Illustrations of NE and MSNE

Lecture 4

Revision

- Mixed strategy - player assigns probabilities $p_1, p_2, \ldots, p_N$ to all of her actions and she is playing her actions randomly according to these probabilities
- Expected utility: preferences over lotteries can be represented as expected value of a utility(payoff) function over deterministic outcomes
- MSNE: The mixed strategy profile α^* in a static game with vNM preferences is a **mixed strategy Nash equilibrium (MSNE) if, for each player i and every mixed strategy** β_i of player i, the expected utility(payoff) to player i of α^* is at least as large as the expected utility(payoff) to player i of (β_i, α^*_{-i}) according to a utility(payoff) function whose expected value represents player i's preferences over lotteries.

Revision: How to check MSNE

- *A mixed strategy profile* α^*
  - *the expected utility(payoff), given* α^*_{-i}, *to every action to which* α^*_i *assigns positive probability is the same*
  - *the expected utility(payoff), given* α^*_{-i}, *to every action to which* α^*_i *assigns zero probability is at most the expected utility(payoff) to any action to which* α^*_i *assigns positive probability.*

You should check for MSNE all combinations. That is, you should check whether there are equilibria, in which one player chooses a pure strategy and the other mixes; equilibria, in which both mix; and equilibria in which neither mixes. Note, that the mixtures need not be over the entire strategy spaces, which means you should check every possible subset.

Restaurants and smoking

It is often claimed by liberalists in CR that smoking in restaurants should not be regulated and that every owner may decide whether to allow or forbid smoking in his restaurant.

Let's have a look on this problem from the game theory point of view. *We will see that the game of restaurant's owners and their customers have two Nash Equilibria in pure strategies. Therefore, similarly like in the case of Island Samoa to change the side of driving on the road, without the governmental intervention the owners and consumers would not have any incentive to change from one NE to another.*

Imagine we have the population of N people of some small village and one restaurant in the village. The village is so small that the existence of more than one restaurant is economically impossible.

Restaurants and smoking

There are K people who will always go to the restaurant and L who will never go there. Further there are X people who will go there only if it is not smoking and Y people who will go there only if it is smoking.

$K+L+X+Y=N$

When the restaurant changes its smoking policy, those who come to the restaurant would learn it immediately. However, people who don't go to the restaurant will realize the change after quite a long time. So, the restaurant would have to cease to operate having just K customers.

