DYNAMIC GAMES with incomplete information

Lecture 12

Revision – weak sequential eq.

- Nash Equilibrium
  - consists only of strategies (plan of action for every information set) of all players
  - given these strategies nobody has any incentive to deviate
- Information partition and information sets
  - defines the amount of information
  - collection of decision nodes (histories after which it is player's turn) such that the player does not know which node in the information set has been reached
- Beliefs - assign probability distribution over the decision nodes (histories) in each information set
- Behavioral strategy - PLAN OF ACTION - assigns action for each information set at which it is the player's turn

Revision – weak sequential eq.

- Weak sequential eq. - Refinement of Nash Equilibrium
  - Consists of strategies and beliefs (probabilities assigned to every decision node in every information set)
  - 2 conditions – consistency of beliefs and sequential rationality
  - consistency of beliefs – beliefs reflect the strategies (in stable state the beliefs should reflect the frequencies of occurrence of particular decision nodes)
    - If 1st player have two choices – LEFT and RIGHT and is playing mixed strategy 50% LEFT, 50% RIGHT, the 2nd player cannot in equilibrium believe that occurrence of history LEFT is 10% and RIGHT 90%, she has to have beliefs 50% and 50%
  - sequential rationality – given the consistent beliefs, nobody has any incentive to deviate in any stage of the game

Revision – weak sequential eq.

Definition: A weak sequential equilibrium consists of behavioral strategies and beliefs system satisfying following conditions 1-2

1. Sequential rationality - Each players' strategy is optimal whenever she has to move, given her belief and the other players' strategies.
2. Consistency of beliefs with strategies – Each player's belief is consistent with strategy profile (behavioral strategies of all players)

Signaling games

Signaling game is a dynamic game of incomplete information involving two players: a Sender (S), and a Receiver (R). The timing of the game is as follows:

1) Chance (Nature) draws a type t_i for the Sender from a set of feasible types $T=\{t_1,\ldots t_N\}$ according to a probability distribution $\{p_1,\ldots, p_N\}$ such that $p_1+\ldots+p_N=1$

2) The sender observes her type and then chooses a message (signal) m_i from a set of feasible messages $M=\{m_1,\ldots m_J\}$

3) The Receiver observes m_j (but not t_i) and then chooses an action a_k from a set of feasible actions $A=\{a_1,\ldots a_K\}$

Example 1: Signaling game – 2 types

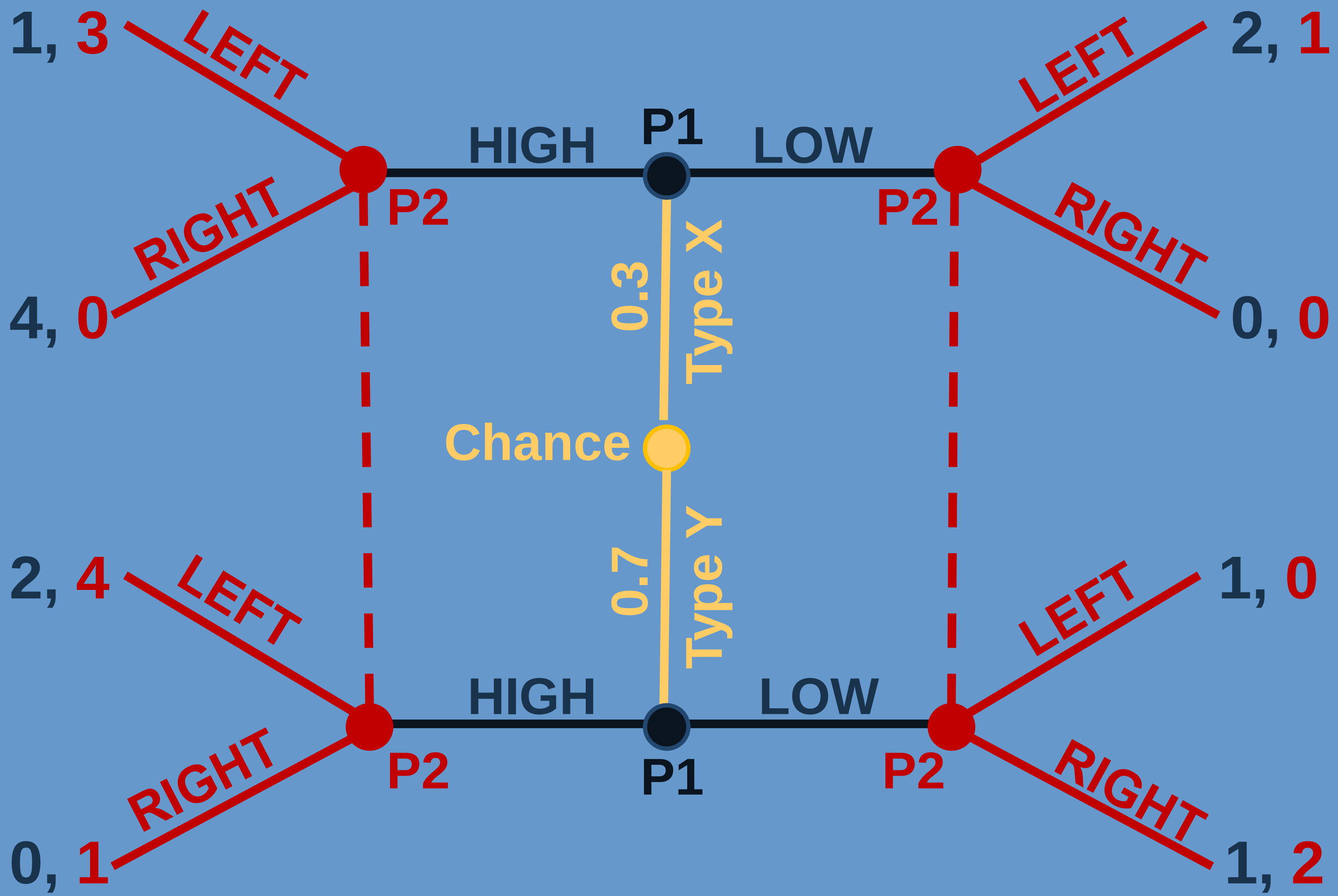
Signaling game is a dynamic game of incomplete information involving two players: a Sender (S), and a Receiver (R). The timing of the game is as follows:

1) Chance (Nature) draws a type t_i for the Sender from a set of feasible types T={X, Y} according to the probability distribution such that $p_X+p_Y=1$ ($p_X=0.3$; $p_Y=0.7$)

2) The sender observes her type and then chooses a message (signal) m_i from a set of feasible messages M={High,Low}

3) The Receiver observes m_j (but not t_i) and then chooses an action a_k from a set of feasible actions A={Left,Right}

Find all pure strategy weak sequential equilibria



Finding weak sequential equilibria

1) If the game has any subgame → No subgame
2) Identify all possible strategies of all players

   Player 1: {High,High}, {High,Low}, {Low,High}, {Low,Low} – the first action is when P1 is type X, the second action when he is type Y
   Player 2: {Left,Left}, {Left,Right}, {Right,Left}, {Right,Right} – the first action is when P2 observes High, the second action he observes Low

3) In our quite simple games - start from the beginning by analyzing one after each other strategies of the first player and compute the respective beliefs of the other players, given the strategy of the first player

Separating vs. Pooling equilibrium

Separating equilibrium: Each type of sender chooses a different action, so that upon observing the sender's action, the receiver knows the sender 's type

(for example : equilibria, where P1's strategy is {High, Low} or {Low, High} – first P1 type X, second P1 type Y)

Pooling equilibrium: All types of the sender choose the same action, so that the sender's action gives the receiver no clue to the sender's type.

(for example : equilibria, where P1's strategy is {High, High} or {Low, Low} – first P1 type X, second P1 type Y)

Partially separating/pooling equilibrium: Some types of sender send the same message, while some others sends some other messages. (for example when P1 X plays always High and P2 mixes High(0.5)Low(0.5))

Finding weak sequential equilibria

1) If the game has any subgame → find at first the weak sequential equilibria of the subgame

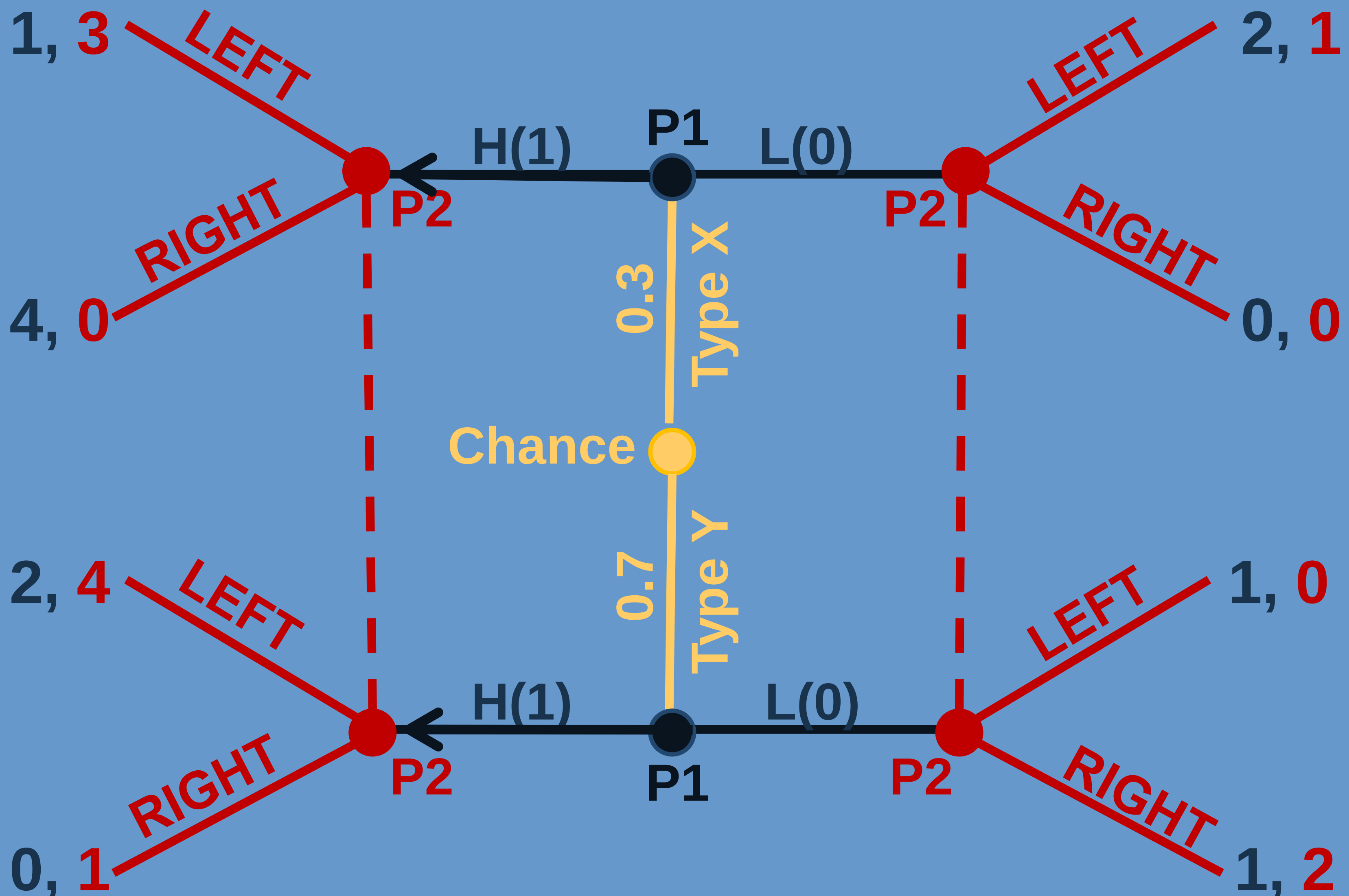
2) Identify all possible strategies of all players

SIMPLE GAMES: (2 players, finite number of actions)

3) In our quite simple games - start from the beginning by analyzing one after each other strategies of the first player and compute the respective beliefs of the other players, given the strategy of first player
4) Continue by finding the optimal strategies of further players, given their beliefs and strategies of the other players.
5) Check for the equilibrium

