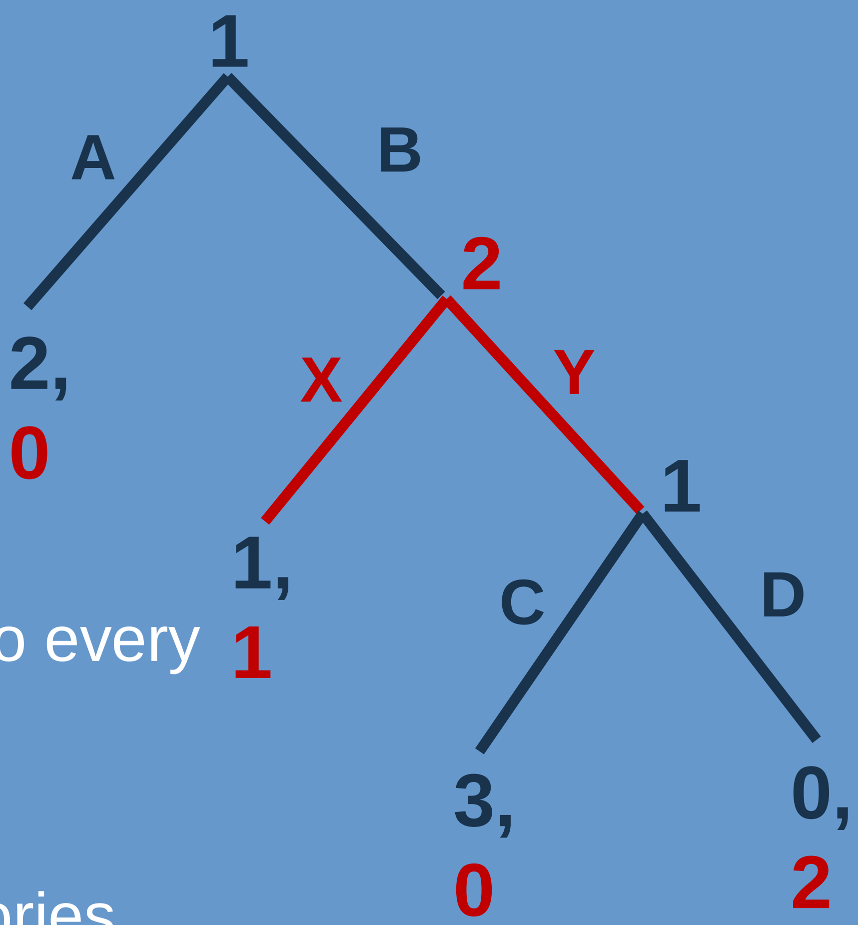
DYNAMIC GAMES with incomplete information

Lecture 11

Revision

Dynamic game:

- Set of players:
- Terminal histories:
  – all possible sequences of actions in the game
- Player function:
  – function that assigns a player to every proper subhistory
- Preferences for the players:
  – Preferences over terminal histories
  – represented by utility (payoff) function



Incomplete inform. – Dynamic games

- NOW we allow for some level of uncertainty also in dynamic games
- We will start with the simple example:

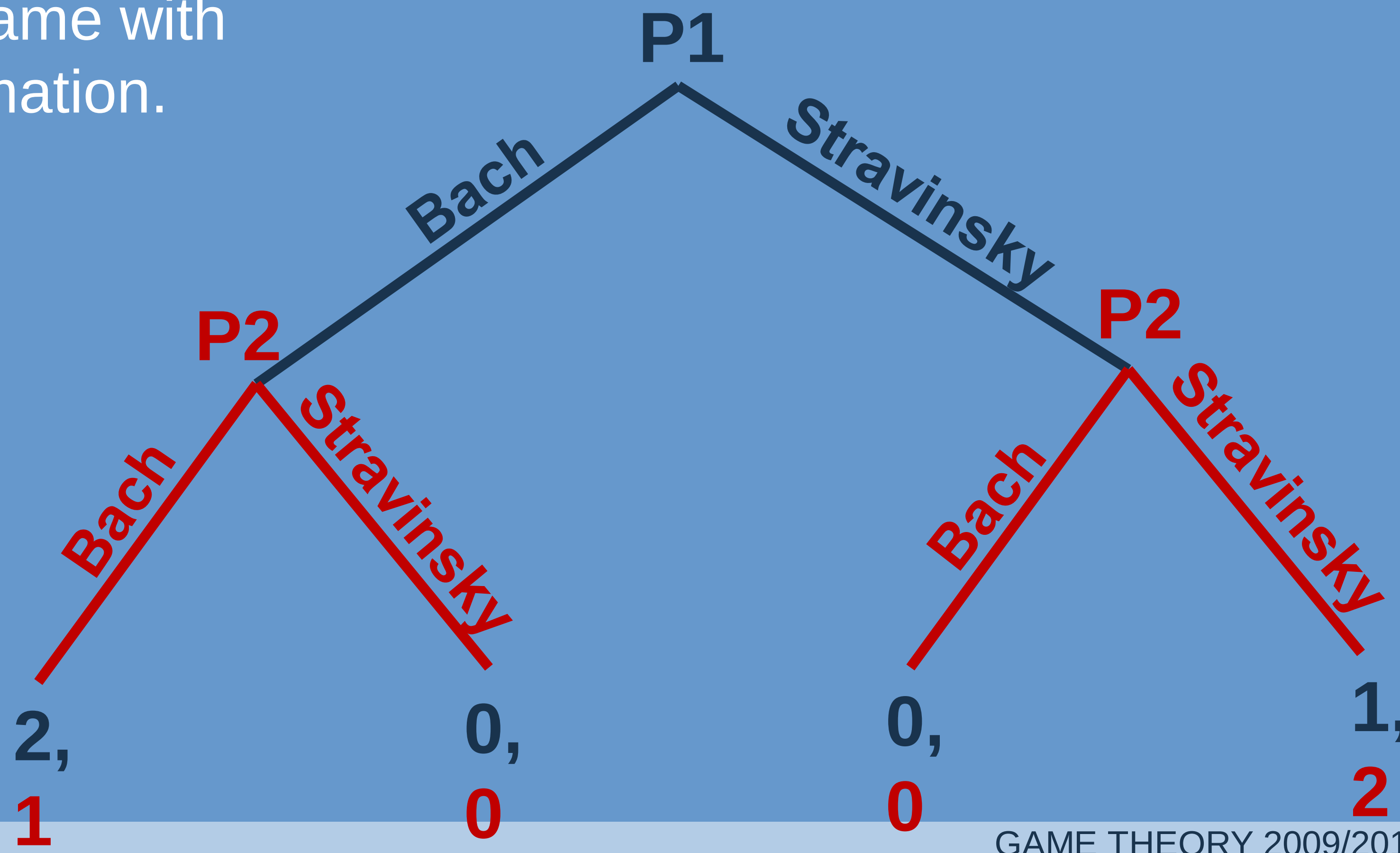
(Variant of Bach or Stravinsky)