Restaurants and smoking

Type K	In the pub	Stay at home
Smoking	1, 1	0, 0
Not smoking	1, 1	0, 0

Type L	In the pub	Stay at home
Smoking	1, 0	0, 1
Not smoking	1, 0	0, 1

Type X	In the pub	Stay at home
Smoking	1, 0	0, 1
Not smoking	1, 2	0, 1

Type Y	In the pub	Stay at home
Smoking	1, 2	0, 1
Not smoking	1, 0	0, 1

Restaurants and smoking

Type Y - smoking

Type X Not smoking

NO BAN	PUB	HOME
PUB	0, 2, Y+X	0, 1, X
HOME	1, 2, Y	1, 1, 0

BAN SMOK.	PUB	HOME
PUB	2, 0, X+Y	2, 1, X
HOME	1, 0, Y	1, 1, 0

Restaurants and smoking

Type Y - smoking

Type X Not smoking

NO BAN	PUB	HOME
PUB	0, 2, X+Y	0, 1, X
HOME	1, 2, Y	1, 1, 0

BAN SMOK.	PUB	HOME
PUB	2, 0, X+Y	2, 1, X
HOME	1, 0, Y	1, 1, 0

Electoral competition

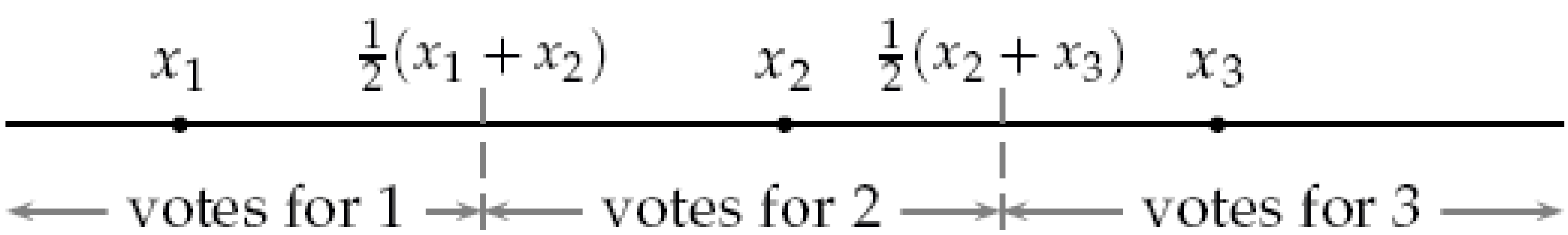
A simple version of this model: players are the candidates and a policy is a number, referred to as a “position”. After the candidates have chosen positions, each of a set of citizens votes for the candidate whose position she likes best. The candidate who obtains the most votes wins. Each candidate cares only about winning. Specifically, each candidate prefers to win than to tie for first place than to lose, and if she ties for first place she prefers to do so with as few other candidates as possible.

Electoral competition

There is a continuum of voters, each with a favorite position. The distribution of these favorite positions over the set of all possible positions is arbitrary. In particular, this distribution may not be uniform: a large fraction of the voters may have favorite positions close to one point, while few voters have favorite positions close to some other point. A position that turns out to have special significance is the *median favorite position: the position m with the property that exactly half of* the voters' favorite positions are at most *m, and half of the voters' favorite positions* are at least *m.*

Electoral competition

Each voter's distaste for any position is given by the distance between that position and her favorite position. In particular, for any value of *k, a voter whose* favorite position is *x^* is indifferent between the positions $x^* - k$ and $x^* + k$.* Under this assumption, each candidate attracts the votes of all citizens whose favorite positions are closer to her position than to the position of any other candidate. An example is shown in Figure



Electoral competition

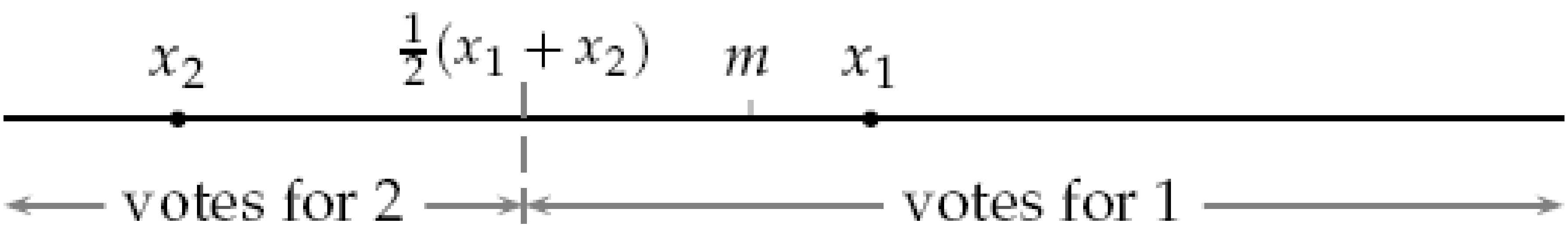
Suppose just 2 candidates

best response of candidate 1 to position x_2 of candidate 2:

if $x_2<m$ then x_1 prefers equally all positions where she wins : positions between x_2 and $\frac{1}{2}(x_1+x_2)=m$

if $x_2>m$ then x_1 prefers equally all positions where she wins : positions between x_2 and $\frac{1}{2}(x_1+x_2)=m$

if $x_2=m$ then x_1 prefers equally all positions where she wins : only if $x_1 = m$ As game is symmetric this is the only NE



War of attrition

Model of a conflict between two animals fighting over prey. Each animal chooses the time at which it intends to give up. When an animal gives up, its opponent obtains all the prey (and the time at which the winner intended to give up is irrelevant). If both animals give up at the same time then they each have an equal chance of obtaining the prey. Fighting is costly: each animal prefers as short a fight as possible.

This game models not only such a conflict between animals, but also many other disputes. The “prey” can be any indivisible object, and “fighting” can be any costly activity—for example, simply waiting.