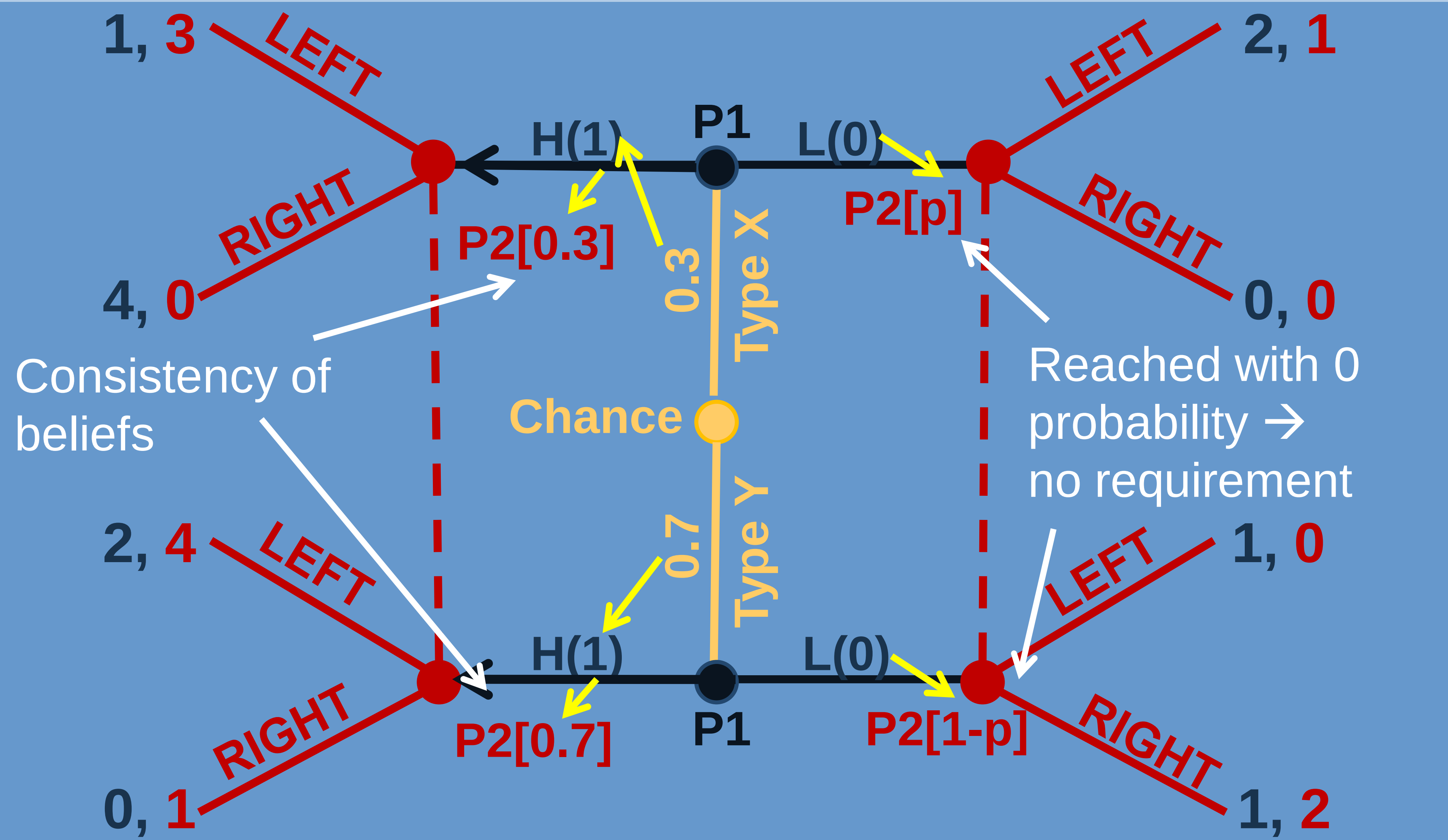
Pooling equilibria? P1: {High, High}

start with some strategy of P1



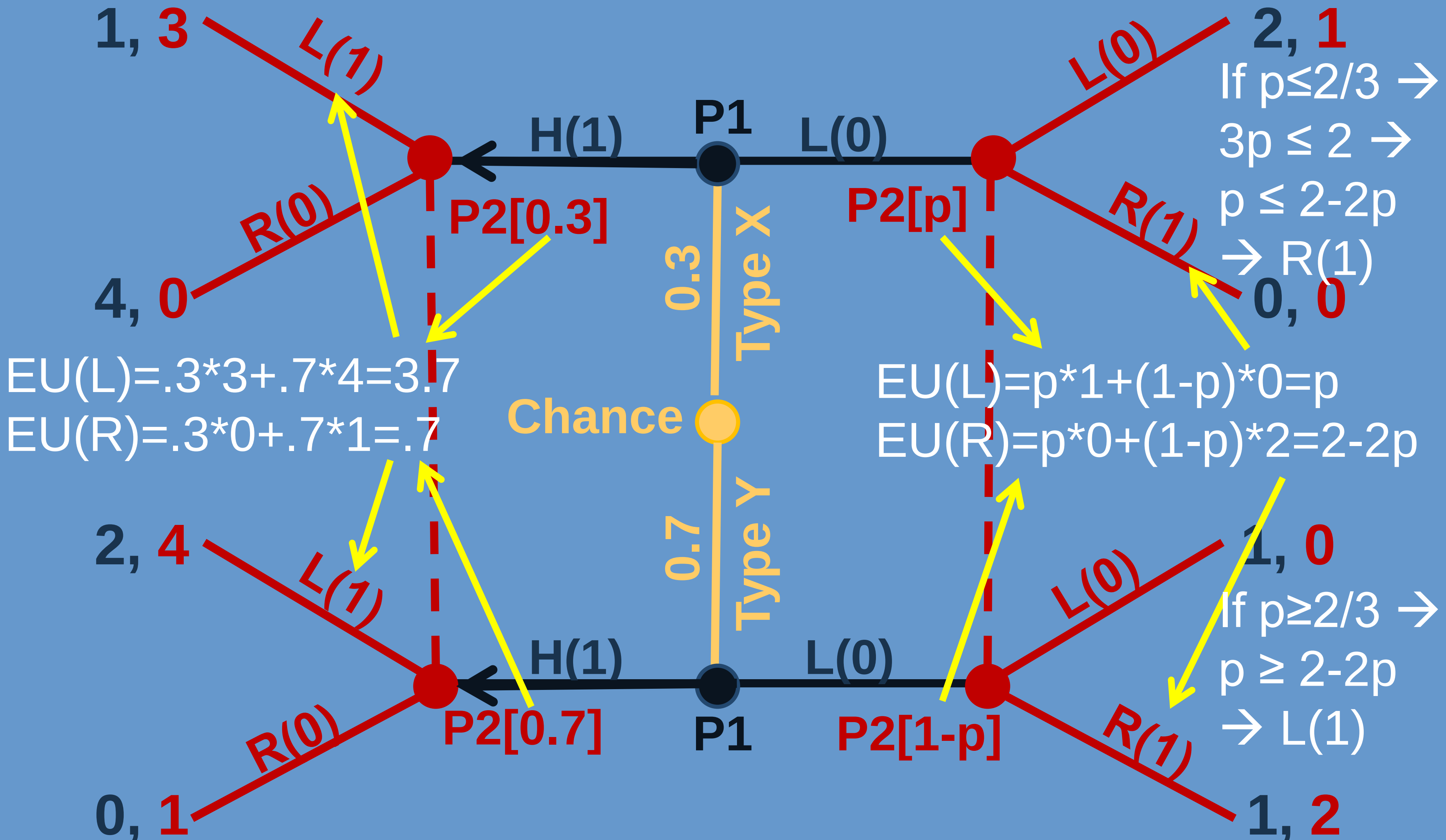
Pooling equilibria? P1: {High, High}

find consistent beliefs of P2



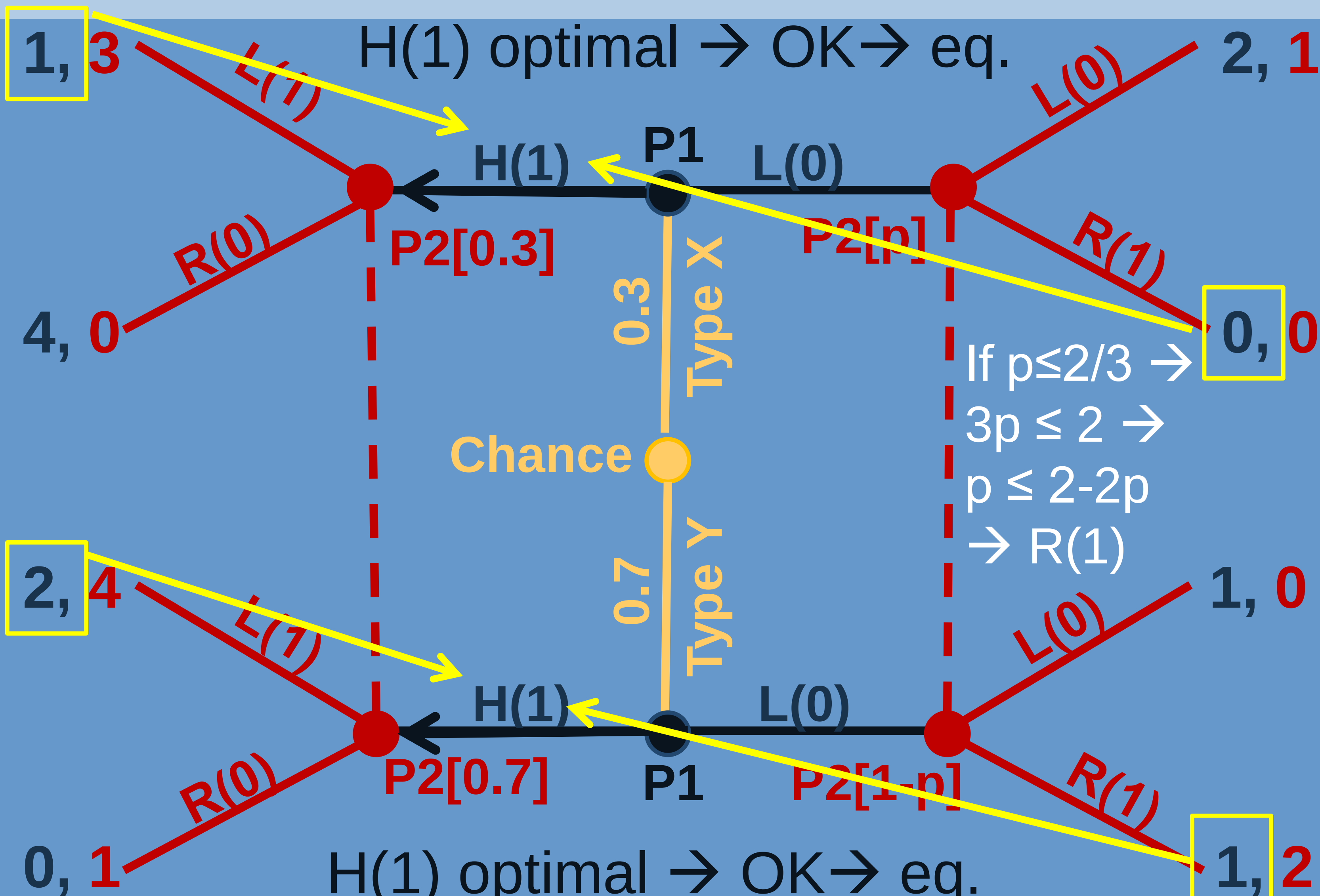
Pooling equilibria? P1: {High, High}

find optimal strategies of P2



Pooling equilibria? P1: {High, High}

check for equilibrium – whether strategy of P1 is optimal



Pooling equilibria? P1: {High, High}

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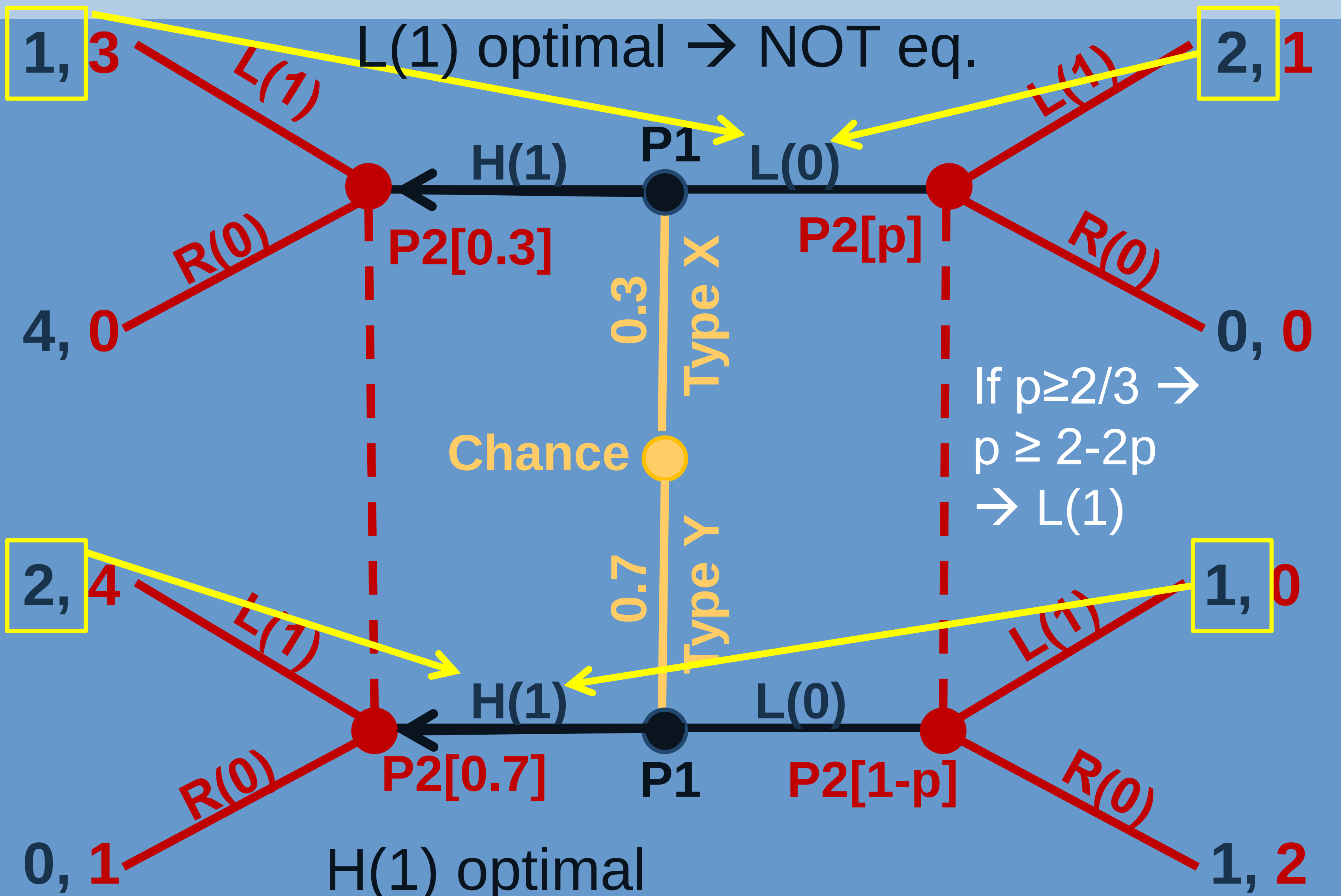
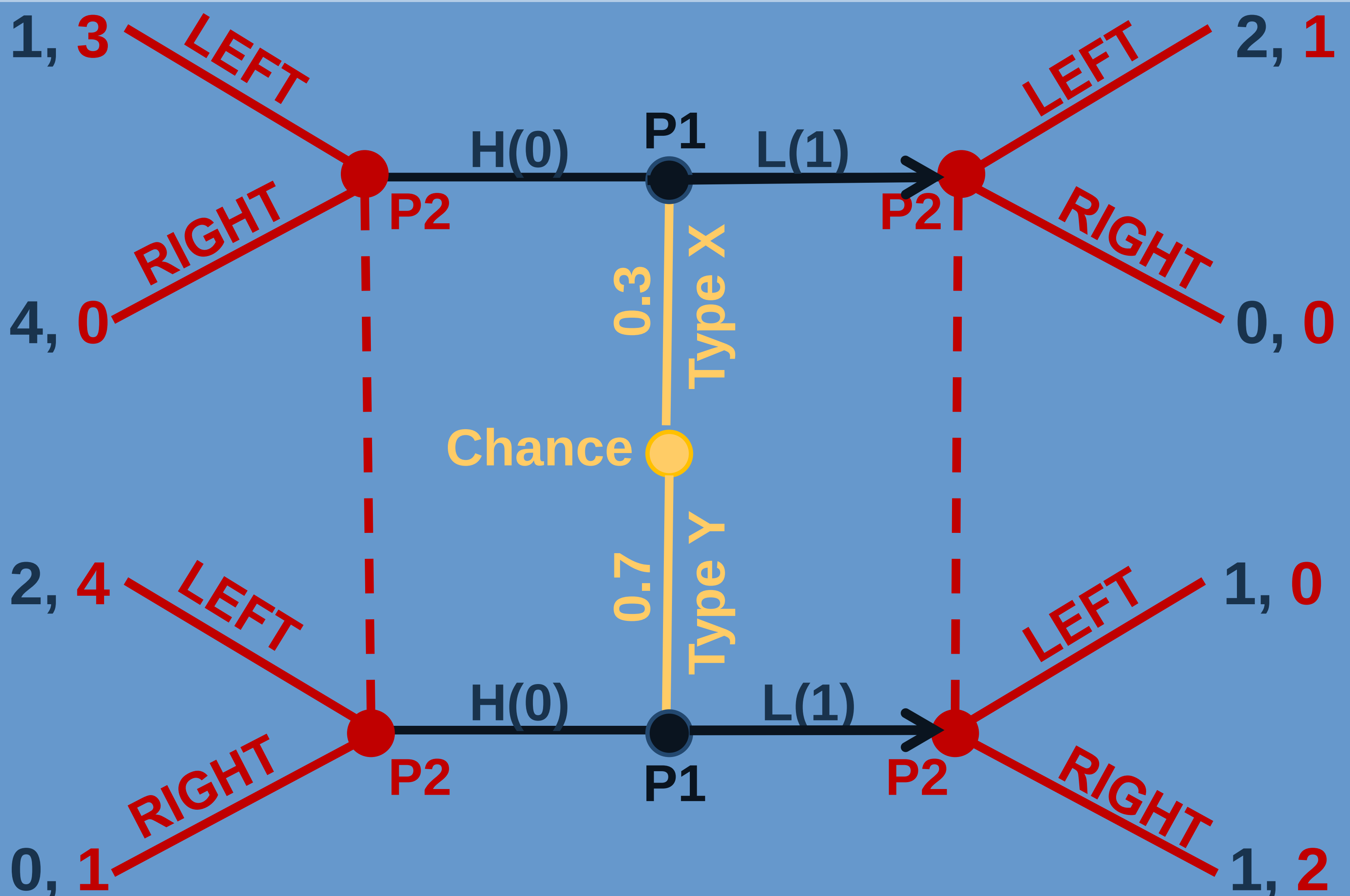