Two people wish to go out together. Two concerts are available: one of music by Bach, and one of music by Stravinsky. One person prefers Bach and the other prefers Stravinsky. If they go to different concerts, each of them is equally unhappy listening to the music of either composer.

But now, they are choosing the concert sequentially. After the first person make a choice, the second person receives a signal (information) about the choice of first person and makes her choice according to this signal.

B or S: perfect information

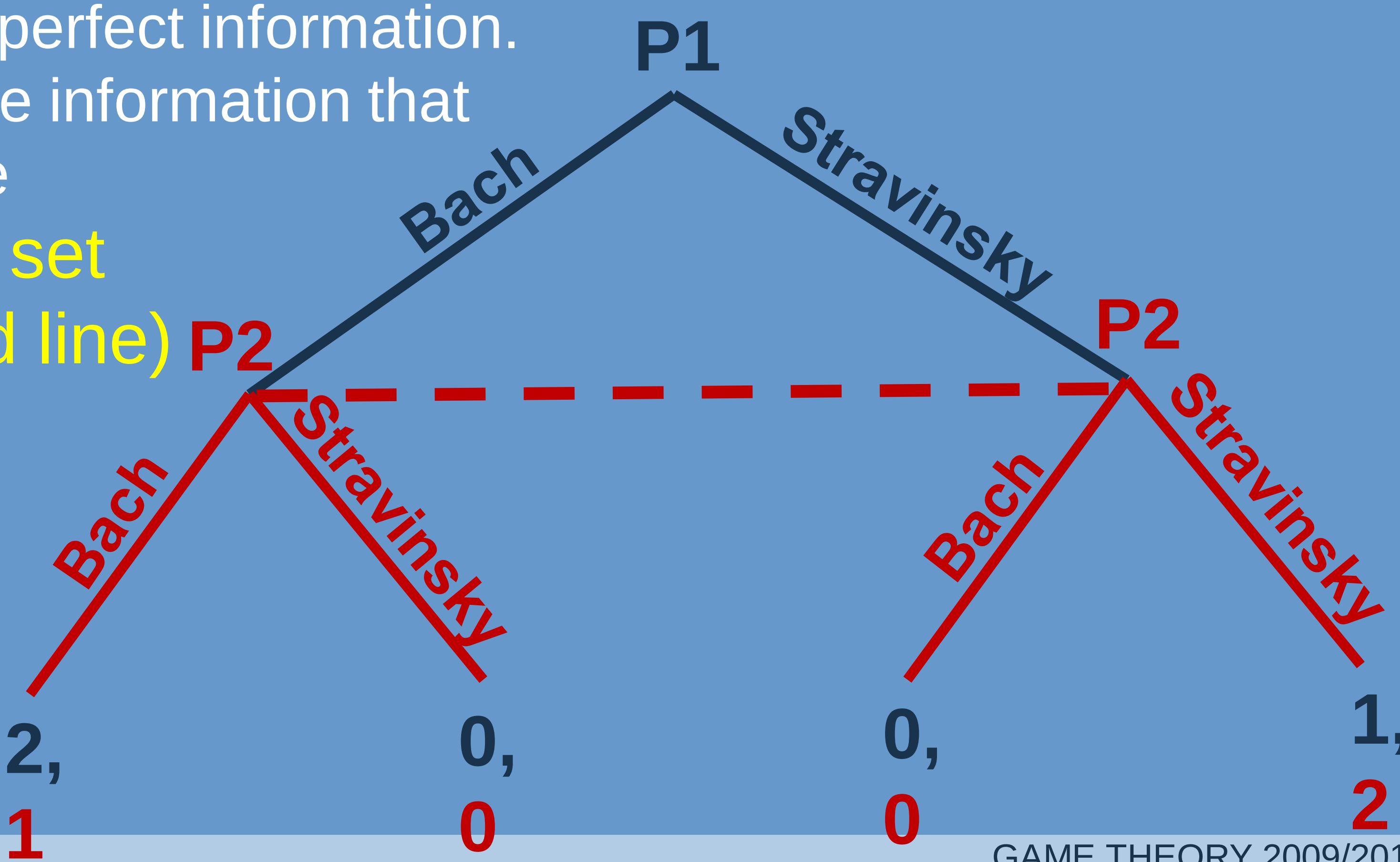
If the signal that the second person receives has different values after P1 plays Bach and after P1 plays Stravinsky, the second person is perfectly informed about the P1’s choice and the game is dynamic game with perfect information.