War of attrition

time t - continuous variable that starts at 0 and runs indefinitely

$v_i > 0$ - value party i attaches to the object in dispute

$\frac{1}{2} v_i$ - value party i attaches to a 50% chance of obtaining the object

Each unit of time that passes before the dispute is settled costs each party one unit of payoff.

if player *i gives up* first, at time t_i, *her payoff is* $-t_i$ *(she spends* t_i *units of time and does not obtain* the object).

If the other player gives up first, at time *tj, player i's payoff is* $v_i - t_j$ (she obtains the object after *tj units of time).*

If both players give up at the same time, player *i's payoff is* $\frac{1}{2}v_i - t_i$

War of attrition

Players: two opponents

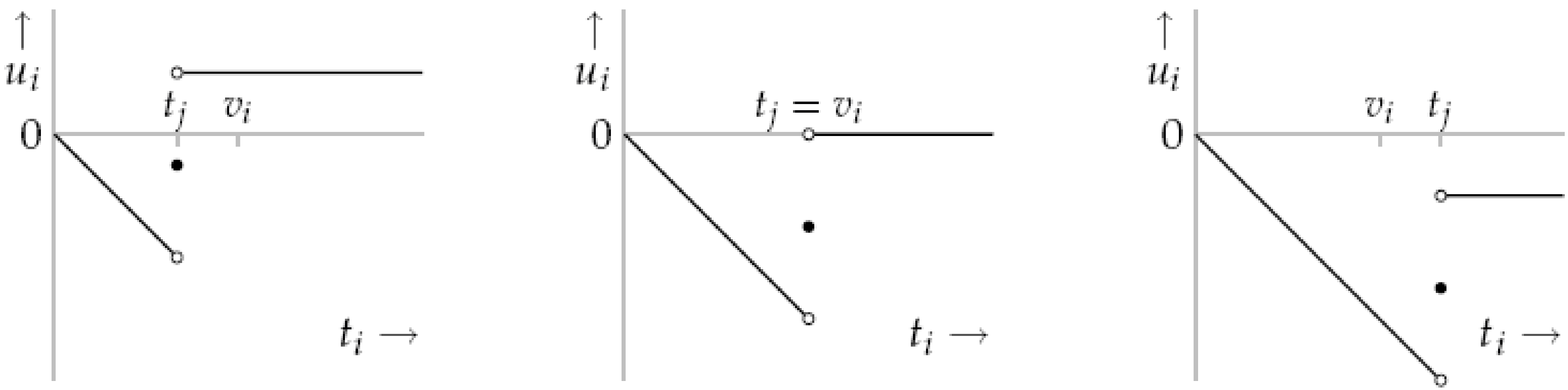
Actions:
for each player time till they give up

Preferences:
represented by payoff function

$$u_i(t_1,t_2) = \begin{cases} -\,t_i & \text{if } t_i < t_j \\ \tfrac{1}{2} v_i - t_i & \text{if } t_i = t_j \\ v_i - t_i & \text{if } t_i > t_j \end{cases}$$

War of attrition

Best response function: Intuitively, if player *j's intended giving up time* is early enough (t_j *is small) then it is optimal for player i to wait for player j to* give up. That is, in this case player *i should choose a giving up time later than* t_j*; any such time is equally good. By contrast, if player j intends to hold out for a* long time (t_j *is large) then player i should give up immediately.*