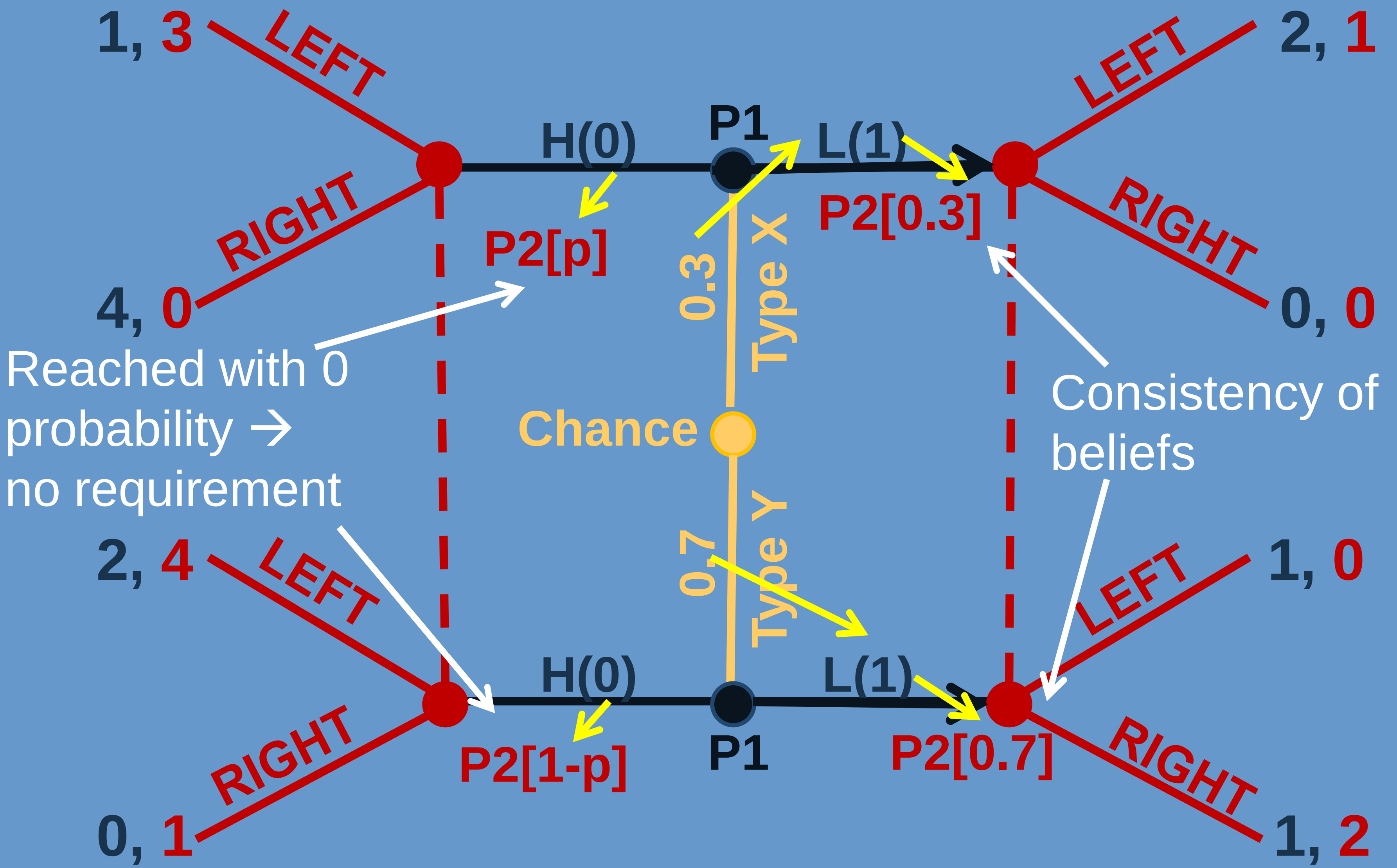
Pooling equilibria? P1: {Low, Low}

start with some strategy of P1



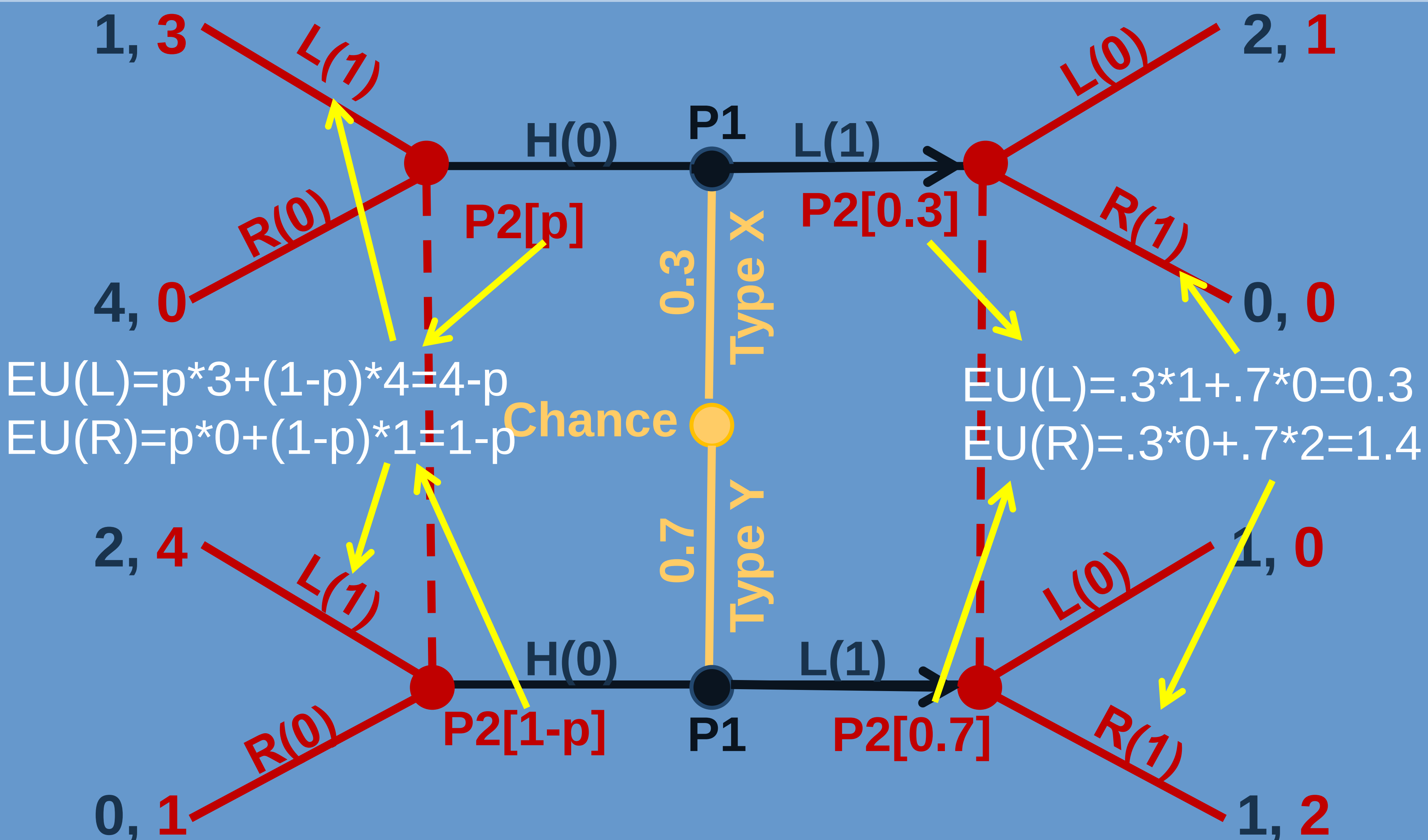
Pooling equilibria? P1: {Low, Low}

find consistent beliefs of P2



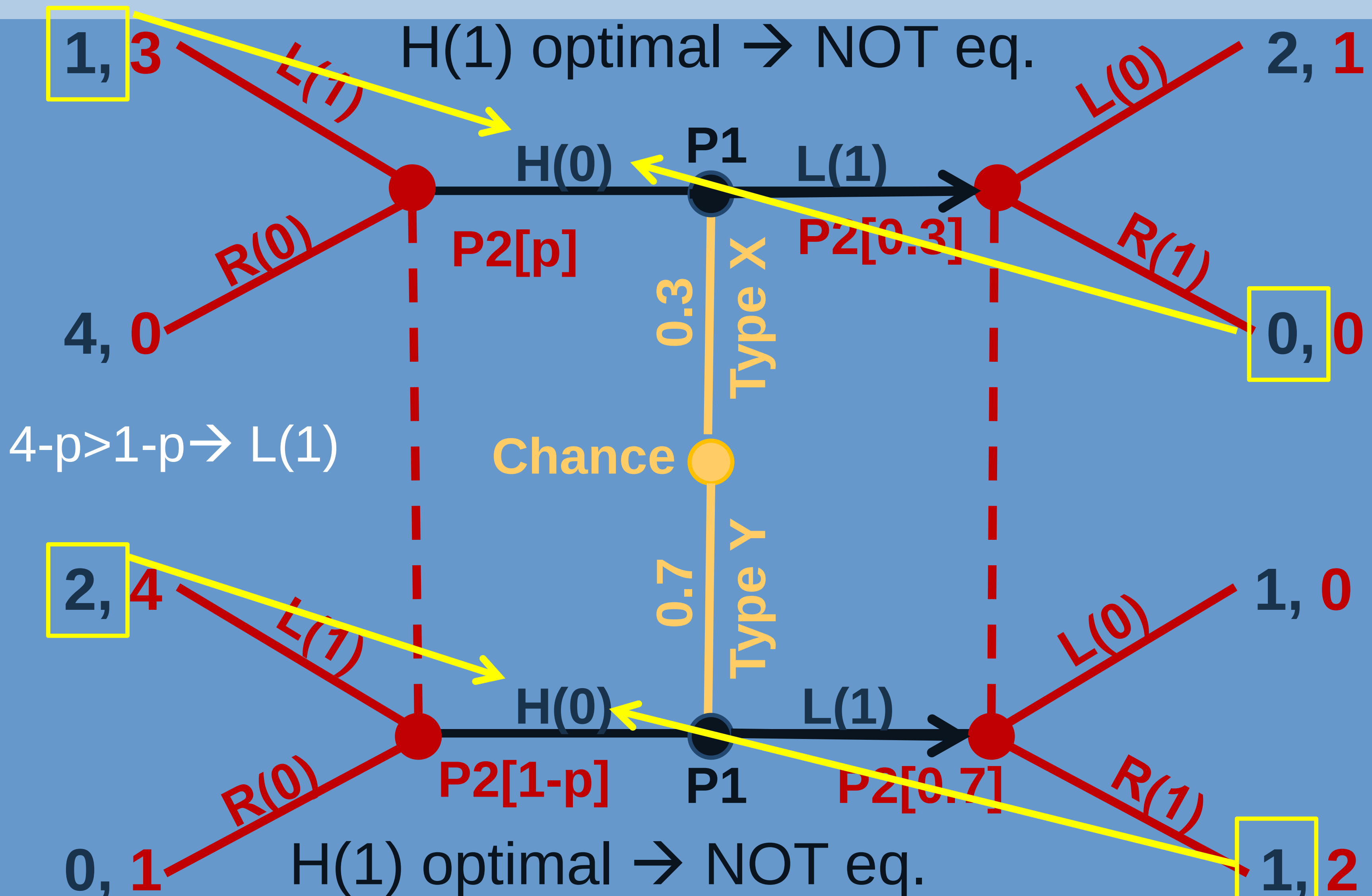
Pooling equilibria? P1: {Low, Low}

find optimal strategies of P2



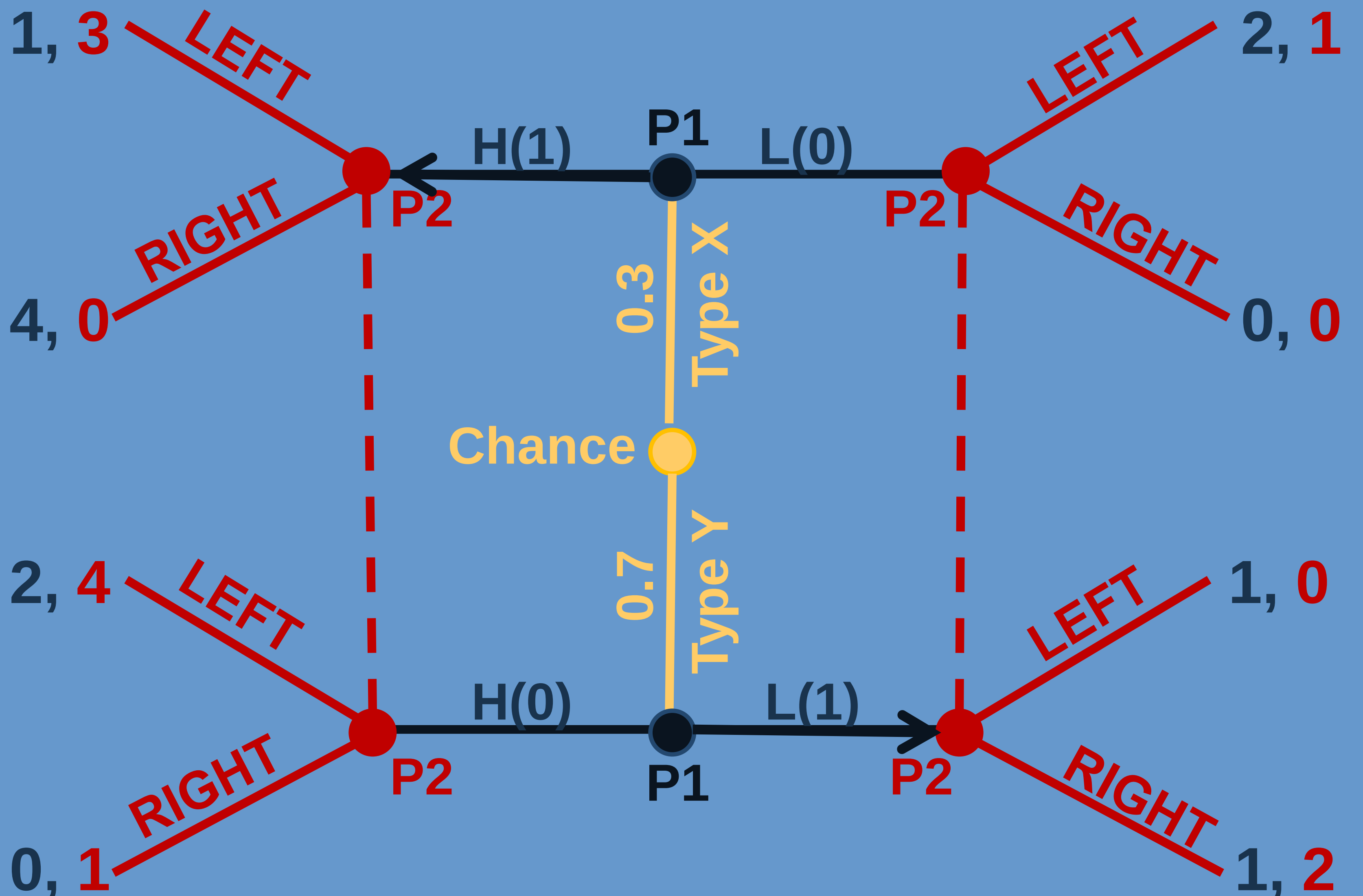
Pooling equilibria? P1: {Low, Low}

check for equilibrium – whether strategy of P1 is optimal



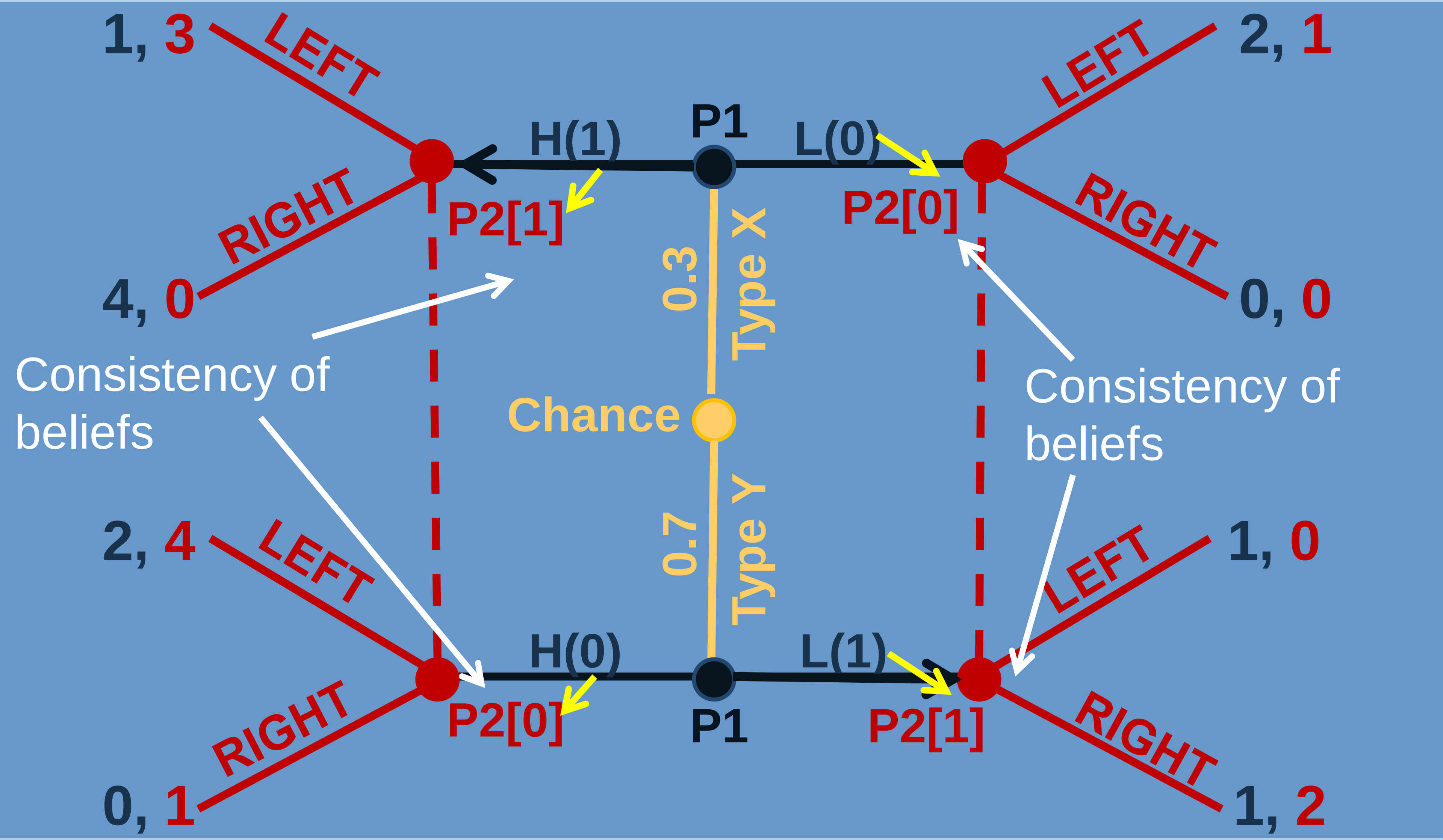
Separating equilibria? P1: {High, Low}

start with some strategy of P1



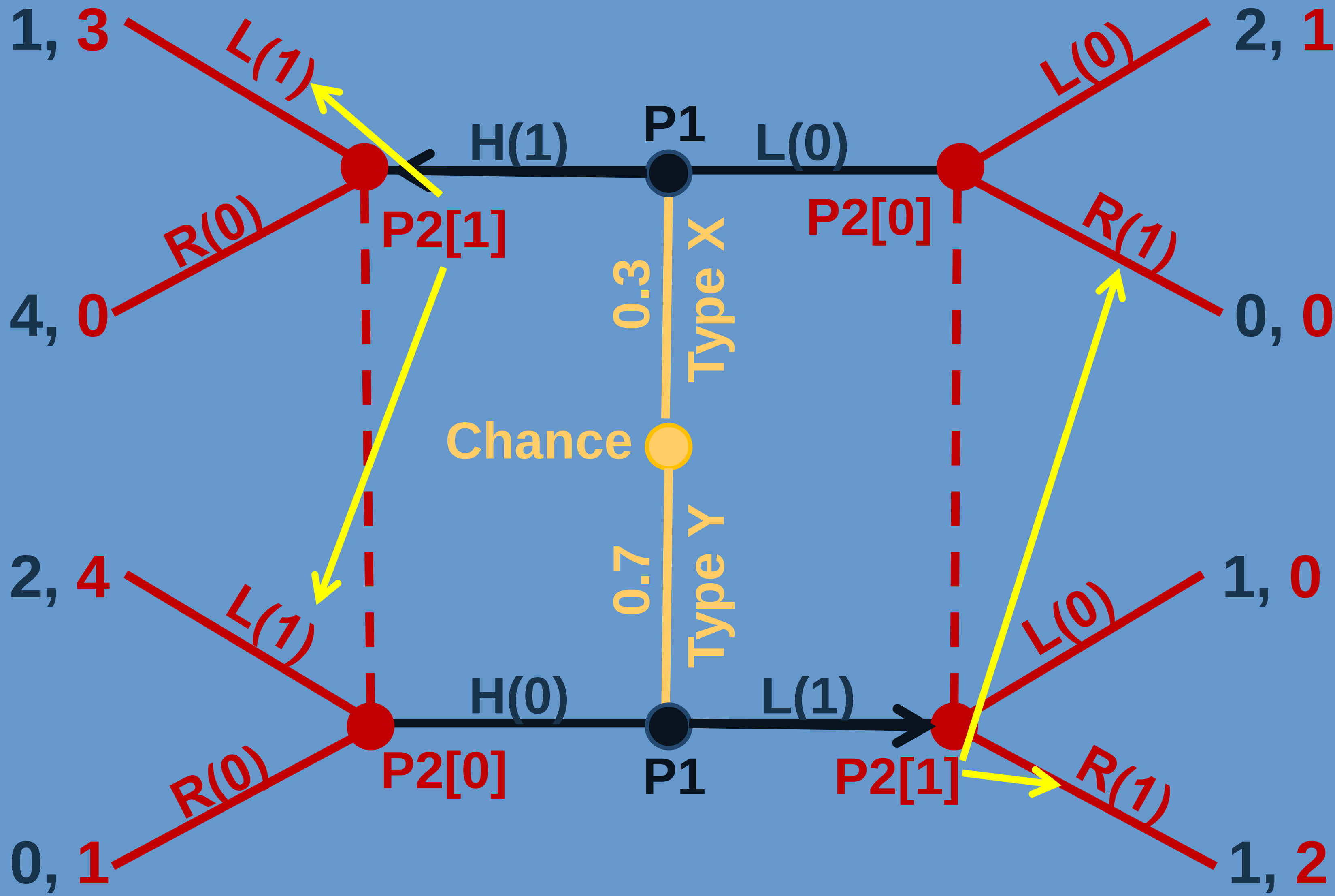
Separating equilibria? P1: {High, Low}

find consistent beliefs of P2



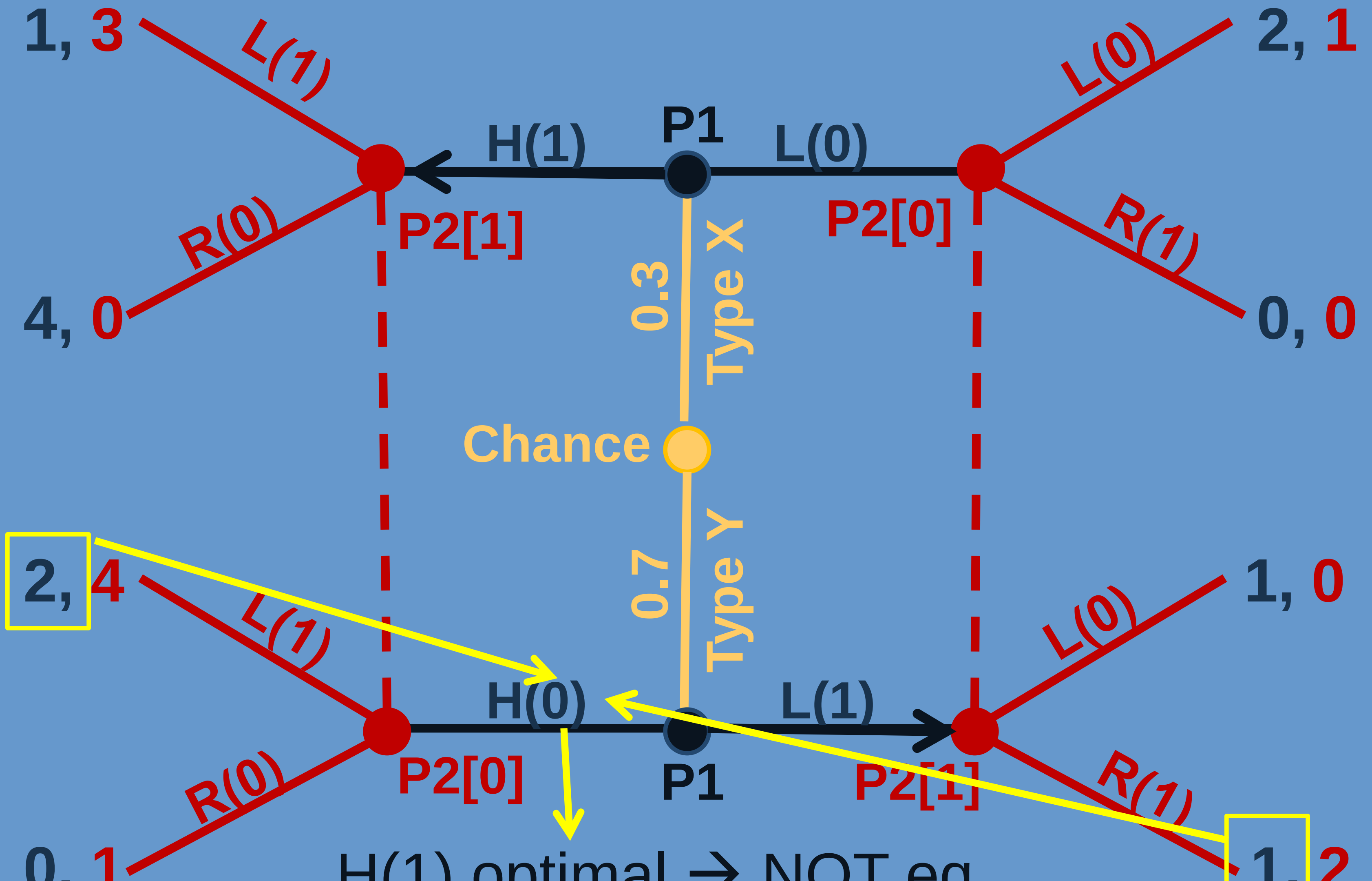
Separating equilibria? P1: {High, Low}

find optimal strategies of P2



Separating equilibria? P1: {High, Low}

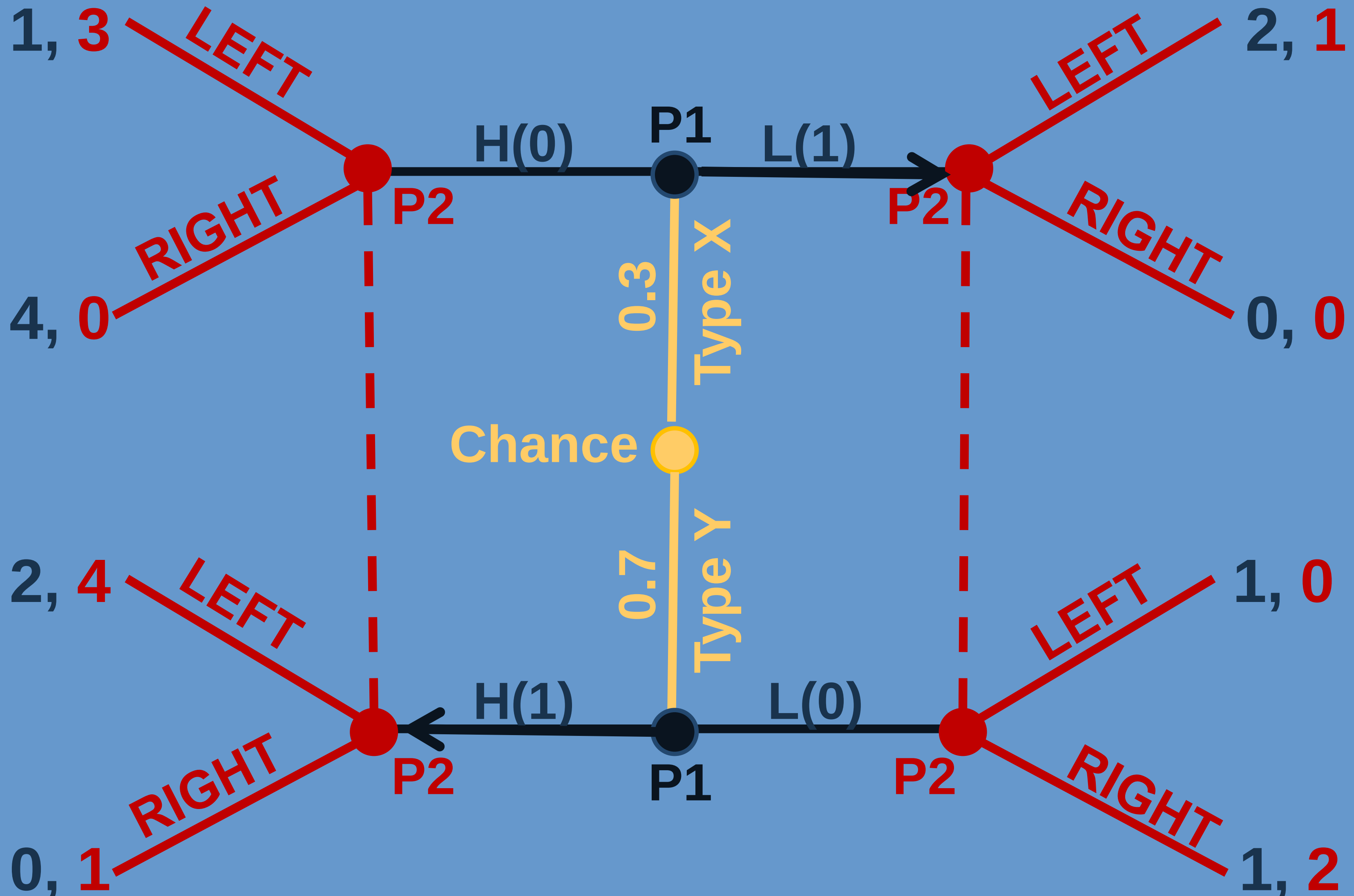
check for equilibrium – whether strategy of P1 is optimal



H(1) optimal → NOT eq.

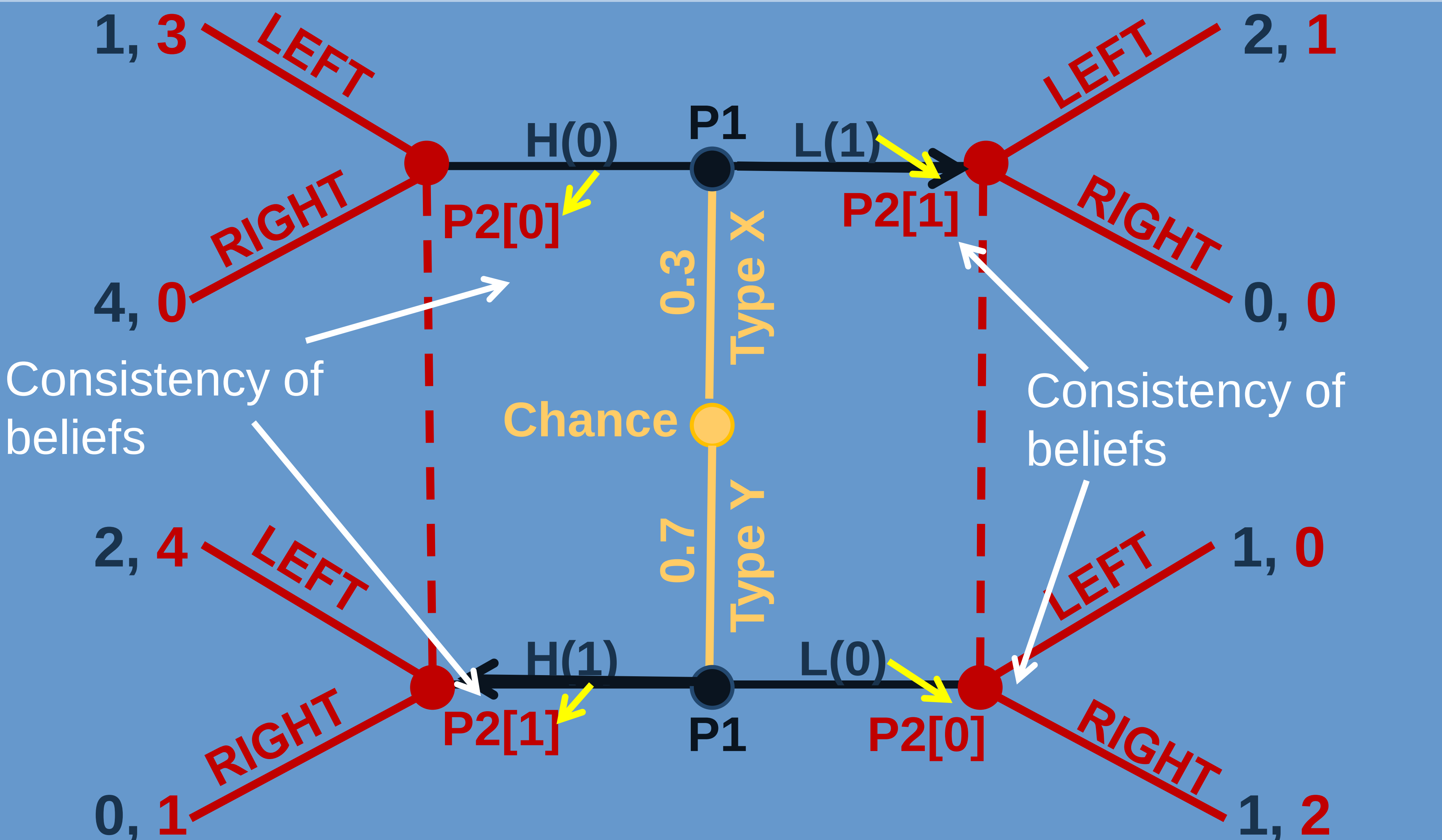
Separating equilibria? P1: {Low, High}