B or S: imperfect information

If the signal that the second person receives has same value after P1 plays Bach and after P1 plays Stravinsky, the second person is not informed about the P1’s choice and the game is the dynamic game with imperfect information.

We denote the information that P2 has by the information set (dashed red line)

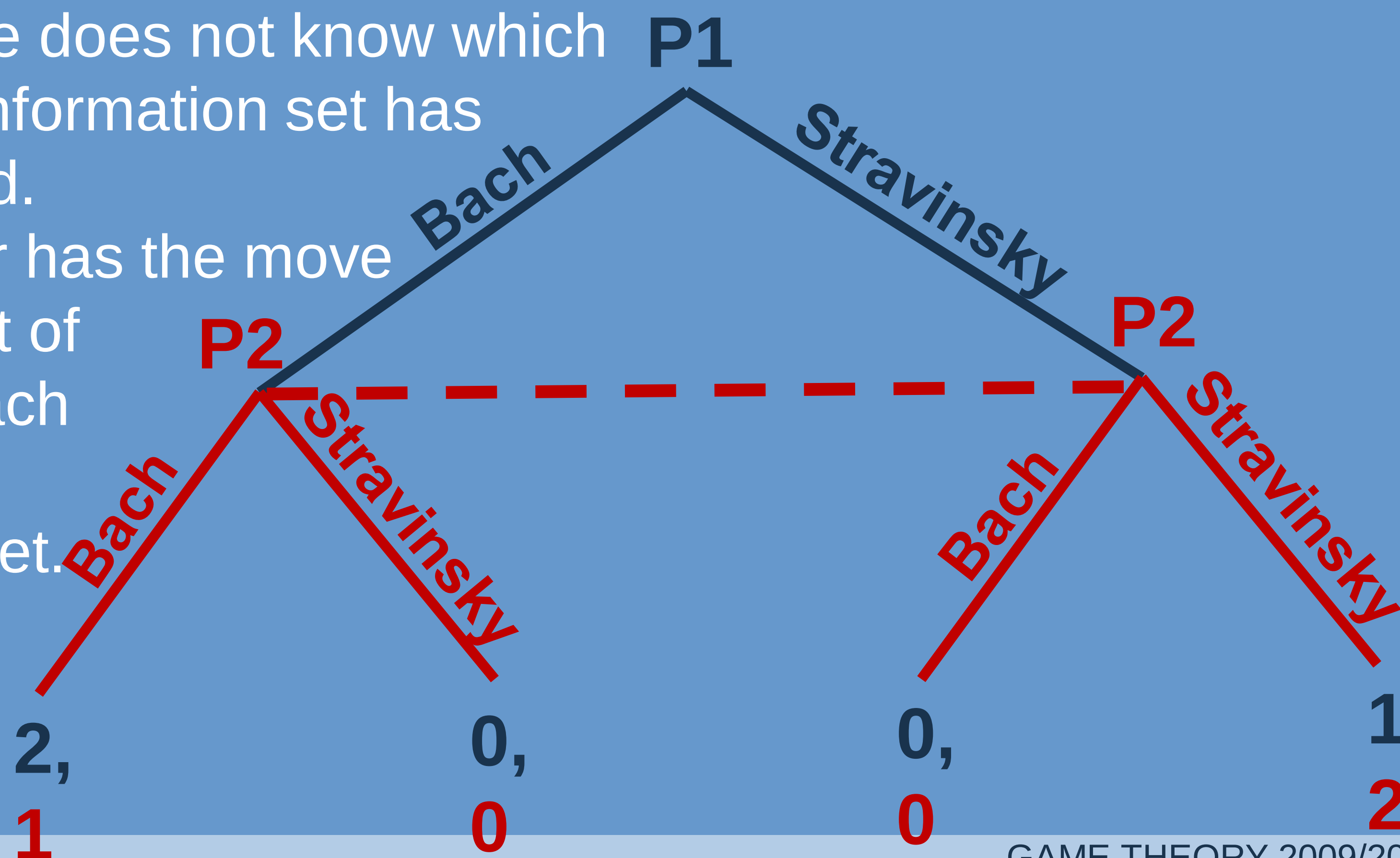


B or S: imperfect information

information set (dashed red line) – collection of decision nodes (histories after which it is player’s turn) such that:

a) when the play reaches a node in the information set, the player with the move does not know which node in the information set has been reached.

b) The player has the move and same set of choices at each node in the Information set.