$t_j < v_i$ $\quad$ $t_j = v_i$ $\quad$ $t_j > v_i$

War of attrition

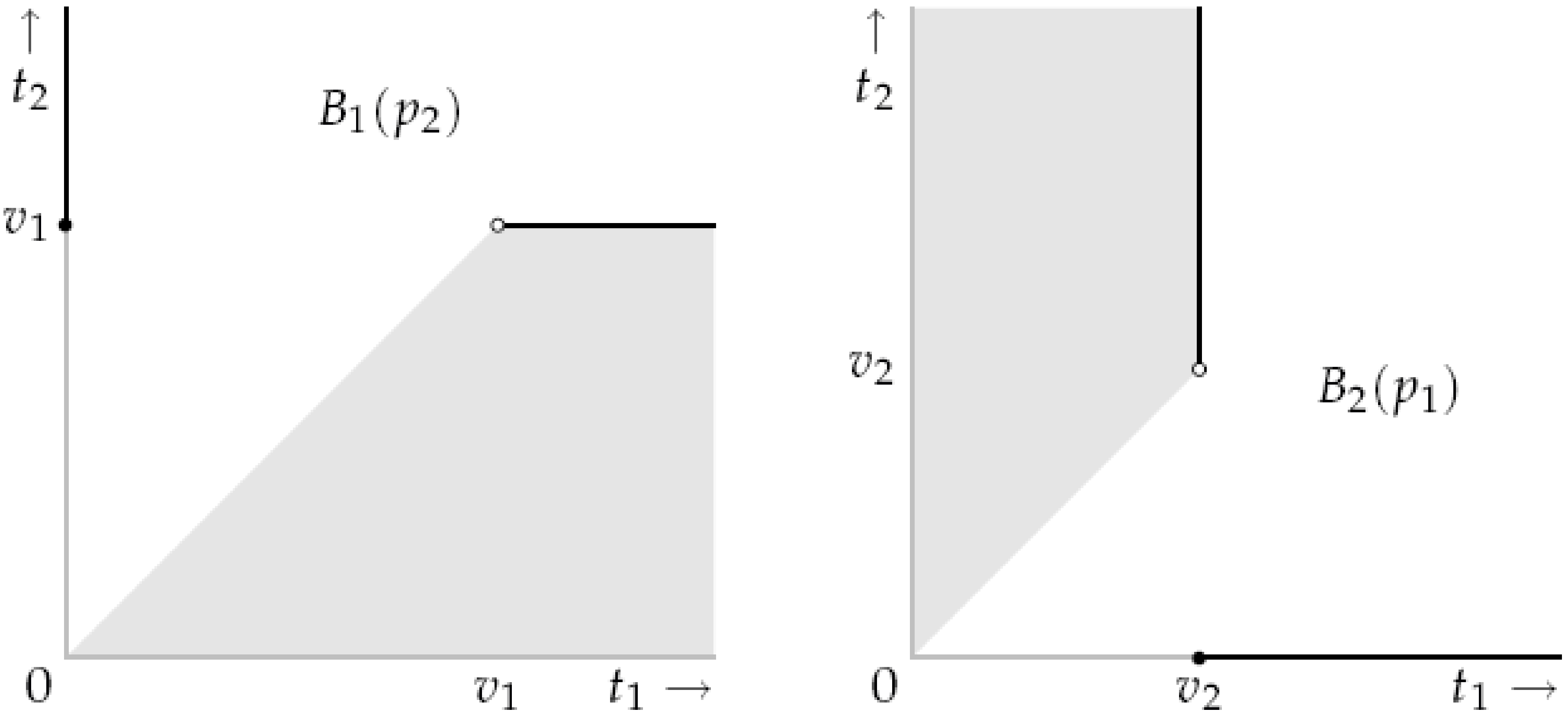
Best response functions:

$$B_i(t_j) = \begin{cases} t_i : t_i > t_j & \text{if } t_j < v_i \\ t_i : t_i = 0 \text{ or } t_i > t_j & \text{if } t_j = v_i \\ 0 & \text{if } t_j > v_i \end{cases}$$

War of attrition

Two Nash Equilibria:

$t_1 = 0 \; t_2 \geq v_1$ and $t_2 = 0 \; t_1 \geq v_2$



War of attrition

Three features of the equilibria are notable.

First, in no equilibrium is there any fight: one player always gives up immediately.

Second, either player may give up first, regardless of the players' valuations. In particular, there are always equilibria in which the player who values the object more gives up first.

Third, the equilibria are *asymmetric (the players' actions are different), even when* $v_1 = v_2$, in which case the game is symmetric—the players' sets of actions are the same and player 1's payoff to (t_1, t_2) *is the same as player 2's payoff to* (t_2, t_1) Given this asymmetry, the populations from which the two players are drawn must be distinct in order to interpret the Nash equilibria as action profiles compatible with steady states.

Equilibrium in a single population

A Nash equilibrium of a strategic game corresponds to a steady state of an interaction between the members of several populations, one for each player in the game, each play of the game involving one member of each population. Sometimes we want to model a situation in which the members of a *single homogeneous* population are involved anonymously in a symmetric interaction.