start with some strategy of P1



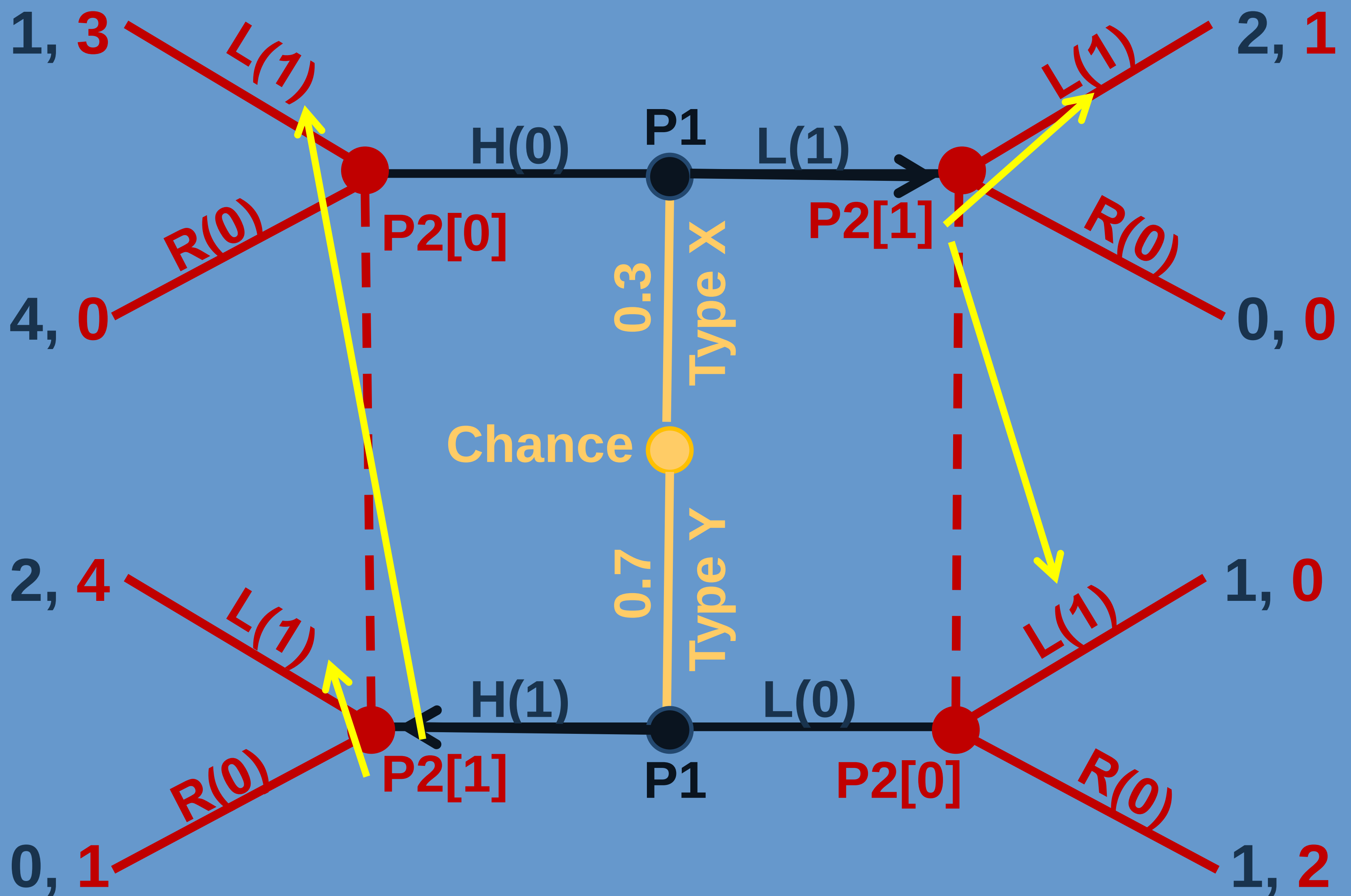
Separating equilibria? P1: {Low, High}

find consistent beliefs of P2



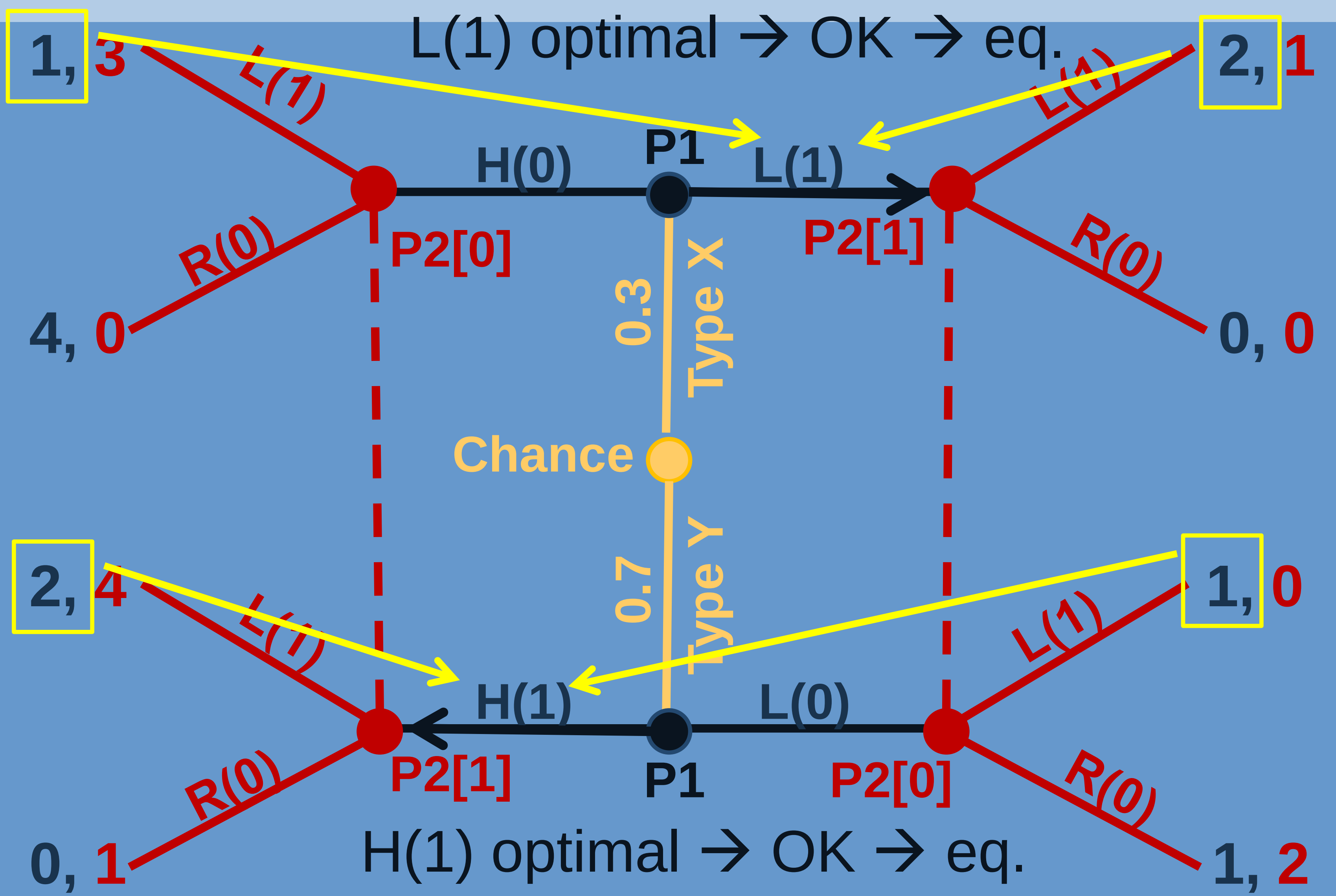
Separating equilibria? P1: {Low, High}

find optimal strategies of P2



Separating equilibria? P1: {Low, High}

check for equilibrium – whether strategy of P1 is optimal



Example 1: Signaling game – 2 types

- The game has two pure strategy weak sequential equilibria:

Pooling eq.:{P1X – High, P1Y - High,

P2 – Left if observes High, P2 Right if observes Low,

P2 believes after High: Type X(0.3) Type Y(0.7),

P2 believes after Low: Type X (p) Type Y(1-p) ,$0 \leq p \leq 2/3$}

Separating eq.: {P1X – Low, P1Y - High,

P2 – Left if observes High, P2 Left if observes Low,

P2 believes after High: Type X(0) Type Y(1),

P2 believes after Low: Type X (1) Type Y(0)}

Example 2: Simple poker game

Two people are playing a following card game, each of them having just 2 dollars: At the beginning of the game each player has to put one dollar into the pot (mandatory bet).