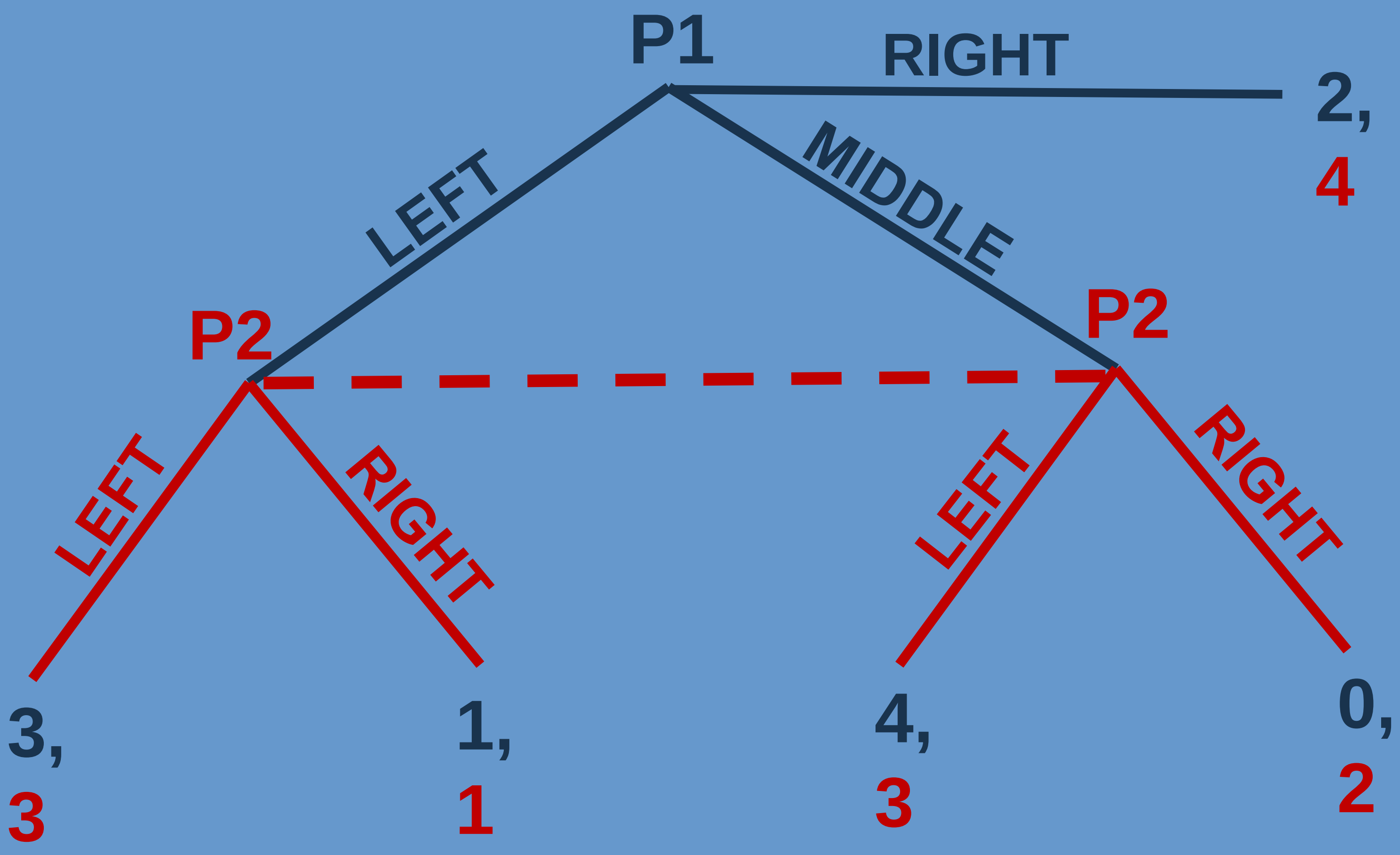
Dynamic games – Incomplete inform.

- Set of players
- Terminal histories: all possible ways(sequences of actions) how we can get at some ending node in the tree diagram
- Player function: function that assign to each history (not terminal history) either player or "chance"
- A function that assign to each history after which it is "chance" turn a probability distribution over the actions available after that history
- Information partition – division of histories (decision nodes) of the player that has the turn into information sets
- Preferences for the players – preferences over the set of lotteries over terminal histories represented by the vNM preferences (expected utility theory)

Dynamic games – Incomplete inform.

- Set of players: person 1, person 2
- Terminal histories: {B,B}, {B,S}, {S,B}, {S,S}
- Player function: P(ø)=1; P(B)=2, P(S)=2
- “Chance” - None
- Information partition – Player 1’s information partition contains one information set – {ø}, Player 2’s information partition contains also one information set – {B,S}.
- (in the case of perfect information, P2’s information partition contains two information sets – {B}, {S})
- Preferences for the players – represented by utility function (payoffs)

Example 1



Example 1

- Set of players: person 1, person 2
- Terminal histories: {LEFT,LEFT}, {LEFT,RIGHT}, {MIDDLE,LEFT}, {MIDDLE,RIGHT}, {RIGHT}
- Player function: P(ø)=1; P(LEFT)=2, P(MIDDLE)=2
- “Chance” - None
- Information partition – Player 1’s information partition contains one information set – {ø}, Player 2’s information partition contains two information set – {LEFT,MIDDLE}, {RIGHT}
- Preferences for the players – represented by utility function (payoffs)

Example 2: Simple poker game

Two people are playing a following card game each of them having just 2 dollars: At the beginning of the game each player has to put one dollar into the pot (mandatory bet).

