Mixed Strategies

Lecture 3

Revision

- Nash equilibrium (NE)
  - A *NE is an action profile* a^* *with the property that no* player *i can do better by choosing an action different from* a^*_i *, given* that every other player *j adheres to* a^*_j

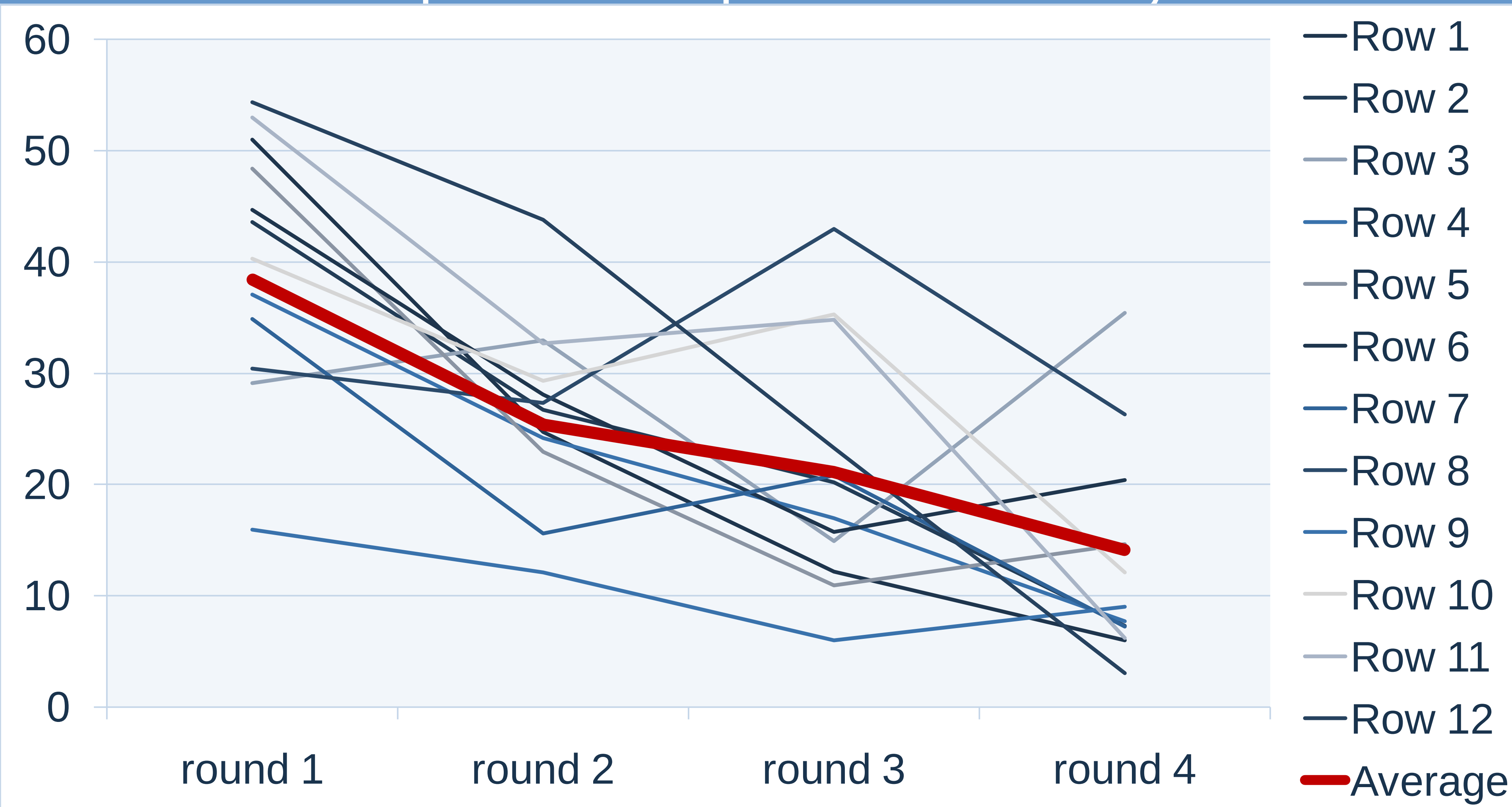
- Best response - set of actions $B_i(a_{-i})$ that gives the player the highest possible payoff given the other players' actions a_{-i}
- The action profile a^* is a Nash equilibrium if and only if every player's action is a best response to the other players' actions