Definition: A two-player strategic game with vNM preferences is **symmetric if the players' sets** of actions are the same and the players' preferences are represented by the expected values of payoff functions u_1 *and* u_2 *for which* $u_1(a_1, a_2) = u_2(a_2, a_1)$ *for* every action pair (*a1, a2).*

Equilibrium in a single population

Definition: A profile α^* of mixed strategies in a strategic game with vNM preferences in which each player has the same set of actions is a **symmetric mixed strategy Nash equilibrium if it is a mixed** strategy Nash equilibrium and α^*_i is the same for every player i.

If we have symmetric game and asymmetric NE or MSNE then we have to have some social norm which will resolve who will take which action or strategy:

For example, the social norm in which the oldest person takes one action or similar. Otherwise asymmetric NE or MSNE would not satisfy the assumptions for the steady state.

Reporting a crime

A crime is observed by a group of n people. Each person would like the police to be informed, but prefers that someone else make the phone call. Specifically, suppose that each person attaches the value v to the police being informed and bears the cost c if she makes the phone call, where $v > c > 0$. Then the situation is modeled by the following strategic game with vNM preferences.

- Players: n people
- Actions: for each player {Call, Don't call}
- Preferences: *are represented by the expected value* of a payoff function that assigns 0 to the profile in which no one calls, $v - c$ to any profile in which she calls, and *v to any profile in which at least one* person calls, but she does not

Reporting a crime

The game has n pure Nash equilibria, in each of which exactly one person calls. (If that person switches to not calling, her payoff falls from $v - c$ to 0; if any other person switches to calling, her payoff falls from v to $v-c$.)

NE – NOT SYMMETRIC

If the members of the group differ in some respect, then these asymmetric equilibria may be compelling as steady states. For example, the social norm in which the oldest person in the group makes the phone call is stable.

If the members of the group either do not differ significantly or are not aware of any differences among themselves then there is no way for them to coordinate, and a symmetric equilibrium is more compelling.

Reporting a crime

No symmetric Nash equilibrium in pure strategies

Symmetric MSNE where each person calls with positive probability $p < 1$

In such equilibrium the expected payoffs of actions played with positive probability have to be equal

EU(person calls) = v – c (does not matter what the others will do, the payoff is still v – c)

EU(person does not call) = 0.P(no one else calls) + v . P (at least one other person calls)

Reporting a crime

$$v - c = 0 \cdot P(\text{no one else calls}) + v \cdot P(\text{at least one other person calls})$$

$$v - c = v \cdot (1 - P(\text{no one else calls}))$$

$$\frac{c}{v} = P(\text{no one else calls})$$

$$p \ldots\ldots P(\text{person will call})$$

$$P(\text{no one else calls}) = (1 - p)^{n-1}$$

$$\frac{c}{v} = (1 - p)^{n-1} \rightarrow p = 1 - \left(\frac{c}{v}\right)^{1/(n-1)}$$

Reporting a crime

Unique symmetric MSNE:

each person will call with probability $p = 1 - \left(\frac{c}{v}\right)^{1/(n-1)}$

That is, there is a steady state in which whenever a person is in a group of *n people facing the situation modeled by the game, she calls with given* probability.

When n is increasing, the probability that no one will call is increasing: As *n increases, 1/(n − 1) decreases, so that (c/v)1/(n−1) increases*

$$P(no\ one\ calls) = P(person\ idoes\ not\ call) \cdot P(no\ one\ else\ calls)$$

$$P(no\ one\ calls) = \left(\frac{c}{v}\right)^{1/(n-1)} \frac{c}{v}$$

Reporting a crime

That is, the larger the group, the less likely the police are informed of the crime! The condition defining a mixed strategy equilibrium is responsible for this result. For any given person to be indifferent between calling and not calling this condition requires that the probability that no one else calls be independent of the size of the group. Thus each person's probability of not calling is larger in a larger group, and hence the probability that no one calls is larger in a larger group.

In a larger group no individual is any less concerned that the police should be called, but in a steady state the behavior of the group drives down the chance that the police are notified of the crime.

Expert diagnosis

Something about which I am relatively ill-informed (my car, my computer, my body) stops working properly. I consult an expert, who makes a diagnosis and recommends an action. I am not sure if the diagnosis is correct - the expert, after all, has an interest in selling her services. I have to decide whether to follow the expert's advice or to try to fix the problem myself, put up with it, or consult another expert.