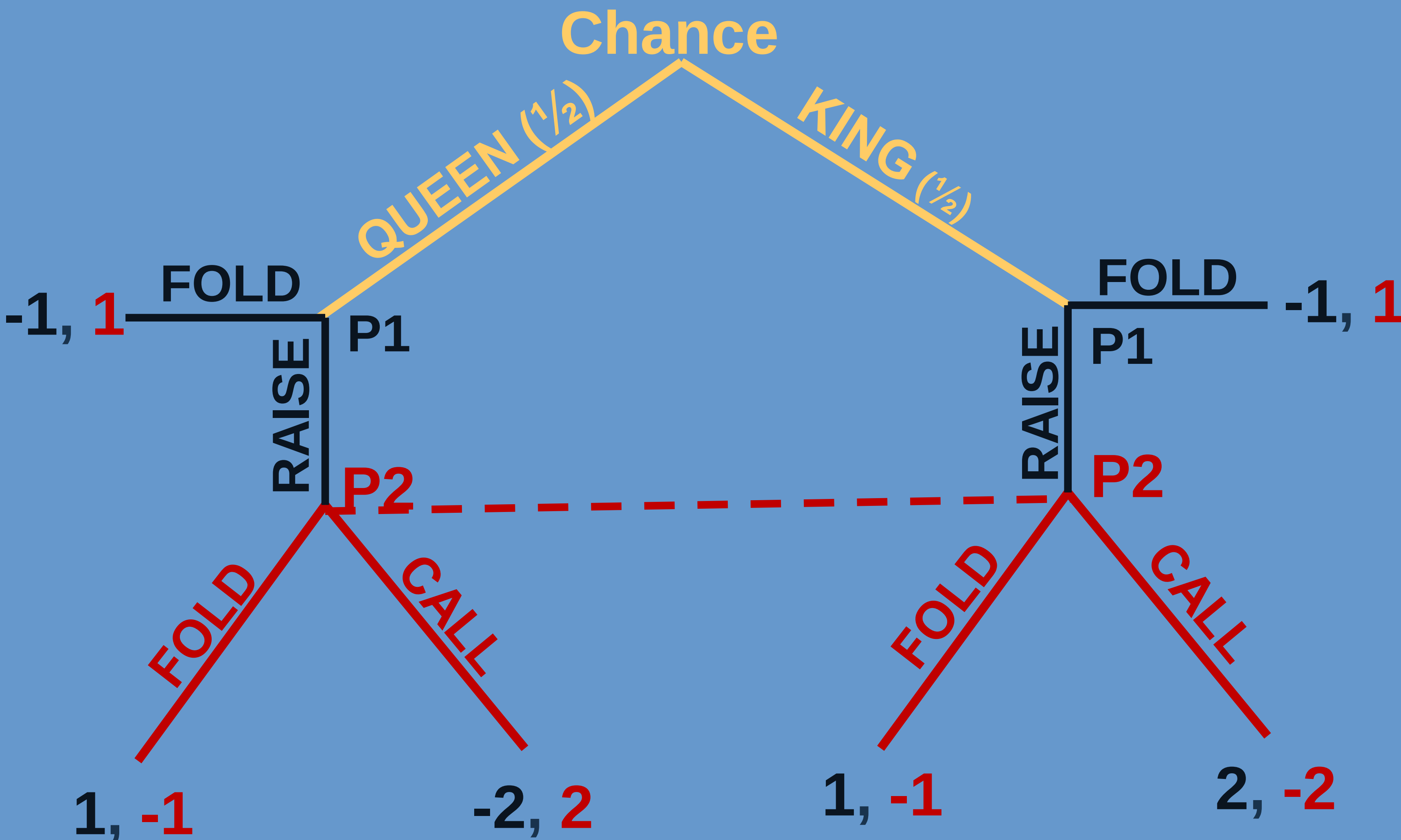
Then the first player (dealer) draws a card from a deck which contains only KINGS and QUEENS. With probability 0.5 player 1 draws KING and with probability 0.5 player 1 draws QUEEN.

After the player 1 privately observes her own card, she moves by either FOLDING or RAISING. FOLD means that the game ends and player 1 lose one dollar, player 2 earns one dollar. RAISE means that she adds an additional dollar to the pot.

After RAISE 2nd player either FOLD (loosing one dollar) or CALL (add additional dollar to the pot). Folding ends the game.

If the 2nd player CALLs player 1 wins the pot is she has KING and loose if she has QUEEN.

Example 2: Simple poker game



Example 2: Simple poker game

- Set of players: player 1, player 2, (chance)
- Terminal histories: {Queen,Fold}, {King,Fold}, {Queen,Raise,Fold}, {Queen,Raise,Call}, {King,Raise,Fold}, {King,Raise,Call},
- Player function: P(ø)=Chance; P(Queen)=1, P(King)=1; P(Queen, Raise)=2, P(King, Raise)=2
- "Chance" – Queen = ½ ; King = ½
- Information partition – Player 1's information partition contains two information set – {Queen}, {King} Player 2's information partition contains two information set – {{Queen,Raise},{King,Raise}}, {{Queen,Fold},{King,Fold}}, .
- Preferences for the players – represented by utility function (payoffs)

Example 3: Signaling games

Signaling game is a dynamic game of incomplete information involving two players: a Sender (S), and a Receiver (R). The timing of the game is as follows:

1) Chance (Nature) draws a type t_i for the Sender from a set of feasible types $T=\{t_1,\ldots t_N\}$ according to probability distribution $\{p_1,\ldots, p_N\}$ such that $p_1+\ldots+p_N=1$

2) The sender observes her type and then chooses a message (signal) m_i from a set of feasible messages $M=\{m_1,\ldots m_J\}$

3) The Receiver observes m_j (but not t_i) and then chooses an action a_k from a set of feasible actions $A=\{a_1,\ldots a_K\}$