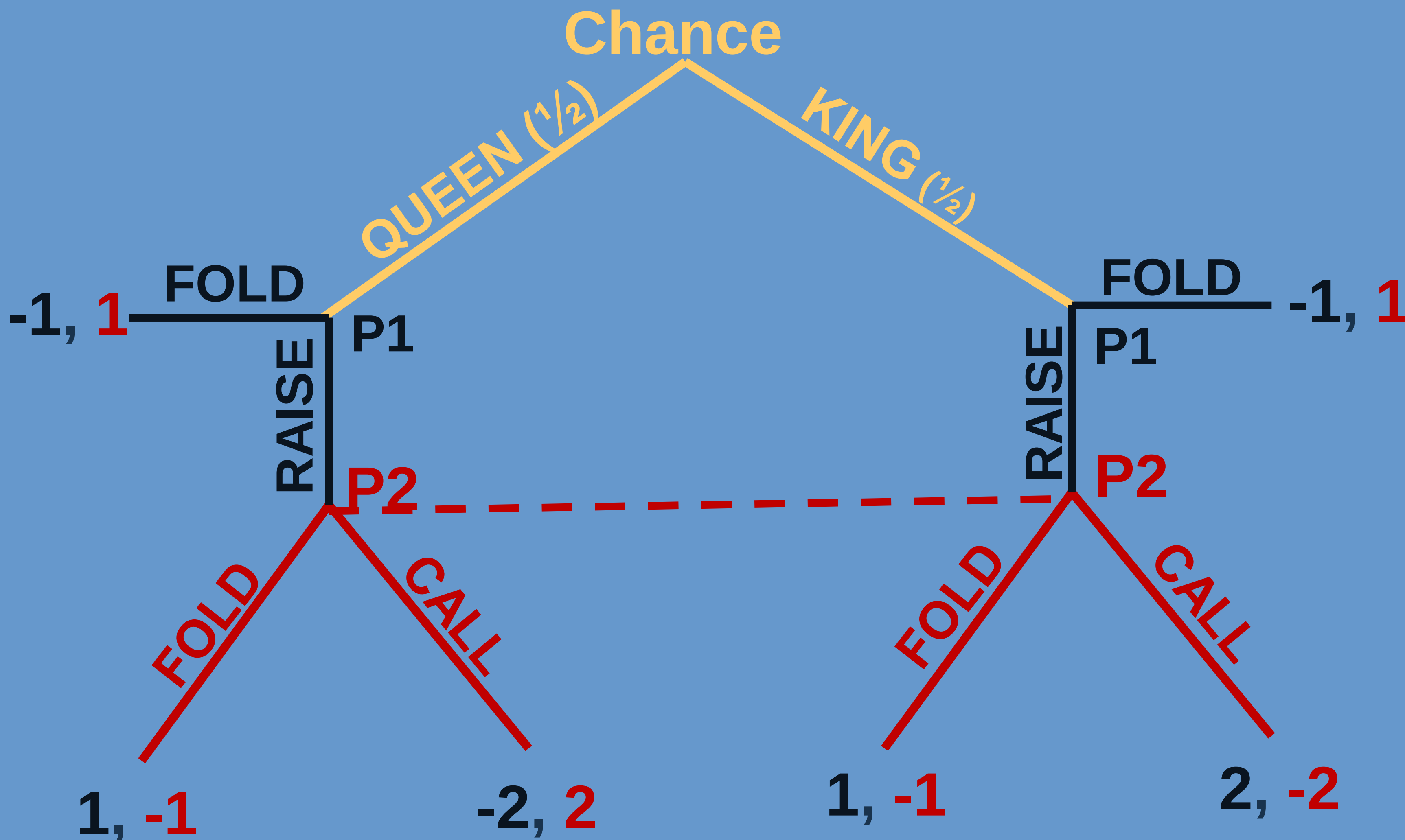
Then player 1(dealer) draws a card from a deck which contains only KINGS and QUEENS. With probability 0.5 player 1 draws a KING and with probability 0.5 player 1 draws a QUEEN.

After the player 1 privately observes her own card, she moves by either FOLD or RAISE. FOLD means that the game ends and player 1 loses one dollar => player 2 earns one dollar. RAISE means that she adds an additional dollar to the pot.

After RAISE player 2 either FOLDs (loosing one dollar) or CALLs (add additional dollar to the pot). Folding ends the game.

If the player 2 CALLs, player 1 wins the pot if she has a KING and looses if she has a QUEEN.

Example 2: Simple poker game



Finding weak sequential equilibria

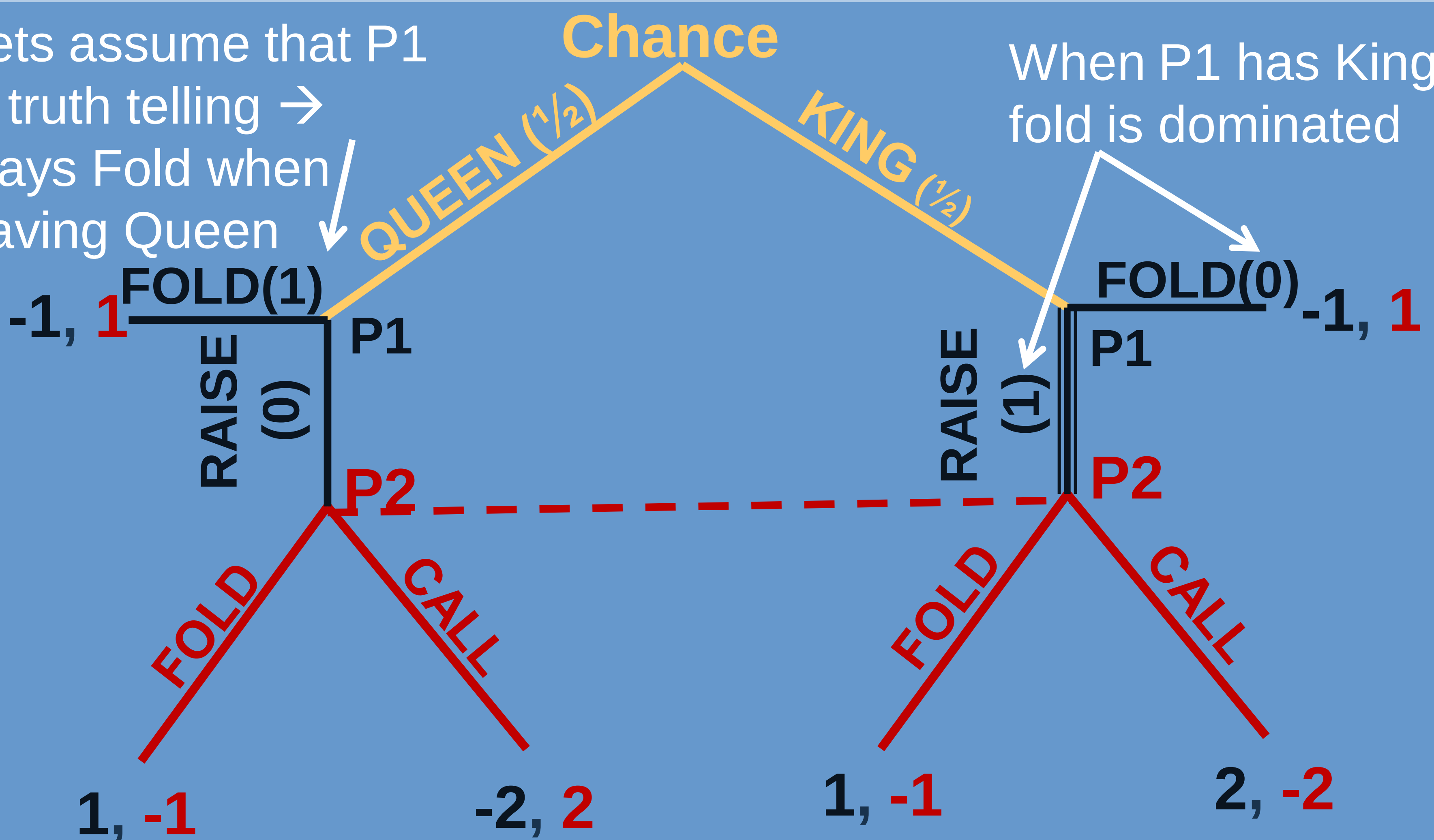
1) If the game has any subgame: NO SUBGAME
2) Identify all possible strategies of all players
   Player 1: {Fold,Fold}, {Fold,Raise}, {Raise,Fold}, {Raise,Raise} – the first action is when P1 having Queen, the second action when having King
   Player 2: {Fold}, {Call} – P2 plays just after Raise

Example 2: Any pure strategy weak seq.eq?

start with some strategy of P1

Lets assume that P1 is truth telling → plays Fold when having Queen

When P1 has King fold is dominated



Example 2: Any pure strategy weak seq.eq?

find consistent beliefs of P2

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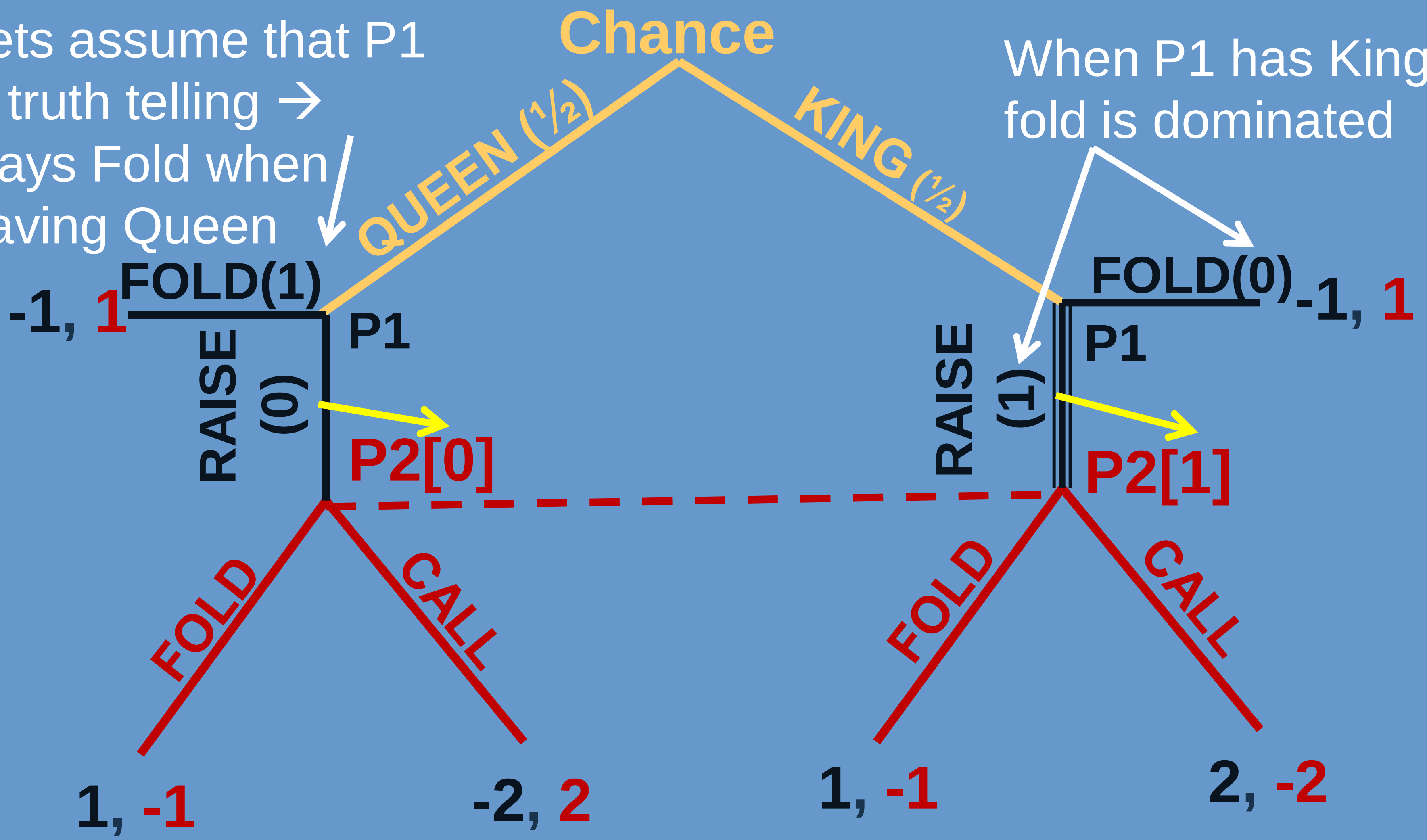

Example 2: Any pure strategy weak seq.eq?

find optimal strategies of P2

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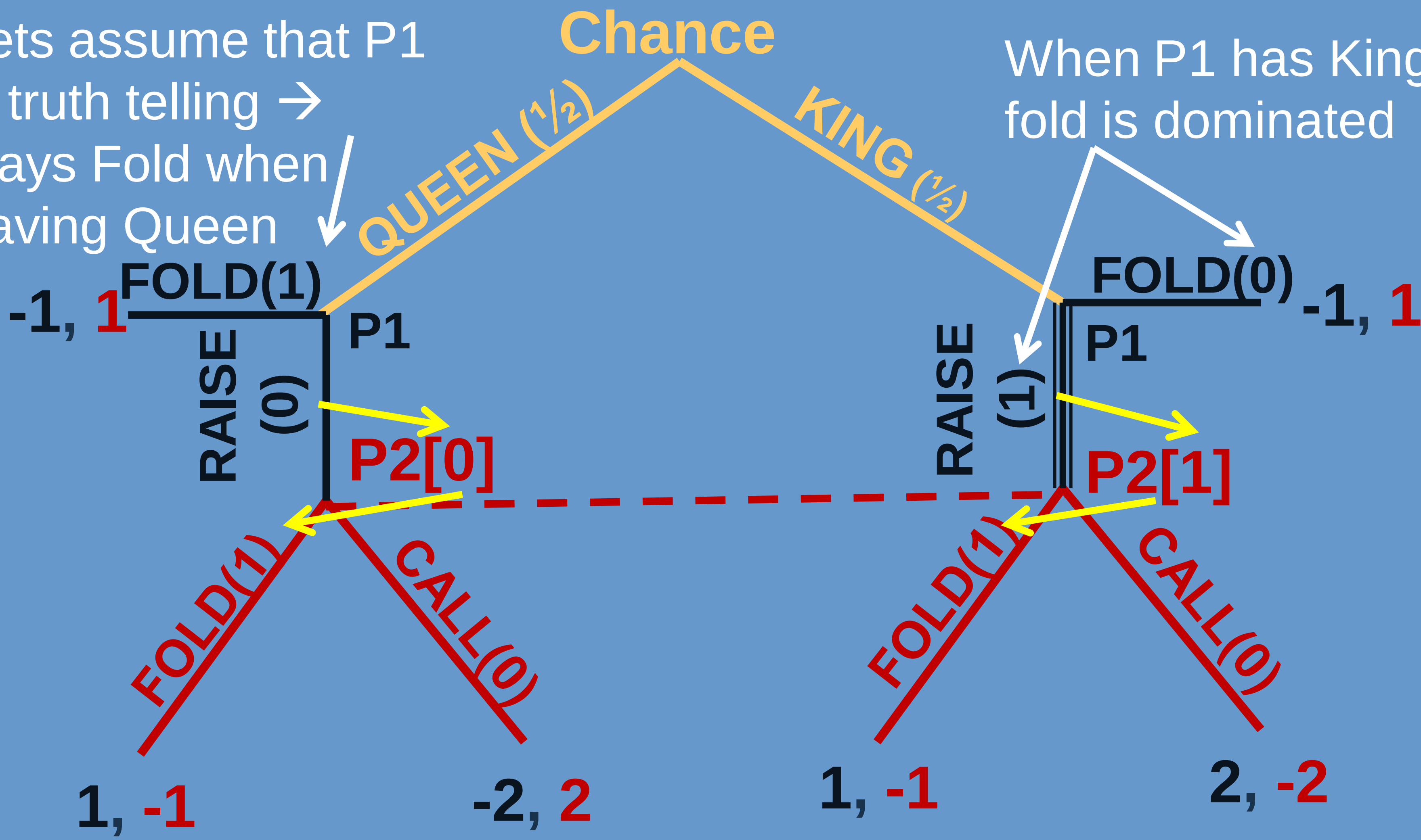