Two type of problems: minor, major;

r… fraction of major problems $0 < r < 1$

Expert recognize the type of the problem, consumer knows only the probability r

Expert diagnosis

Players: Expert and Consumer

Actions: Expert – report minor or major problem (do not have to be truth telling)

Consumer – accept recommendation or seek another remedy

Expert recognize the type of the problem, consumer knows only the probability r. A major repair fixes both a major problem and a minor one. A minor repair fixes just minor problem. Therefore Expert will never recommend minor repair to major problem. Further assume that Expert get same profit from selling minor repair to consumer with minor problem and major repair to consumer with major problem. However, he will get higher profit when selling major repair to minor problem.

Expert diagnosis

Players: Expert and Consumer

Actions: Expert – report minor or major problem (do not have to be truth telling)

Consumer – accept recommendation or seek another remedy

A consumer pays an expert E for a major repair and I < E for a minor one; the cost she effectively bears if she chooses some other remedy is E’ > E if her problem is major and I’ > I if it is minor. Further assume that E > I’.

Expert diagnosis

Under these assumptions the players have both basically just two reasonable actions:

Expert: Honest (recommend a minor repair for a minor problem and a major repair for a major problem)

Dishonest (recommend a major repair for both types of problem)

Consumer: Accept (buy whatever repair the expert recommends)

Reject (buy a minor repair but seek some other remedy if a major repair is recommended)

Expert diagnosis

Assume that each player's preferences are represented by her expected monetary payoff :

(*H*, *A*): *With probability r the consumer's problem is major, so she pays E, and* with probability 1 − *r it is minor, so she pays I.*

EU = −r E − (1 − r) I. The expert's profit is π.

(*D*, *A*): *The consumer's payoff is −E. The consumer's problem is major with* probability *r, yielding the expert π, and minor with probability 1 − r, yielding* the expert *π', EU=r π + (1 − r) π'.*

Expert diagnosis

Assume that each player's preferences are represented by her expected monetary payoff :

(H, R): *The consumer's cost is E' if her problem is major (in which case she rejects* the expert's advice to get a major repair) and *I if her problem is minor, EU=−r E' −(1 −r) I. The expert obtains a payoff only if the* consumer's problem is minor, in which case she gets *π; thus her expected* payoff is EU = (1 − *r) π.*

(D, R): *The consumer never accepts the expert's advice, and thus obtains the expected* payoff *−r E' − (1 − r) I'. The expert does not get any business, and* thus obtains the payoff of 0

Expert diagnosis

Consumer

Expert

	Accept(q)	Reject(1-q)
Honest(p)	**π, -rE-(1-r)I**	**$(1-r)\pi$, -rE'-(1-r)I**
Dishonest (1-p)	**$r\pi+(1-r)\pi'$, -E**	**0, -rE'-(1-r)I'**

No NE in pure strategies, only in mixed strategies

Expert diagnosis

Best response functions:

Expert: q=0 → p=1 ($0<(1-r)\pi$)

q=1 → p=0 ($r\pi < \pi + (1-r)\pi'$)

q: he is indifferent

$$q\pi + (1 - q)(1 - r)\pi = q[r\pi + (1 - r)\pi']$$

$$q = \pi / \pi'$$

Expert diagnosis

Best response functions:

Consumer: p=0

if $-E>-rE'-(1-r)I'$ then q=1 else q=0

p=1

q=1 ($-rE-(1-r)I>-rE'-(1-r)I$)

p: he is indifferent ($E>rE'+(1-r)I'$)