Example 3: Signaling game – 2 types

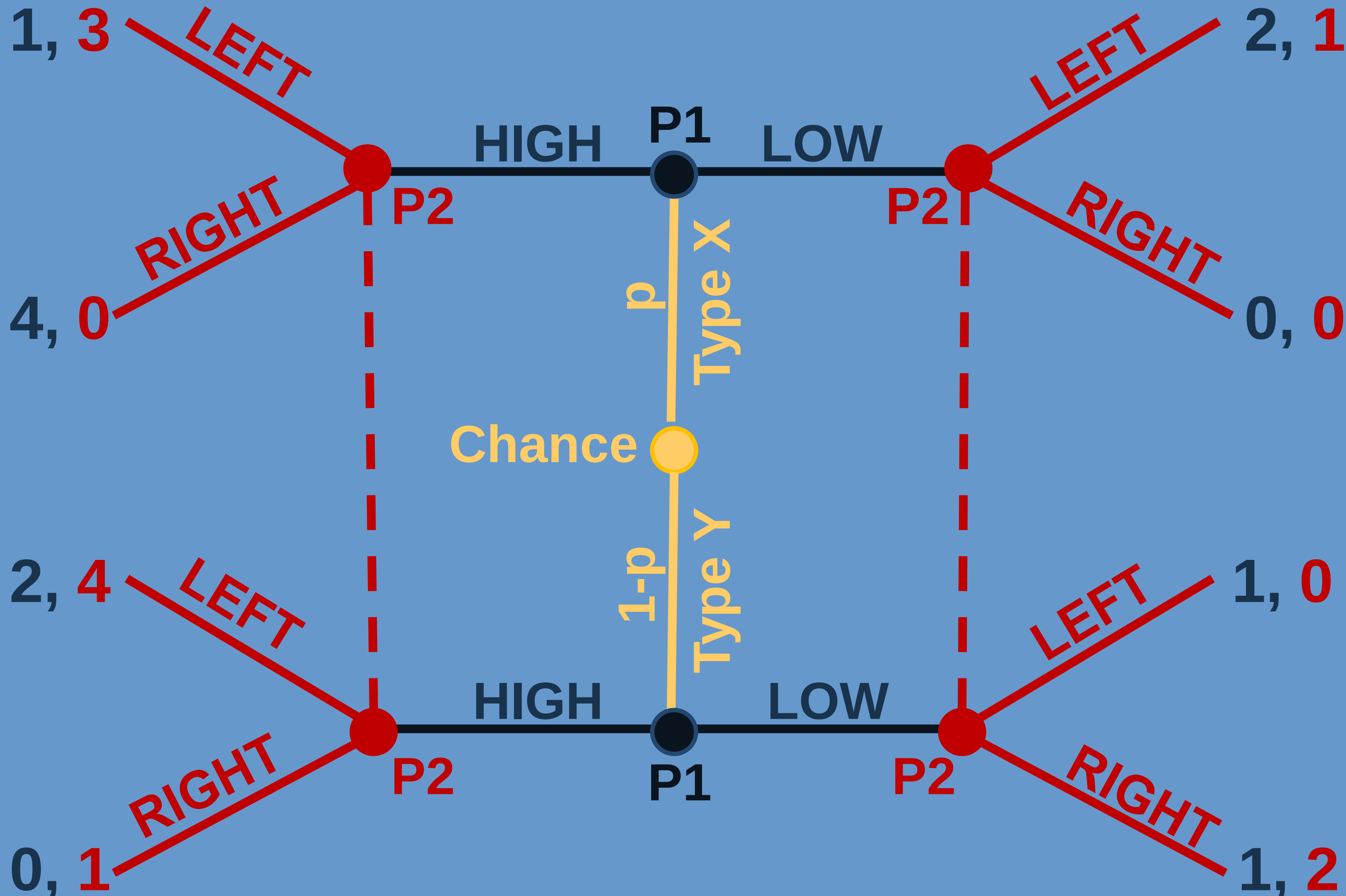
Signaling game is a dynamic game of incomplete information involving two players: a Sender (S), and a Receiver (R). The timing of the game is as follows:

1) Chance (Nature) draws a type t_i for the Sender from a set of feasible types T={X, Y} according to probability distribution such that $p_X+p_Y=1$ ($p_X=p$; $p_Y=1-p$)

2) The sender observes her type and then chooses a message (signal) m_i from a set of feasible messages M={High,Low}

3) The Receiver observes m_j (but not t_i) and then chooses an action a_k from a set of feasible actions A={Left,Right}

Signaling game – 2 types



Example 3: Signaling game – 2 types

- Set of players: player 1, player 2, (chance)
- Terminal histories: {TypeX,High,Left}, {TypeX,High,Right}, {TypeX,Low,Left}, {TypeX,Low,Right},

  {TypeY,High,Left}, {TypeY,High,Right}, {TypeY,Low,Left}, {TypeY,Low,Right},
- Player function: P(ø)=Chance; P(TypeX)=1, P(TypeY)=1; P(TypeX, High)=2, P(TypeX, Low)=2, P(TypeY, High)=2, P(TypeY, Low)=2,
- “Chance” – TypeX = p ; TypeY = 1-p
- Information partition – Player 1’s information partition contains two information set – {TypeX}, {TypeY} Player 2’s information partition contains also two information set – {{TypeX,High},{TypeY,High}}, {{TypeX,Low},{TypeY,Low}}