Revision – Experiment

- You should choose a number in the interval [0–100]. You are allowed to choose 0 and 100. The winning number is the number closest to 2/3 of the average of all the numbers chosen by your group
- Iterative elimination of strictly dominated actions
  - Choosing number higher than 66 is dominated (100 is strictly dominated, the others weakly)
  - If all players knows that nobody will play above 66 than in next step number higher than 44 is dominated…
  - Following this way of thinking everybody should play 0
- NE
  - If players are experienced with their typical opponents they should all play 0 – the only NE in the game

Revision – Experiment

experience X perfect rationality



Mixed strategies

- Each player may chose not only one of his actions but also strategy in which she assigns probability to each of her actions:
  - 2 players, P1 actions: left, right
    P2 actions: left, middle, right
  - 6 action profiles: (L,L), (L,M), (L,R), (R,L), (R,M), (R,R)
  - Both players may not only choose one of their actions but also mixed strategy – assign probability to every action:
  - P1 plays left with probability ½ and right with ½
  - P2 plays left with probability ¼, middle with ¼ and right with ½
  - PROBABILITIES HAVE TO SUM TO 1!!

Mixed strategies

- Each player may chose not only one of his actions but also strategy in which she assigns probability to each of her actions:
- Mixed strategy - player assigns probabilities $p_1, p_2, \ldots, p_N$ to all of her actions and she is playing her actions randomly according to these probabilities
  - May model also population of several types of players who are playing different actions and players are drawn randomly from the population
- Pure strategy – players assigns probability 1 to one of her actions
  - PROBABILITIES HAVE TO SUM TO 1!!

Mixed strategies – preferences

- if P2 is playing mixed strategy then P1 have to decide whether she prefers
- ¼ (L,L)+ ¼ (L,M)+ ½ (L,R) when she is playing L
- or ¼ (R,L)+ ¼ (R,M)+ ½ (R,R) when she is playing R
- or p(¼ (L,L)+ ¼ (L,M)+ ½ (L,R)) +
- + (1-p)(¼ (R,L)+ ¼ (R,M)+ ½ (R,R))
  - When she is playing L with probability p and R with (1-p)
- Ordinal preferences are not enough to represent preferences over lotteries
- von Neumann-Morgenstern (vNM) preferences – represented by expected value of utility (payoff) function

Expected utility theory

Preferences over lotteries can be represented as expected value of a utility(payoff) function over deterministic outcomes:

there exist utility function u such that

- Player 1 prefers ¼ (L,L)+ ¼ (L,M)+ ½ (L,R) over
- ¼ (R,L)+ ¼ (R,M)+ ½ (R,R) if and only if

¼ u(L,L)+ ¼ u(L,M)+ ½ u(L,R)> ¼ u(R,L)+ ¼ u(R,M)+ ½ u(R,R)

!!!NOW the differences between payoffs does MATTER!!!