Example 2: Any pure strategy weak seq.eq?

check for equilibrium – whether strategy of P1 is optimal

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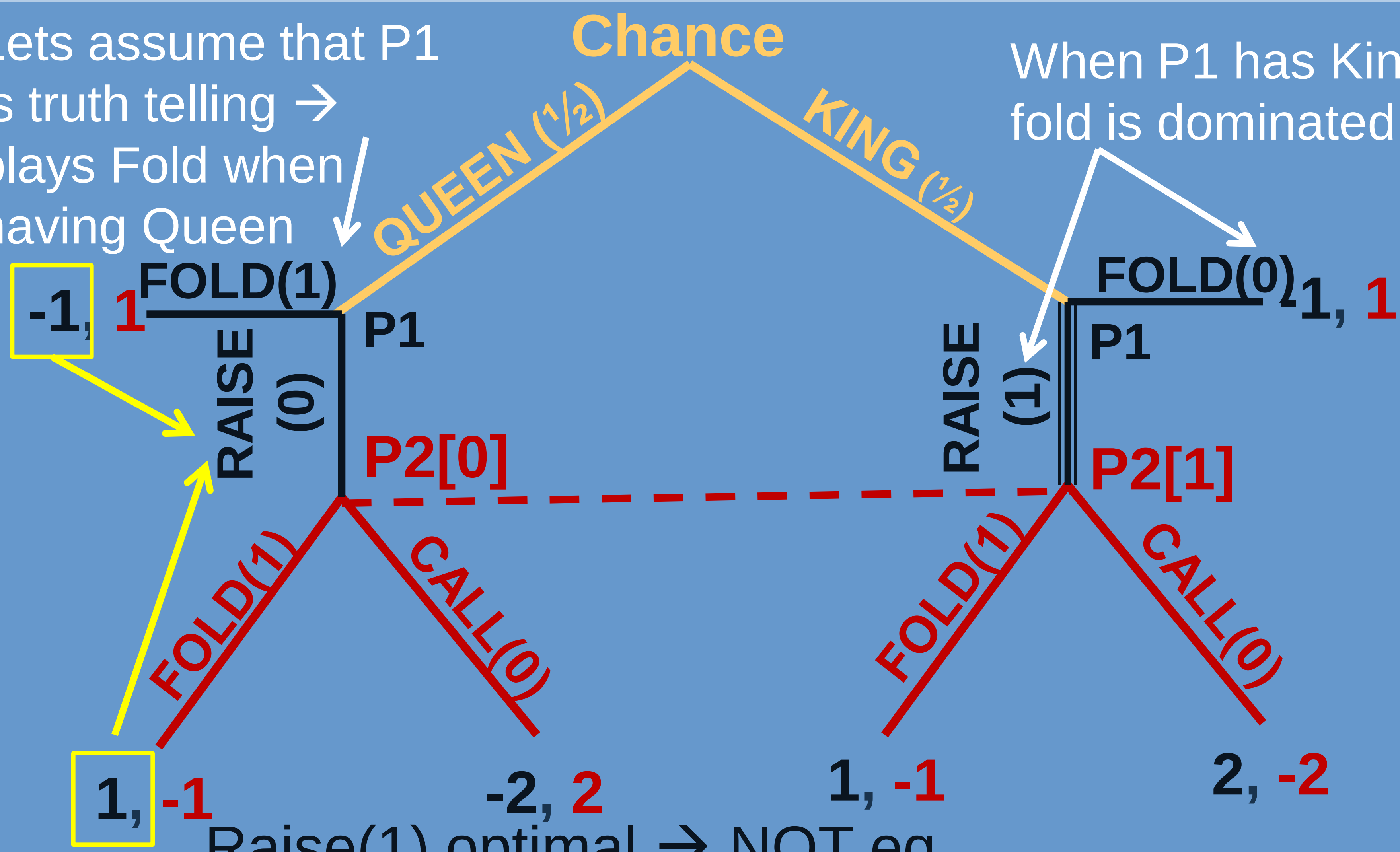
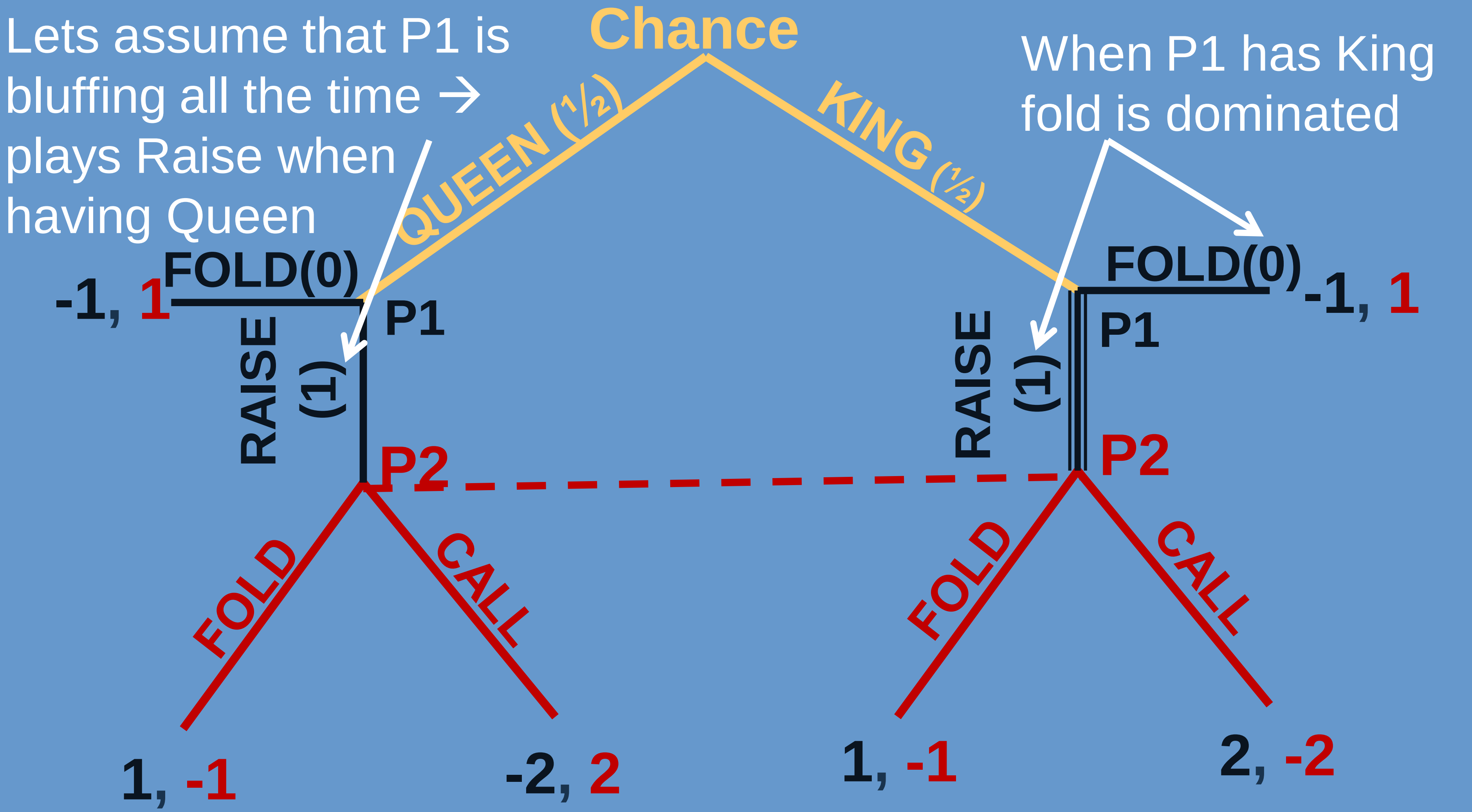


Raise(1) optimal → NOT eq.

Example 2: Any pure strategy weak seq.eq?

start with some strategy of P1

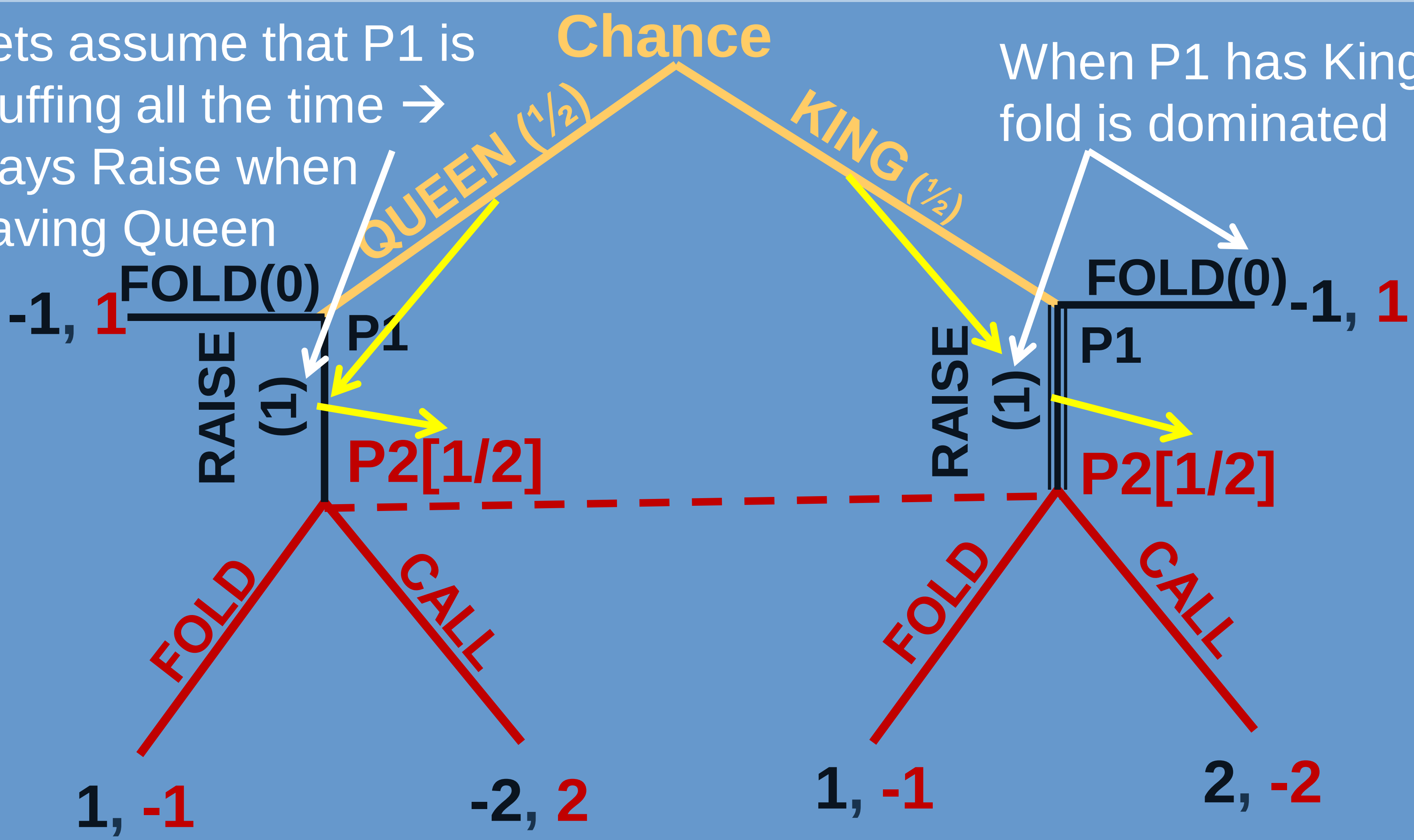


Example 2: Any pure strategy weak seq.eq?

find consistent beliefs of P2

Lets assume that P1 is bluffing all the time → plays Raise when having Queen

When P1 has King fold is dominated



Example 2: Any pure strategy weak seq.eq?

find optimal strategies of P2

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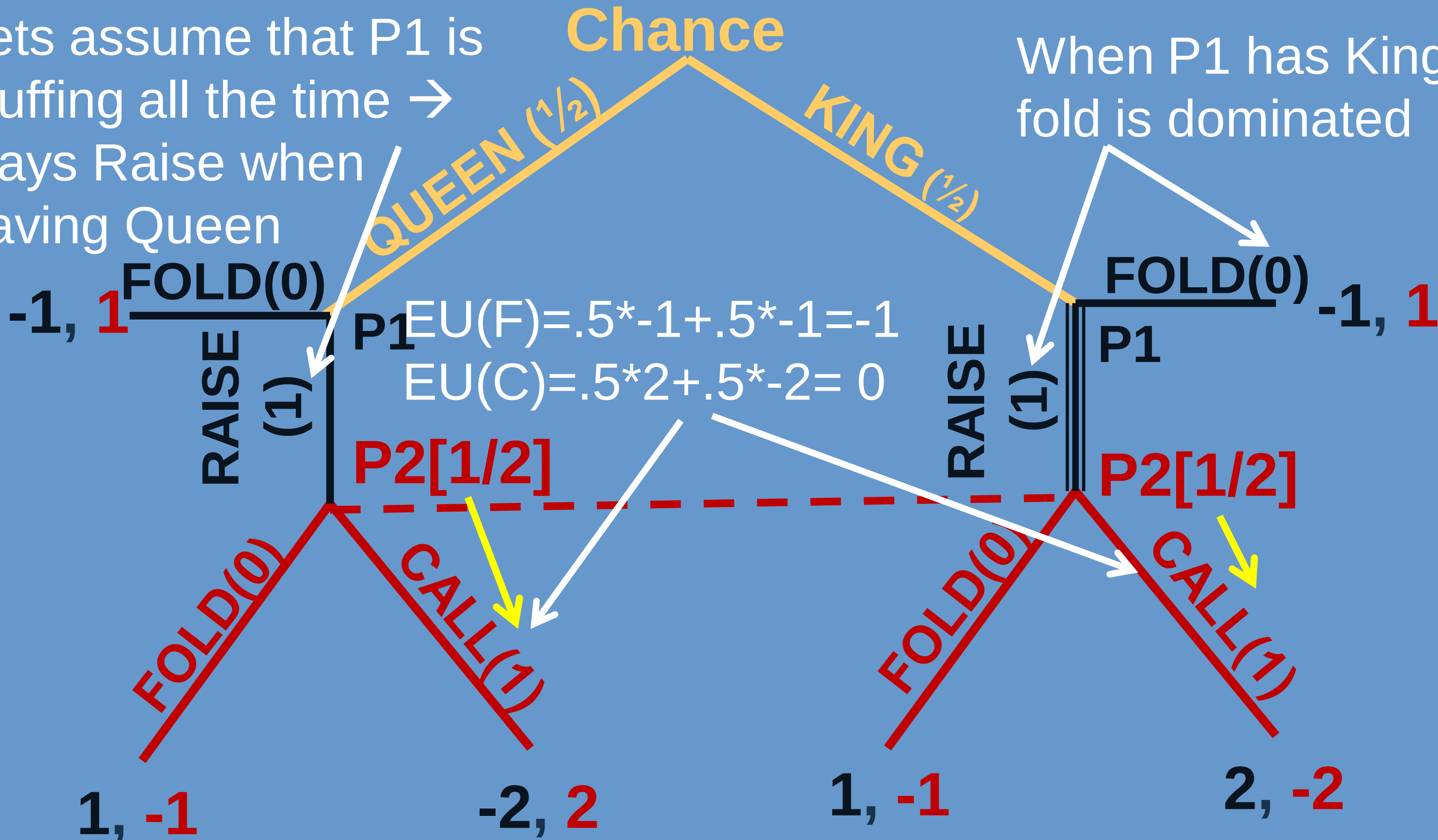

Example 2: Any pure strategy weak seq.eq?

check for equilibrium – whether strategy of P1 is optimal

Lets assume that P1 is bluffing all the time → plays Raise when having Queen

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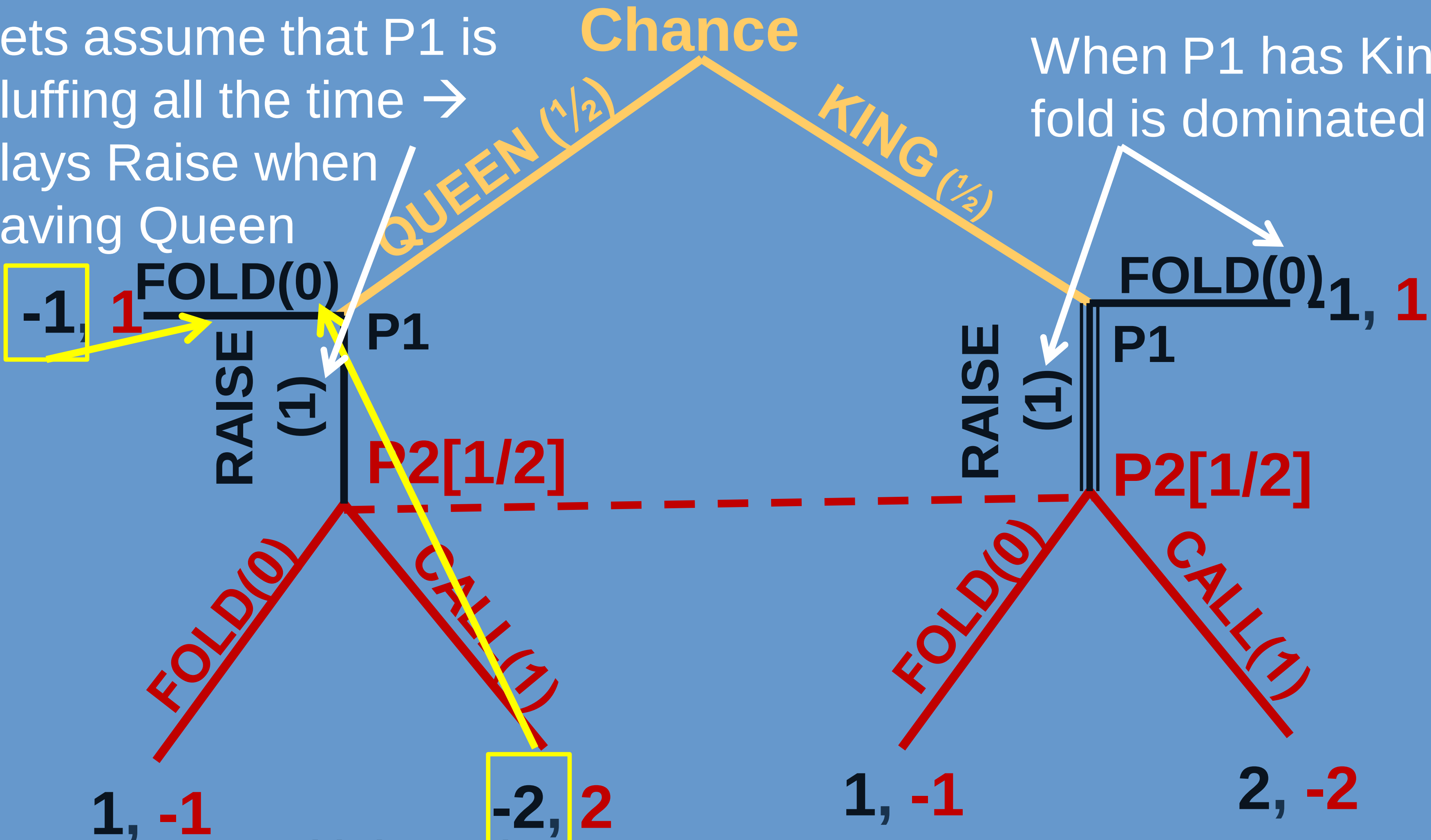


Fold(1) optimal → NOT eq.

