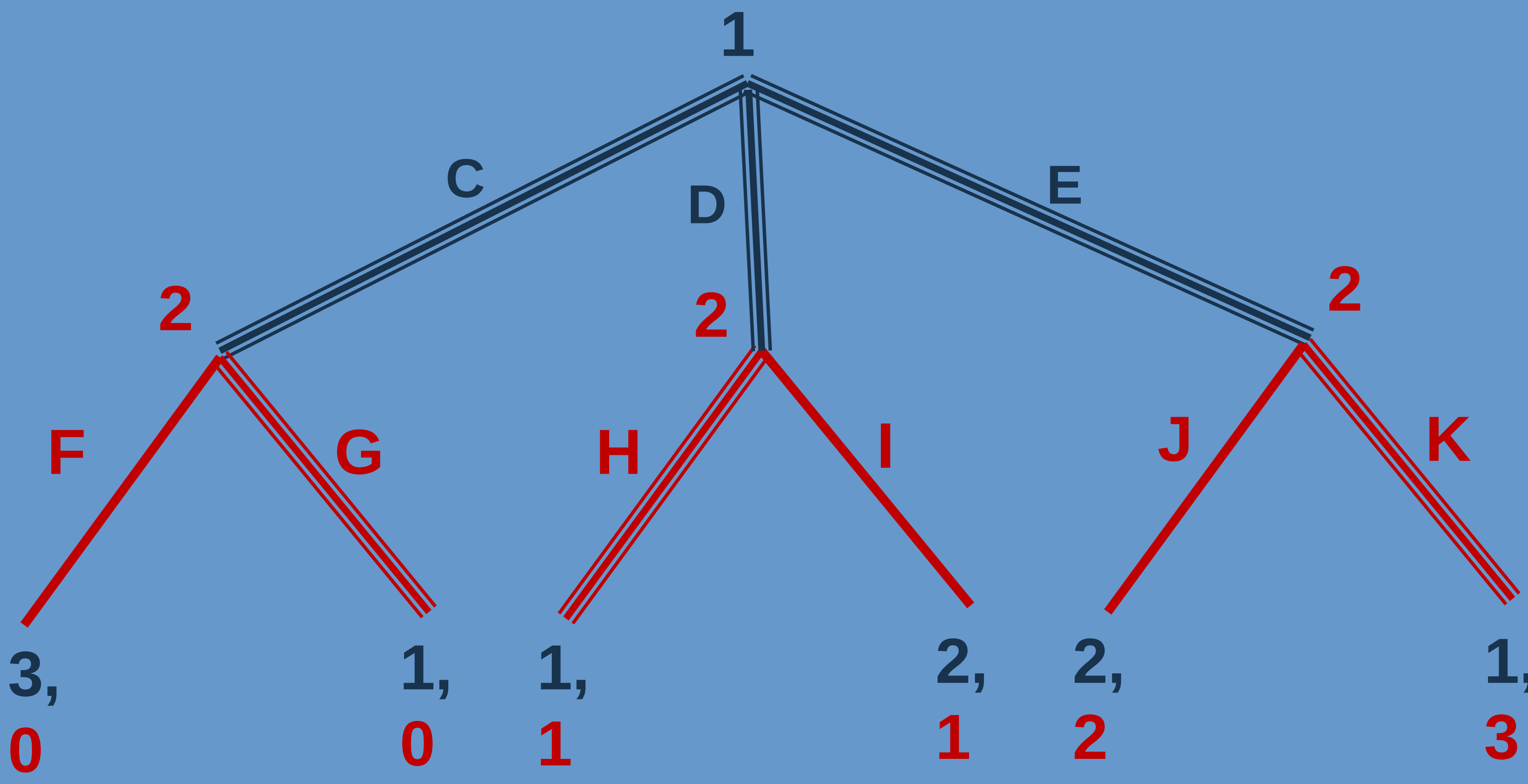
For each combination of these actions find optimal actions in subgames of length 2