Example: P1 preferences u(L,L) = 2, u(L,M)=2, u(L,R)=1

u(R,L)= 0, u(L,M)=0, u(R,R)=2

¼*2+ ¼*2+ ½*1 = 1 ½ > 1 = ¼*0+ ¼*0+ ½*2

Expected utility theory

- vNM preferences - significantly stronger assumption than just ordinal preferences over deterministic outcomes
- Roughly capture the key essence of decisions and preferences of people in many situations under uncertainty
  - dominant theory in standard economic theory
  - experimental evidence that people not always behave according to the theory
  - Following standard game theory we will use it for the rest of the course
- Several other theories
  - Cumulative prospect theory, Rank-dependent utility theory
    - Assume that people weight probabilities non-linearly

Notation

- $\mathbf{a_i}$ - particular action of i^{th} player
- **a** - action profile = set of actions of all players
- $\alpha_i = (p_1, p_2, p_3, \ldots, p_N)$ – particular mixed strategy of i^{th} player $\quad p_1+p_2+p_3+ \ldots +p_N = 1$

  if $p_k=1$ then $\alpha_i = a_i$ - mixed strategies incorporate also pure strategies
- $\boldsymbol{\alpha}$ – mixed strategy profile = set of mixed strategies of all players (includes pure strategies)

  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \ldots, \alpha_{N-2}, \alpha_{N-1}, \alpha_N)$
- $\boldsymbol{\alpha_{-i}}$ - action profile of mixed strategies of all players except i^{th} player – again including both pure and mixed strategies

Static game of complete inf.

- Set of players
  - firms, political candidates, bidders, etc.
- For each player set of actions
  - each action may affect also other players
  - $a_1, \ldots, a_N$ – different choices of behavior for each player
- For each player set of preferences over the set of action profiles and regarding lotteries over action profiles that may be represented by the expected value of utility (payoff) function over action profiles

MSNE: Matching Pennies

Person 2

Person 1

	Head(q)	Tail(1-q)
Head(p)	**1, -1**	**-1, 1**
Tail(1-p)	**-1, 1**	**1, -1**

No NE in pure strategies, only in mixed strategies

MSNE: Matching Pennies

- **P1 best response** – if person 2 is playing Head with probability q and Tail with (1-q)
- **Expected utility(payoff) of Person 1 playing:**
- **Head:** $1q+(-1)(1-q) = 2q-1$
- **Tail:** $(-1)q+1(1-q) = 1-2q$

If $q>½$ best response is playing Head if $q<½$ best response is playing Tail

- **Head with p and Tail with (1-p)**

$p(2q-1)+(1-p)(1-2q)=2pq-p+1-2q-p+2pq =$

$= 4pq-2p-2q+1$

MSNE: Matching Pennies

- if person 2 is playing Head with probability ½ and Tail with ½
- **Expected utility(payoff) of Person 1 playing:**
- **Head:** $1q+(-1)(1-q) = 2q-1=2*½-1=0$
- **Tail:** $(-1)q+1(1-q) = 1-2q=1-2*½=0$
- **Head with p and Tail with (1-p)**

  $p(2q-1)+(1-p)(1-2q)=2pq-p+1-2q-p+2pq =$

  $= 4pq-2p-2q+1=4p*½ -2p-2*½+1=2p-2p+0=0$

  Every action or mixed strategy is best response when person 2 is playing mixed strategy with q=½