$$p[\, rE + (1-r)I \,] + (1-p)E =$$

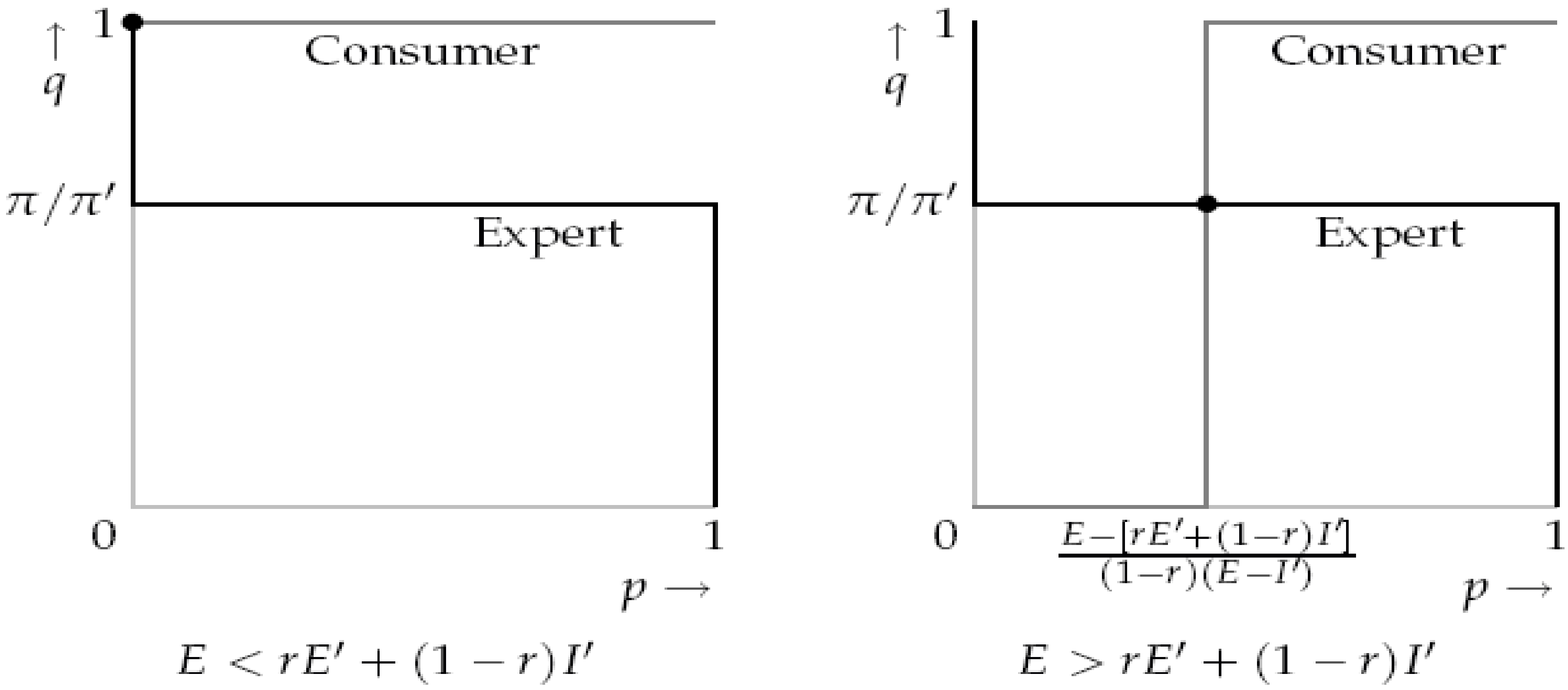
$$p[\, rE' + (1-r)I \,] + (1-p)[\, rE' + (1-r)I' \,]$$

$$p= [\, E - [\, rE' + (1-r)I' \,] \,] / [\, (1-r)(E-I') \,]$$

Expert diagnosis

NE: if E< rE'+(1-r)I' → (D,A)

MSNE: q = π / π' *and* $p = [\,E - [\,rE' + (1 - r)I'\,]\,] / [\,(1 - r)(E - I')\,]$



Contributing to a public good

Each of *n people chooses whether* or not to contribute a fixed amount toward the provision of a public good. The good is provided if and only if at least *k people contribute, where 2 ≤ k ≤ n; if* it is not provided, contributions are not refunded. Each person ranks outcomes from best to worst as follows: (*i) any outcome in which the good is provided and* she does not contribute, (*ii) any outcome in which the good is provided and she* contributes, (*iii) any outcome in which the good is not provided and she does not* contribute, (*iv) any outcome in which the good is not provided and she contributes.* Formulate this situation as a strategic game and find its Nash equilibria. (Is there a Nash equilibrium in which more than *k people contribute? One in which k people* contribute? One in which fewer than *k people contribute?*

Summary

- Illustrations of NE and MSNE
- Symmetric games
- Symmetric and Asymmetric NE and MSNE

- Gibbons 1.3; Osborne 3-4

NEXT WEEK:

Dynamic games, Subgame perfect, Backward induction