Information set

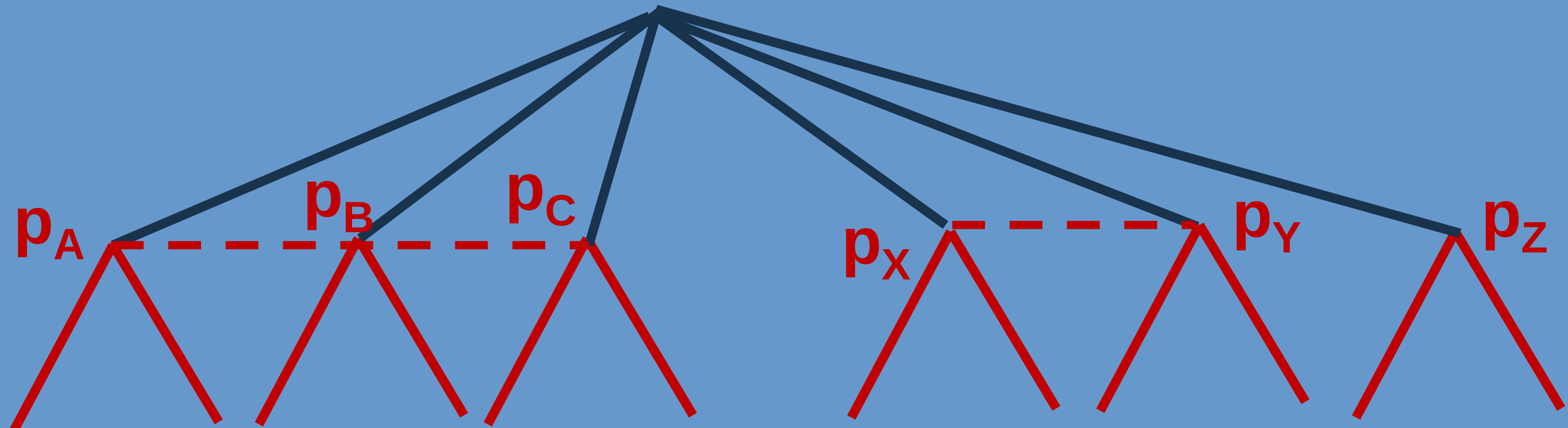
- **definition**:
  - collection of decision nodes (histories after which it is player's turn) such that:

1. when the play reaches a node in the information set, the player with the move does not know which node in the information set has been reached.
2. The player has the move and same set of choices at each node in the Information set

Belief system

We assume that if an information set that contains more than one history, the player whose turn it is to move forms a belief about the history that has occurred.

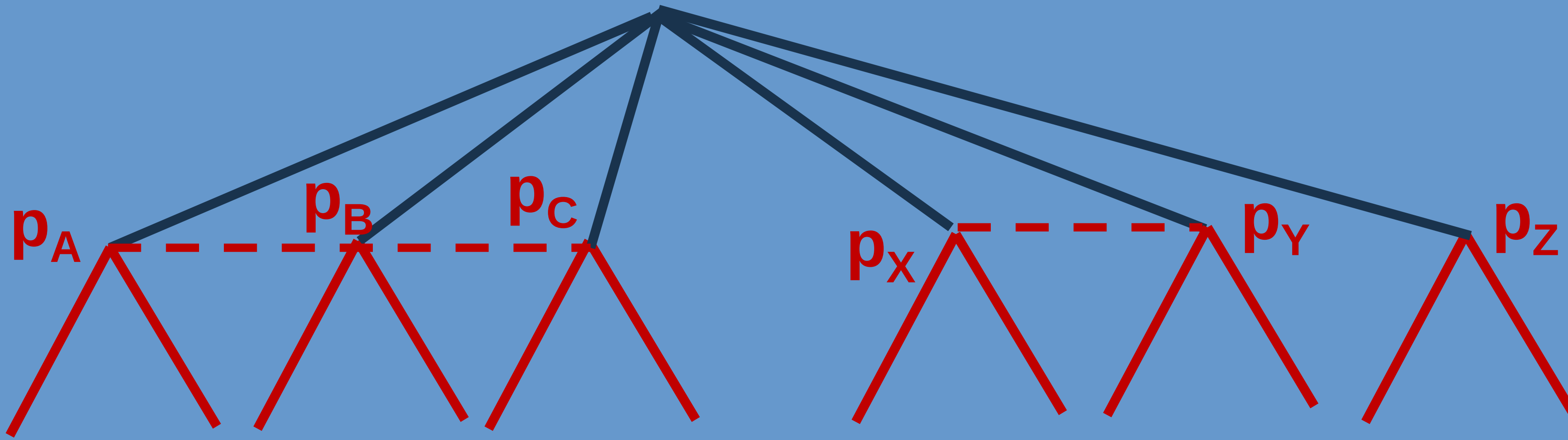
We model this belief as a probability distribution over the histories in the information set.



$p_A+p_B+p_C=1$ $p_X+p_Y=1$ $p_Z=1$

Belief system

Definition: Belief system assigns to each information set a probability distribution over the decision nodes (histories) in that information set.