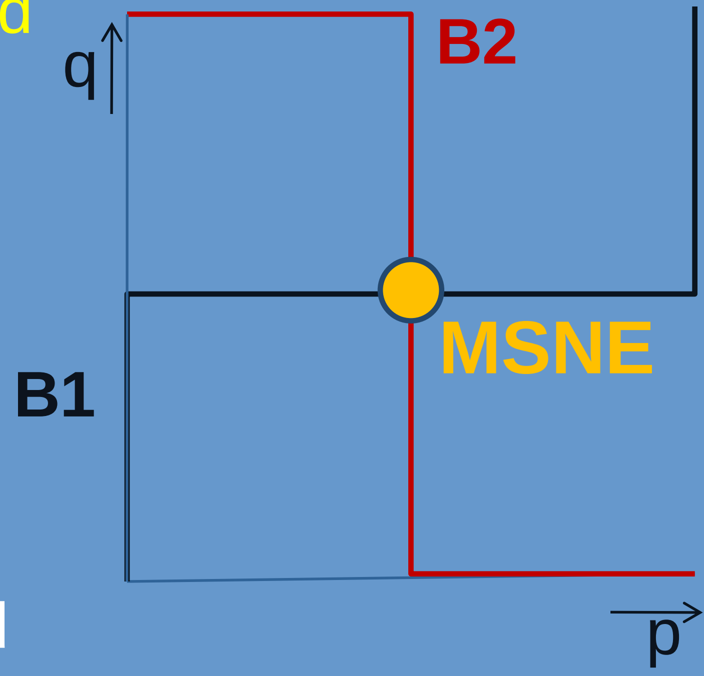
MSNE: Matching Pennies

p - probability of Player 1 playing Head
q - probability of Player 2 playing Head

Best response of player 1 (B1) is to play Tail (Head with p=0) if q<0.5

If q=0.5 all strategies are best response

If q>0.5 best response is to play Head with p=1



Mixed strategy Nash Equilibrium

- Definition:

  The mixed strategy profile α^* in a static game with vNM preferences is a **mixed strategy Nash equilibrium (MSNE)** **if, for each player i and every mixed strategy** β_i of player i, the expected utility(payoff) to player i of α^* is at least as large as the expected utility(payoff) to player i of $(\beta_i , \alpha^*_{-i})$ according to a utility(payoff) function whose expected value represents player i's preferences over lotteries.

Equivalently, for each player i,

$EU_i(\alpha^*) \geq EU_i(\beta_i, \alpha^*_{-i})$ for every mixed strategy β_i of player i, where $EU_i(\alpha)$ is player i's expected utility(payoff) to the mixed strategy profile α.

Mixed strategy Nash Equilibrium

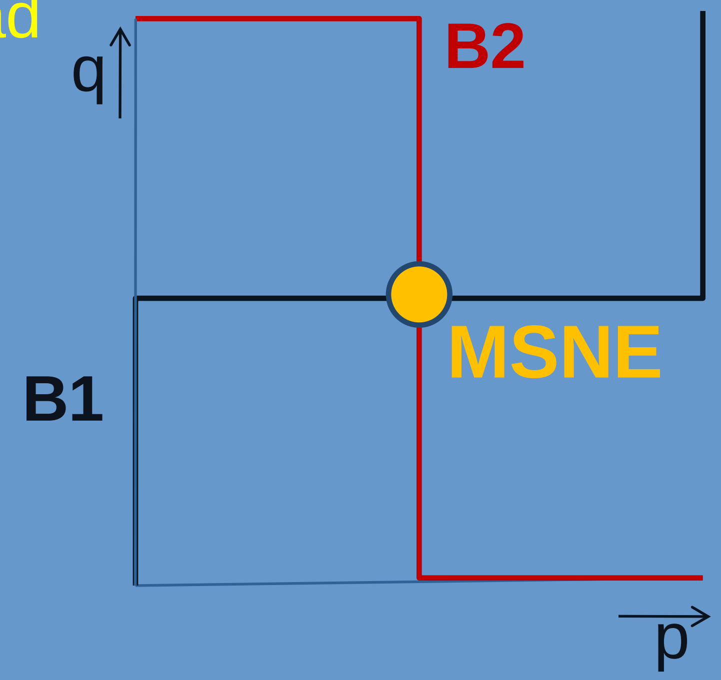
- Definition:

  the mixed strategy profile α^* is a mixed strategy Nash equilibrium if and only if α^*_i is in $B_i(\alpha^*_{-i})$ for every player i

MSNE: Matching Pennies

p - probability of Player 1 playing Head
q - probability of Player 2 playing Head

MSNE: p=0.5 and q=0.5
Player 1: $\alpha^*_1=(0.5;0.5)$
Player 2: $\alpha^*_2=(0.5;0.5)$



MSNE: Bach or Stravinsky?