NOW: GHK



Subgame Perfect Equilibrium

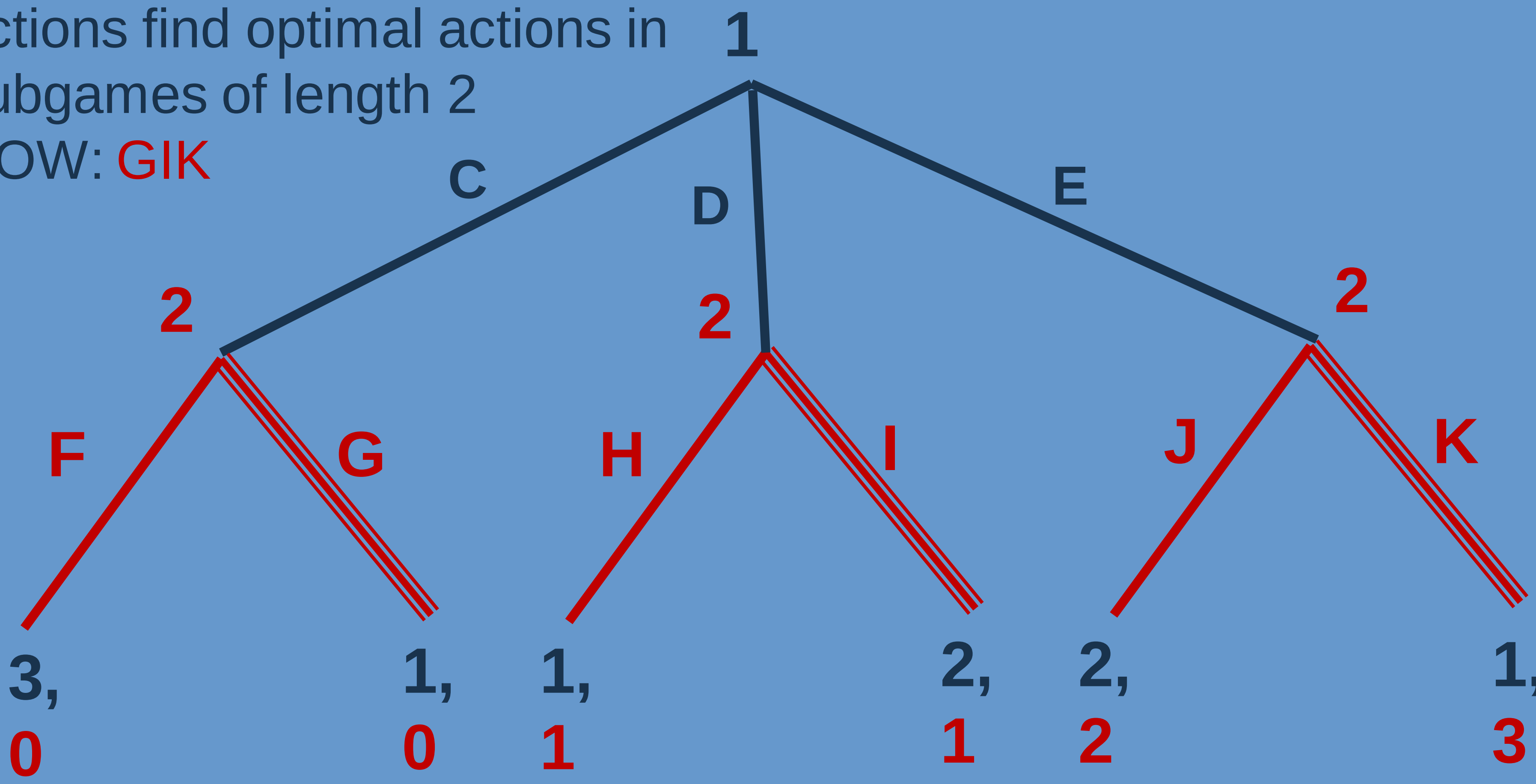
SPNE: C, GHK and D, GHK and E, GHK



Subgame Perfect Equilibrium

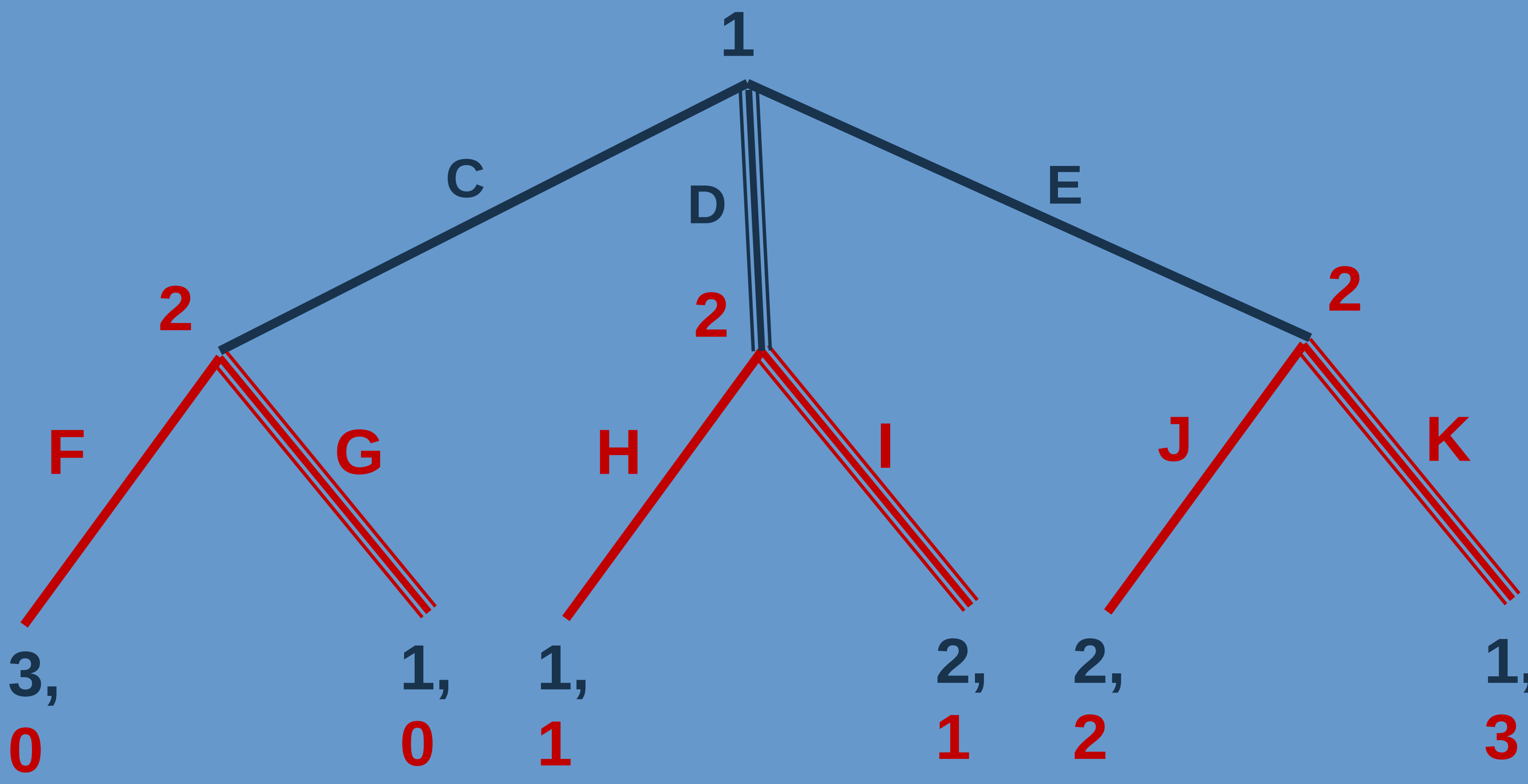
For each combination of these actions find optimal actions in subgames of length 2

NOW: GIK



Subgame Perfect Equilibrium

SPNE: D, GIK



Summary

- Dynamic games
- Backward induction
- Nash equilibrium
- Subgame perfect equilibrium
- Gibbons 2-2.1.D; Osborne 5

NEXT WEEK:

SBNE, Illustrations