Example 2: Any pure strategy weak seq.eq?

- The game has no pure strategy weak sequential equilibria:

  When P1 has a King, Fold is dominated strategy. When P1 has a Queen and would all the time play Raise, P2 will all the time Fold when he would after a while anticipate P1 strategy…

  But then P1Q would react on that and play all the time Fold and again, when P2 will anticipate this, he will start playing Call.

  So it will be again optimal for P1Q to Raise. Bringing us back to the beginning of the analysis. → no stable state → just one player reacts on the other player

  NO Pooling or Separating equilibrium →

  Partially separating/pooling equilibrium?

Example 2: Any pure strategy weak seq.eq?

- Semi-separating equilibria typically involve mixed strategies
  - observing a RAISE by P1 allows P2 to update the probability of a King from 1/2 to p>1/2, but not all the way to 1
- Simple poker vs. standard signaling models (job market etc.)
  - Standard models → one type of sender wants to signal his type truthfully, whereas the other type does not (high ability worker wants the employer to know his ability)
  - Simple poker → P1 never wants P2 to know his card type (if he has a king he would like P2 to think he has a Queen)
  - Real-world versions of poker are much more complicated
  - But has some of this flavor - player with a weak hand may sometimes wish to bluff to get others to fold and a player with a strong hand may sometimes wish to "slow play" to keep others in the game and win more money from them

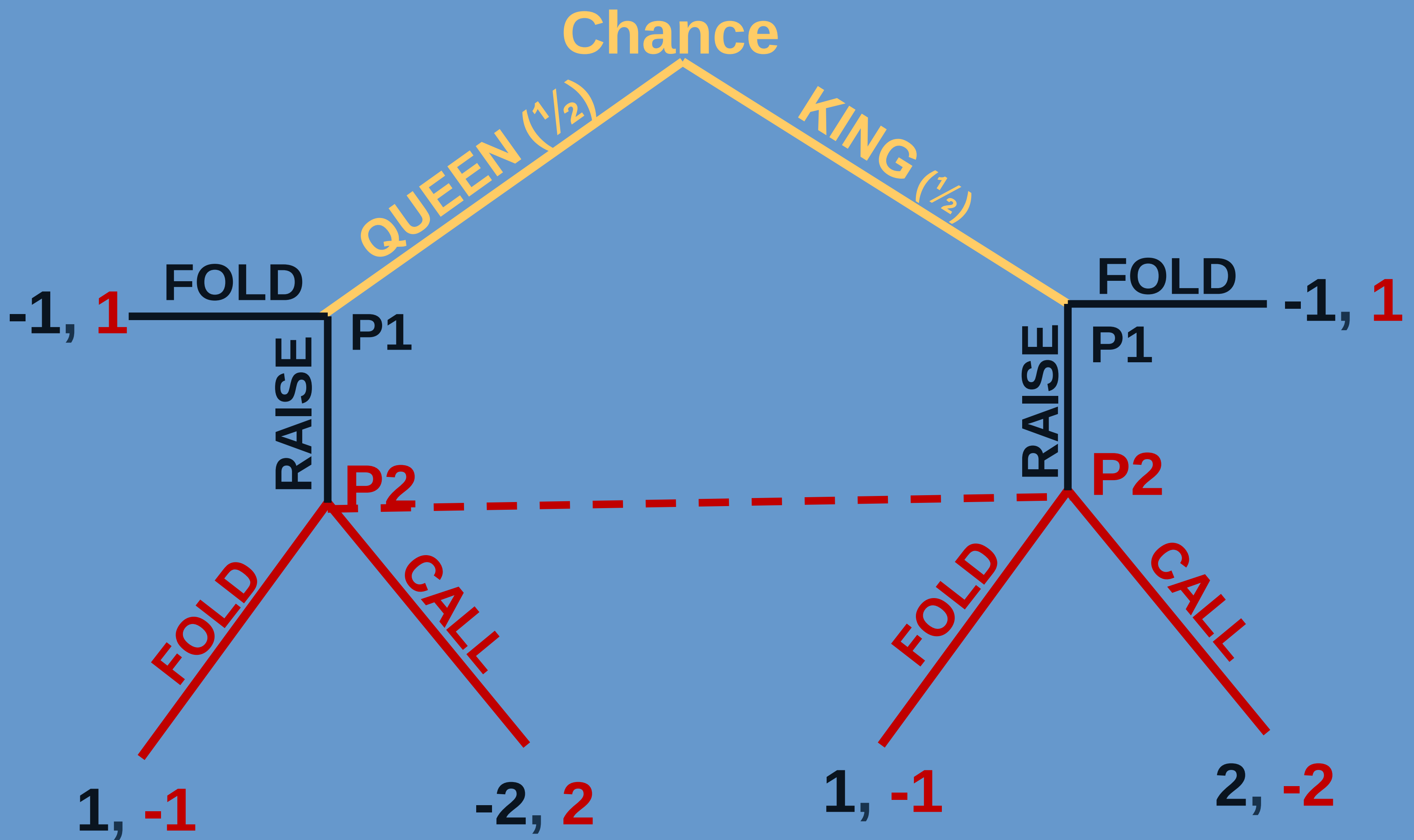
Finding all weak sequential equilibria

1) If the game has any subgame: NO SUBGAME
2) Identify all possible strategies of all players
   Player 1: {Fold,Fold}, {Fold,Raise}, {Raise,Fold}, {Raise,Raise} – the first action is when P1 having Queen, the second action when having King
   Player 2: {Fold}, {Call} – P2 plays just after Raise

When finding also Mixed strategy weak sequential equilibria, it is better to start with finding all MSNE of the game:

3) Find all MSNE (when both players choose strategy at the beginning of the game) of the game – in similar way as in the case of dynamic games – table with all possible strategies
4) Compute the beliefs – consistent with the strategies
5) Check all MSNE whether they satisfies the 2 conditions for the weak sequential equilibrium

Example 2: Simple poker game



Finding all weak sequential equilibria

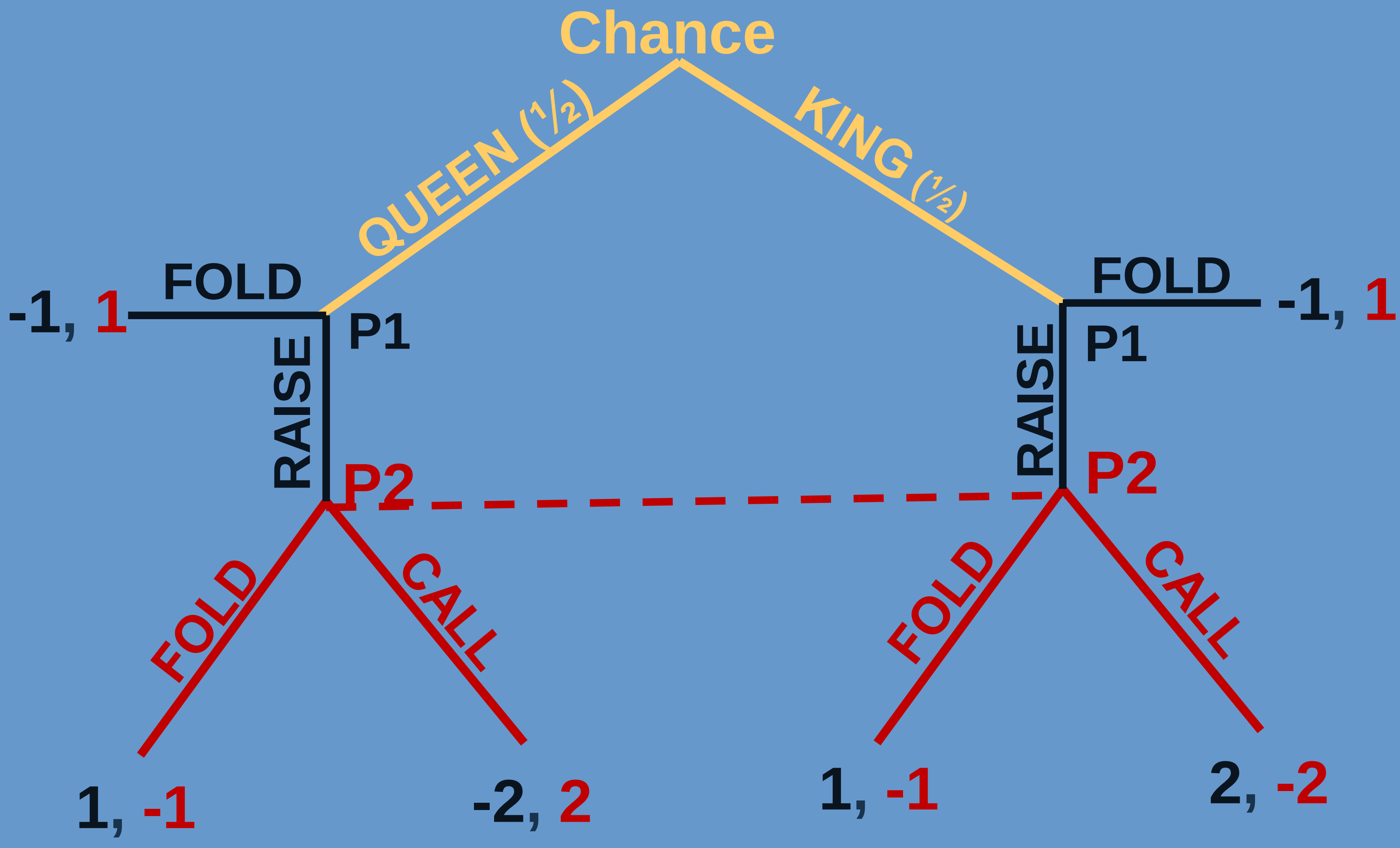
find all MSNE of the game → start with expected utilities

Player 2

P1

	Fold	Call
EU(Fold,Fold)	**½ *-1+½ *-1= -1**	**½ *-1+½ *-1= -1**
EU(Fold,Raise)	**½ *-1+½ *1= 0**	**½ *-1+½ *2= 0.5**
EU(Raise,Fold)	**½ *1+½ *-1= 0**	**½ *-2+½ *-1=-1.5**
EU(Raise,Raise)	**½ *1+½ *1= 1**	**½ *-2+½ *2=0**

Example 2: Simple poker game



Finding all weak sequential equilibria

find all MSNE of the game → start with expected utilities

Player 2

P1		EU(Fold)	EU(Call)
	Fold,Fold	**½ *1+½ *1= 1**	**½ *1+½ *1= 1**
	Fold,Raise	**½ *1+½ *1= 0**	**½ *1+½ *-2= -0.5**
	Raise,Fold	**½ *-1+½ *1= 0**	**½ *2+½ *1= 1.5**
	Raise,Raise	**½ *-1+½ *-1= -1**	**½ *2+½ *-2=0**

Finding all weak sequential equilibria

find all MSNE of the game → start with expected utilities

Player 2

Player 1

	Fold	Call
Fold,Fold	**-1**, **1**	**-1**, **1**
Fold,Raise	**0**, **0**	**0.5**,**-0.5**
Raise,Fold	**0**, **0**	**-1.5**,**1.5**
Raise,Raise	**1**,**-1**	**0**, **0**

Finding all weak sequential equilibria

find all MSNE of the game

Player 2

	Fold	Call
~~Fold,Fold~~	~~**-1**, **1**~~	~~**-1**, **1**~~
Fold,Raise	**0**, **<u>0</u>**	**<u>0.5</u>**,**-0.5**
~~Raise,Fold~~	~~**0**, **0**~~	~~**-1.5**,**1.5**~~
Raise,Raise	**<u>1</u>**,**-1**	**0**, **<u>0</u>**

Player 1

Finding all weak sequential equilibria

find all MSNE of the game

Player 2

Player 1		Fold (q)	Call (1-q)
	Fold,Raise (p)	**0**, **<u>0</u>**	**<u>0.5</u>**,**-0.5**
	Raise,Raise (1-p)	**<u>1</u>**,**-1**	**0**, **<u>0</u>**

No NE in pure strategies

MSNE: condition for P1: 0q+0.5(1-q)=1q+0.(1-q)

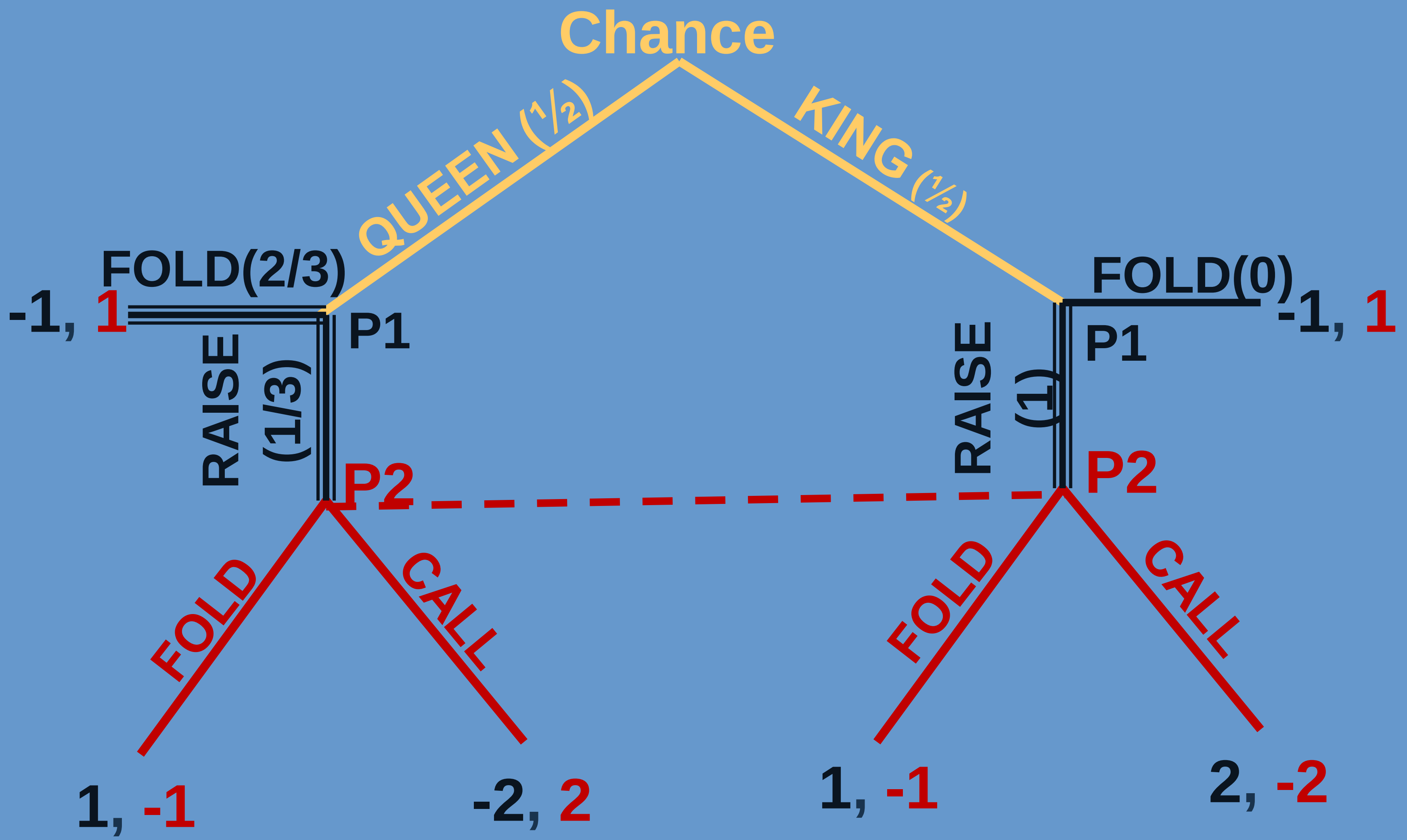
0.5 – 0.5q=q → q=1/3

condition for P2: 0p+(-1)(1-p)=(-0.5)p+0(1-p)