Friend 2

Friend 1

	Bach	Stravinsky
Bach	**2, 1**	**0, 0**
Stravinsky	**0, 0**	**1, 2**

Strict NE

MSNE: Bach or Stravinsky?

Friend 2

Friend 1

	Bach(q)	Stravinsky(1-q)
Bach(p)	**2, 1**	**0, 0**
Stravinsky(1-p)	**0, 0**	**1, 2**

MSNE: Matching Pennies

- if friend 2 is playing Bach with probability q and Stravinsky with (1-q)
- **Expected utility(payoff) of Friend 1 playing:**
- **Bach:** $2q+0(1-q) = 2q$
- **Stravinsky:** $0q+1(1-q) = 1-q$

Bach is best response if $2q>1-q \rightarrow 3q>1 \rightarrow q>1/3$

Stravinsky is best response if $2q<1-q \rightarrow q< 1/3$

All mixed strategies are best response if $q= 1/3$

MSNE: Matching Pennies

- if friend 1 is playing Bach with probability p and Stravinsky with (1-p)
- **Expected utility(payoff) of Friend 2 playing:**
- **Bach:** 1p+0(1-p) = p
- **Stravinsky:** 0p+2(1-p) = 2-2p

Bach is best response if $p>2-2p \rightarrow 3p>2 \rightarrow p>\frac{2}{3}$

Stravinsky is best response if $p<2-2p \rightarrow p< \frac{2}{3}$

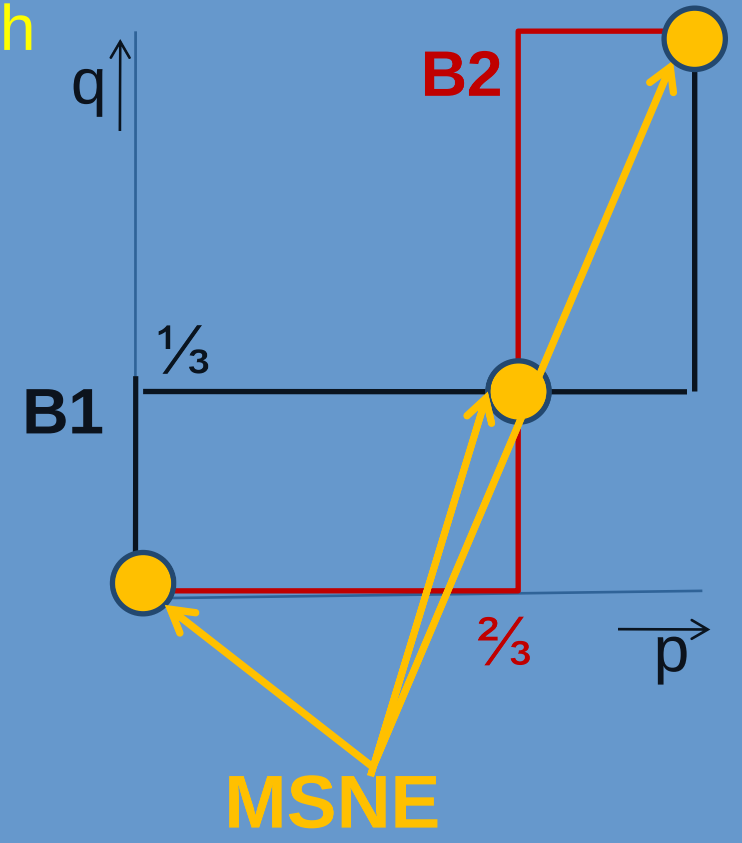
All mixed strategies are best response if $p= \frac{2}{3}$

MSNE: Bach or Stravinsky?

p - probability of Player 1 playing Bach
q - probability of Player 2 playing Bach

$$B1(q) = \begin{cases} 0 & \text{if } q < \frac{1}{3} \\ p: 0 \le p \le 1 & \text{if } q = \frac{1}{3} \\ 1 & \text{if } q > \frac{1}{3} \end{cases}$$

$$B2(p) = \begin{cases} 0 & \text{if } p < \frac{2}{3} \\ q: 0 \le q \le 1 & \text{if } p = \frac{2}{3} \\ 1 & \text{if } p > \frac{2}{3} \end{cases}$$



How to check MSNE

- *A mixed strategy profile α^* in a static game with vNM preferences in which each player has finitely many actions is a mixed strategy Nash equilibrium if and only if, for each player i,*
  - *the expected utility(payoff), given α^*_{-i}, to every action to which α^*_i assigns positive probability is the same*
  - *the expected utility(payoff), given α^*_{-i}, to every action to which α^*_i assigns zero probability is at most the expected utility(payoff) to any action to which α^*_i assigns positive probability.*

How to check MSNE

- *expected utility, given α^*_{-i}, to every action to which α^*_i assigns positive probability is the same*
- *expected utility, given α^*_{-i}, to every action to which α^*_i assigns zero probability is at most the expected utility to any action to which α^*_i assigns positive probability.*

P 2

P 1

	L(0)	C(⅓)	R(⅔)
T(¾)	0 , 2	3 , 3	1 , 1
M(0)	1 , 0	0 , 2	2 , 0
B(¼)	0 , 4	5 , 1	0 , 7

How to find all MSNE

You should check for MSNE all combinations. That is, you should check whether there are equilibria, in which one player chooses a pure strategy and the other mixes; equilibria, in which both mix; and equilibria in which neither mixes. Note that the mixtures need not be over the entire strategy spaces, which means you should check every possible subset.

- 2 *2 two-player game, each player has three possible choices: two in pure strategies* and one that mixes between them:

9 combinations

- 3 *3 two-player game, each player has 7 choices: three pure strategies, one completely* mixed, and three partially mixed

49 combinations

How to find all MSNE

Check for pure strategies

Player 2

Player 1

	B	S	X
B	**4, 2**	**0, 0**	**0, 1**
S	**0, 0**	**2, 4**	**1, 3**

How to find all MSNE

Check one mixing, the other pure strategies

If player 1 is playing pure strategy and player 2 mixed, first condition not satisfied

If player 2 is playing pure strategy and player 1 mixed, first condition not satisfied

Player 2

Player 1

	B	S	X
B	**4, 2**	**0, 0**	**0, 1**
S	**0, 0**	**2, 4**	**1, 3**

How to find all MSNE

If player 1 is playing B with probability p

If player 2 is playing mixed strategy – combination of just two from three of his actions

1) B and S: $2p = 4(1 - p) \geq p + 3(1 - p)$ – *not possible*

2) B and X: $2p = p + 3(1 - p) \geq 4(1 - p)$ *and* $4q = 1 - q$ *:* $p=¾$ $q=1/5$

3) S and X: not possible → p=1

Player 2

	B	S	X
B	**4, 2**	**0, 0**	**0, 1**
S	**0, 0**	**2, 4**	**1, 3**

Player 1

How to find all MSNE

If player 1 is playing B with probability p

If player 2 is playing mixed strategy with all of her actions

$2p = 4(1 - p) = p + 3(1 - p)$ – not possible

		Player 2		
		B	S	X
Player 1	B	4, 2	0, 0	0, 1
	S	0, 0	2, 4	1, 3

Strictly Dominated strategies

- *Strict domination in a static game with vNM preferences*
- Definition: player *i's mixed strategy* ***α_i strictly dominates her action a_i if*** $U_i(\alpha_i, a_{-i})$ **>** $u_i(a_i, a_{-i})$ *for every list a_{-i} of the other players' actions,* where *u_i is a utility(payoff)function that represents player i's preferences over lotteries and $U_i(\alpha_i, a_{-i})$ is player i's expected utility(payoff under u_i when she uses the mixed strategy α_i and the actions chosen by the other players are given by a_{-i}*

Strictly Dominated strategies

- *Strict domination in a static game with vNM preferences*
- Definition: If any action or mixed strategy strictly dominates the action b_i, *we say that* b_i *is* ***strictly dominated***

- Strictly dominated action is never played with positive probability in any MSNE and thus also in pure NE.
- When finding all MSNE it is sometimes useful to start with searching for strictly dominated strategies and use iterative elimination.