p-1=-0.5p → p=2/3

Example 2: Simple poker game

check all the MSNE - start with strategy of P1

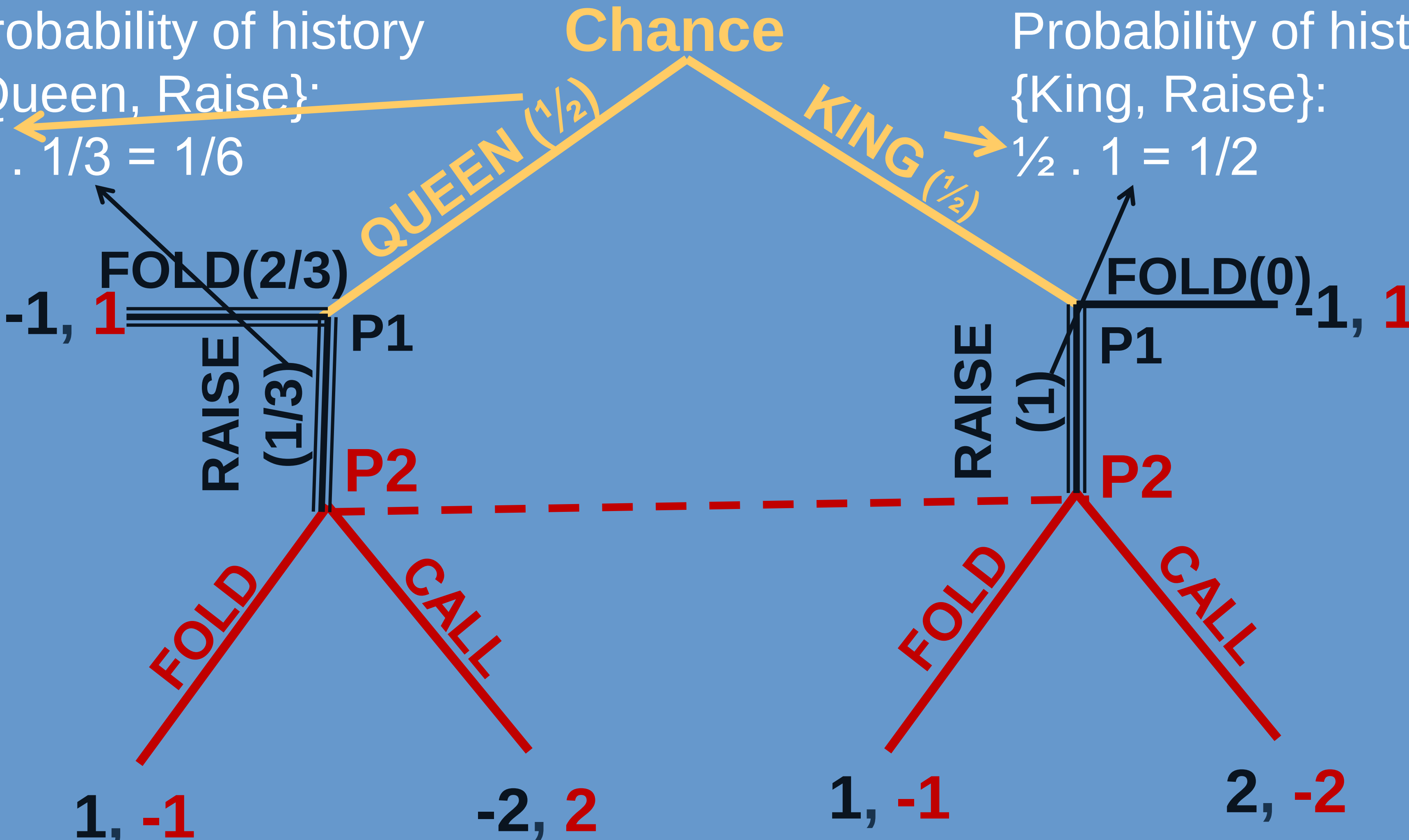


Example 2: Simple poker game

find consistent beliefs of P2

Probability of history {Queen, Raise}:
½ . 1/3 = 1/6

Probability of history {King, Raise}:
½ . 1 = 1/2



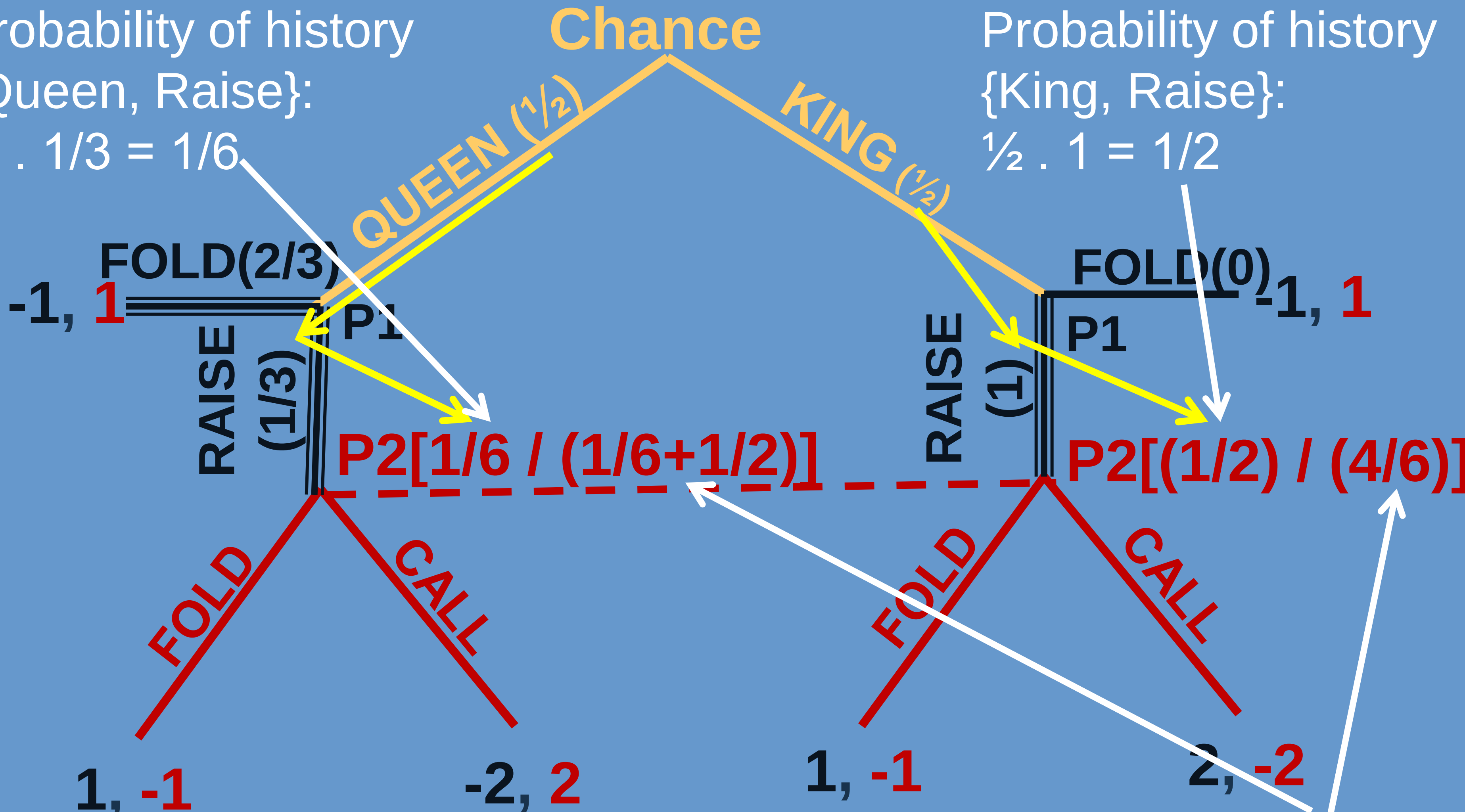
Probability of RAISE: P{Q,RAISE}+P{K,RAISE} = 1/6 + ½ = 4/6

Example 2: Simple poker game

find consistent beliefs of P2

Probability of history {Queen, Raise}: ½ . 1/3 = 1/6

Probability of history {King, Raise}: ½ . 1 = 1/2



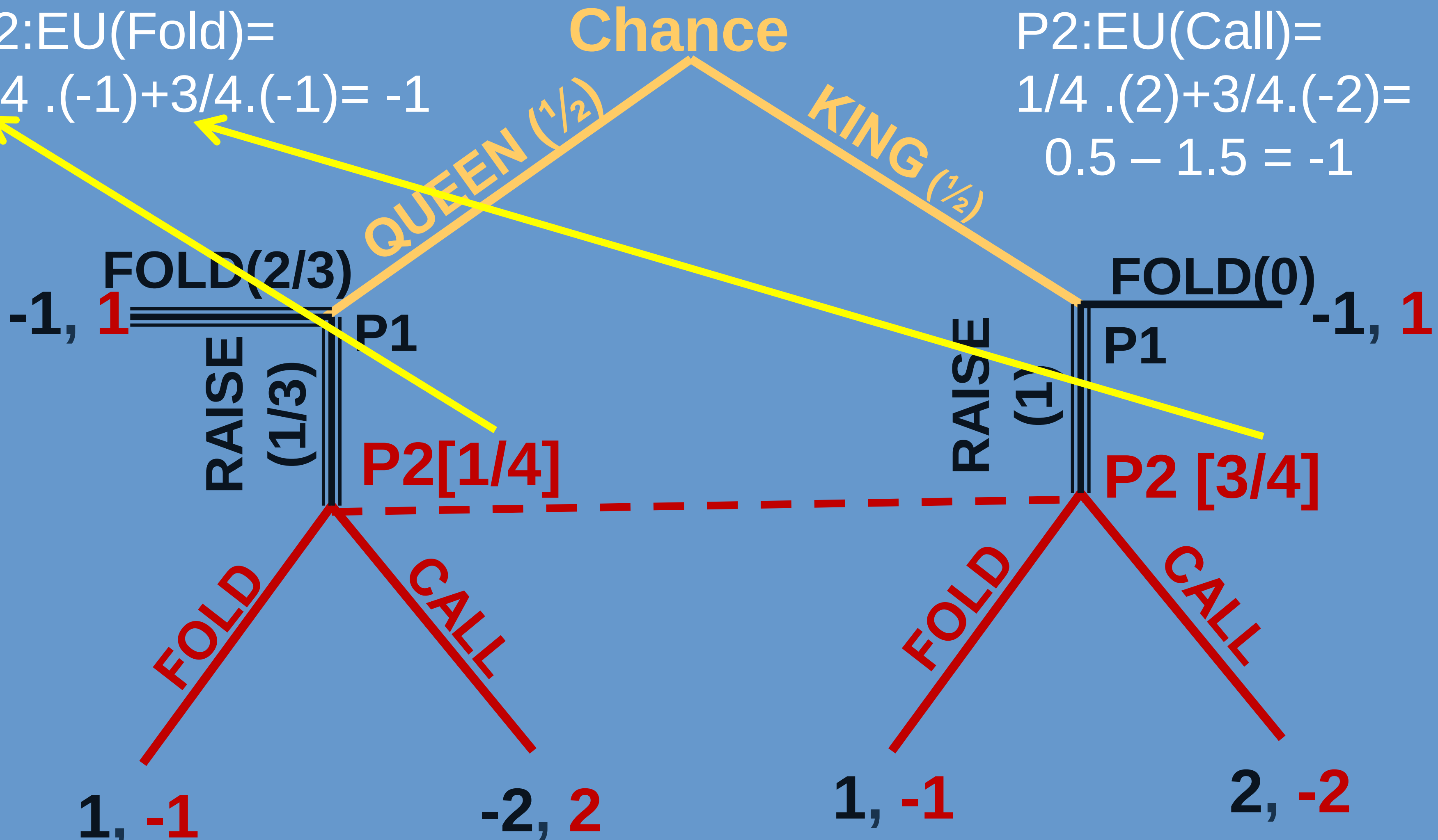
Probability of RAISE: P{Q,RAISE}+P{K,RAISE} = 1/6 + ½ = 4/6

Example 2: Simple poker game

find optimal strategies of P2

P2:EU(Fold)=
1/4 .(-1)+3/4.(-1)= -1

P2:EU(Call)=
1/4 .(2)+3/4.(-2)=
0.5 – 1.5 = -1



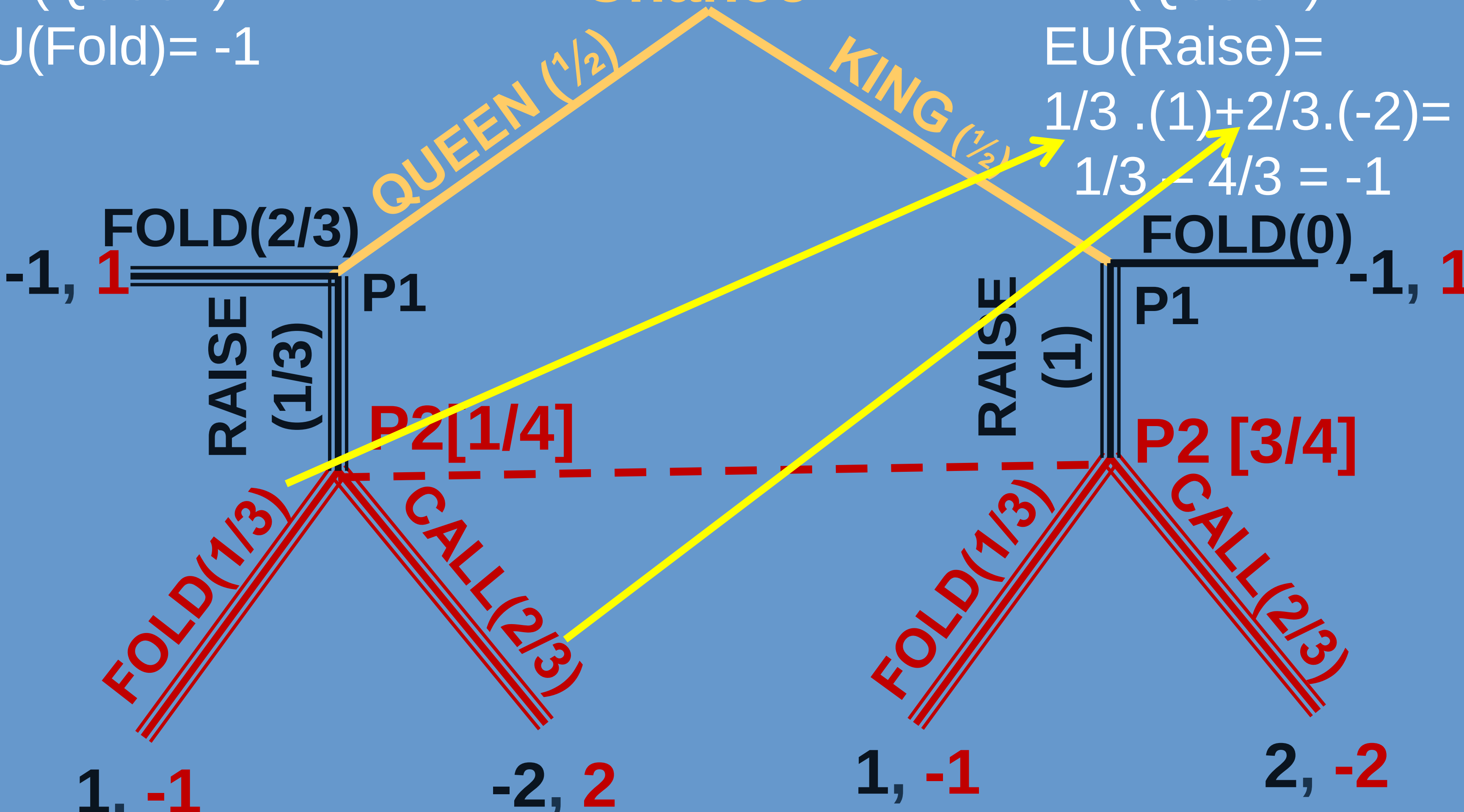
Player 2 is indifferent between FOLD and CALL → may mix

Example 2: Simple poker game

check for equilibrium – whether strategy of P1 is optimal

P1 (Queen) :
EU(Fold)= -1

P1 (Queen):
EU(Raise)=
1/3 .(1)+2/3.(-2)=
1/3 – 4/3 = -1



P1 (Queen) is indifferent between FOLD and RAISE→ may mix

Example 2: Simple poker game

- The game has no NE in pure strategies and one MSNE which coincides with weak sequential equilibrium:

  {P1 – After Queen: Fold(2/3)Raise(1/3), After King Raise, P2 – Fold(1/3)Raise(2/3), P2 believes after Raise: Queen(1/4) King (3/4) }

  In equilibrium, P1 can expect to win \$1/3 from P2 for each round of the game → game is not fair

  Advantage comes exclusively from the private information

  P1 cannot keep bluffing all the time but just in such a way to keep P2 uncertain whether he has King or Queen when P1 Raising…

Summary

- Dynamic games with incomplete information
- Weak sequential equilibrium

(in our simple cases coincides with perfect Bayesian equilibrium in Gibbons)

- Gibbons 2.4.A, 4; Osborne 10

Final exam, make up midterm

- Make-up midterm:
  - 16.12.2009 9:00 NB350 – capacity 20 (priority given to exchange students)
  - 05.01.2010 16:00 RB211 – capacity 30
  - 11.01.2010 9:00 RB210 – capacity 30
- Final exam:
  - 16.12.2009 10:30 NB350 – capacity 15 (priority given to exchange students)
  - 05.01.2010 17:45 Vencovského aula – capacity 100
  - 11.01.2010 11:00 NB D – capacity 50
  - 18.01.2010 10:15 NB D – capacity 50

Make up midterm

Topics:

Static games: actions, action profiles, Iterative elimination of dominated strategies, Nash equilibrium, Mixed strategies, Dominated strategies in mixed strategies, mixed strategy NE, symmetric games and NE

Dynamic games: Backward induction, strategies, NE, SBNE, synergic relationship – NE in static, NE and SBNE in dynamic version, finite sequential bargaining

Will not be in make up midterm: electoral competition, war of attrition, reporting crime, expert diagnosis, sequential bargaining with infinite number of moves (time periods)

Final exam

Topics: Enough for 50% out of 60% from the Final exam:

All the topics for midterm may appear also in Final exam

Dynamic games with simultaneous moves (not International tariffs example),

Bayesian Games (actions, signals, belief about the states consistent with the signal, types of players, Nash equilibrium of a Bayesian game, Finding NE of Bayesian games, Examples - More information may hurt, Infection),

Dynamic games with incomplete information (information set and information partition, belief system, behavioral strategy, weak sequential equilibrium – sequential rationality and consistent beliefs, signaling games, pooling and separating equilibrium, finding weak sequential equilibria in pure strategies)

Final exam

Topics:

Topics for the rest 10% out of 60% from the Final exam:

electoral competition, war of attrition, reporting crime, expert diagnosis, sequential bargaining with infinite number of moves (time periods)

Dynamic games with simultaneous moves (International tariffs example)

Bayesian Games (Cournot's duopoly example, Reporting crime example),

Dynamic games with incomplete information (finding weak sequential equilibria also in mixed strategies)