Weakly Dominated strategies

- Definition: player *i's mixed strategy* α_i ***weakly dominates her action*** a_i ***if*** $U_i(\alpha_i, a_{-i}) \geq u_i(a_i, a_{-i})$ *for every list* a_{-i} *of the other players' actions,* where u_i *is a utility(payoff)function that represents player i's preferences over lotteries and* $U_i(\alpha_i, a_{-i})$ *is player i's expected utility(payoff under* u_i *when she uses the mixed strategy* α_i *and the actions chosen by the other players are given by* a_{-i}

Strictly Dominated strategies

- Find all Nash Equilibria of the game (also in mixed strategies)

P 2

P 1

	L	C	R
T	5 , 8	3 , 4	1 , 3
M	2 , 0	1 , 2	5 , 0
B	0 , 1	5 , 5	10, 8

Strictly Dominated strategies

Find strictly dominated strategies using also mixed strategies and use Iterative elimination of strictly dominated strategies – if mixed strategy using any two actions strictly dominates the other, sum of their payoffs is higher than the other

P 2

P 1

	L	C	R
T	5 , 8	3 , 4	1 , 3
M	2 , 0	1 , 2	5 , 0
B	0 , 1	5 , 5	10, 8

Strictly Dominated strategies

M strictly dominated by P1 playing T with probability ½ and B with probability ½

P 2

P 1	L	C	R
T	5 , 8	3 , 4	1 , 3
~~M~~	~~2 , 0~~	~~1 , 2~~	~~5 , 0~~
B	0 , 1	5 , 5	10, 8

Strictly Dominated strategies

Find strictly dominated strategies using also mixed strategies and use Iterative elimination of strictly dominated strategies - if mixed strategy using any two actions strictly dominates the other, sum of their payoffs is higher than the other

P 2

P 1

	L	C	R
T	5 , 8	3 , 4	1 , 3
B	0 , 1	5 , 5	10, 8

Strictly Dominated strategies

C strictly dominated by P2 playing L with probability ¼ and R with probability ¾

P 2

	L	C	R
T	5 , 8	3 , 4	1 , 3
P 1 B	0 , 1	5 , 5	10, 8

Strictly Dominated strategies

Find all MSNE of the game

P 2

	L	R
T	5 , 8	1 , 3
B	0 , 1	10, 8

P 1

Strictly Dominated strategies

2 NE in pure strategies

P 2

P 1		L	R
	T	5 , 8	1 , 3
	B	0 , 1	10, 8

Strict NE

Strictly Dominated strategies

Find all MSNE of the game – check all combinations

both pure strategies – already done

one pure, the other mixed – not possible

P 2

	L	R
T	5 , 8	1 , 3
B	0 , 1	10, 8

P 1

Strictly Dominated strategies

If player 1 is playing T with probability p and player 2 L with probability q:

$8p+1(1-p)=3p+8(1-p) \rightarrow 12p=7 \rightarrow p = 7/12 \ (1-p) = 5/12$

$5q+1(1-q)=0q+10(1-q) \rightarrow 14q=9 \rightarrow q = 9/14 \ (1-q) = 5/14$

P 2

P 1

	L(q)	R(1-q)
T(p)	5 , 8	1 , 3
B(1-p)	0 , 1	10, 8

Summary

- Mixed strategies
- Mixed strategy Nash equilibrium
- Iterative Elimination of Strictly Dominated strategies using also Mixed strategies
- Gibbons 1.3; Osborne 4

NEXT WEEK:

Examples of NE and MSNE