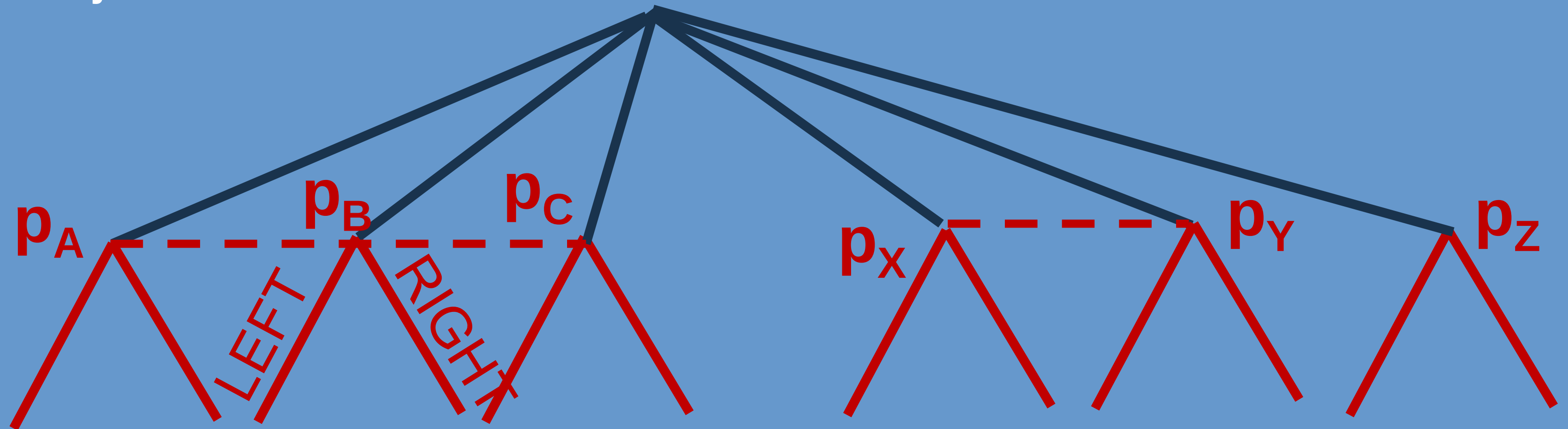


$$p_A+p_B+p_C=1 \qquad p_X+p_Y=1 \qquad p_Z=1$$

Behavioral strategy

In the case of dynamic games with perfect information, the strategy defines action at every node where it is the player's turn.

However, now the player does not know the exact node (history), but just the information set.

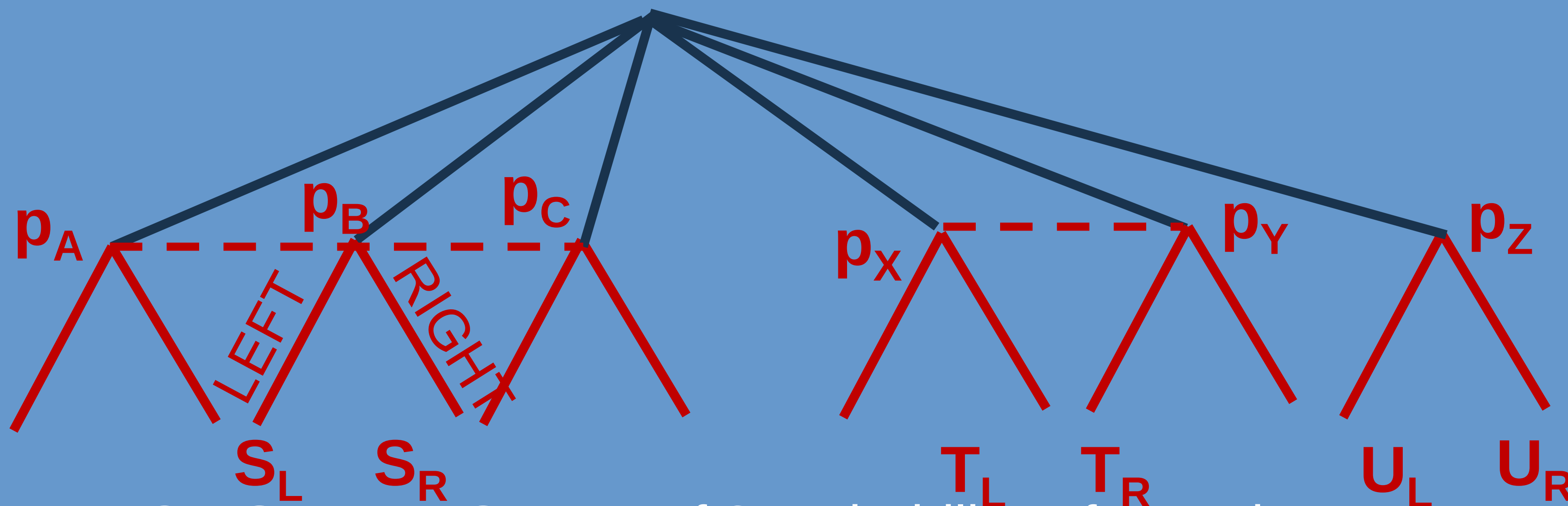


BEHAVIORAL STRATEGY: PLAN OF ACTION - assigns action for each information set at which it is the player's turn

Behavioral strategy

To incorporate both pure and mixed strategies:

BEHAVIORAL STRATEGY: assigns to each information set at which it is the player's turn a probability distribution over all feasible actions of the player in that information set.



BEHAVIOR STRATEGY: set of 2 probabilities for each information set, such that $S_L+S_R=1$ $T_L+T_R=1$ $U_L+U_R=1$

Weak Sequential Equilibrium

More complex and richer games → strengthening equilibrium concept

1. Static games with complete information – Nash equilibrium
2. Dynamic games with complete information – Subgame perfect Nash equilibrium
3. Static games with incomplete information – Bayesian Nash equilibrium
4. Dynamic games with incomplete information – weak sequential equilibrium - refines Bayesian Nash equilibrium in the same sense as Subgame perfect Nash equilibrium refines Nash equilibrium

   Introduces sequential rationality into Bayesian Nash equilibrium

Weak Sequential Equilibrium

Definition: A weak sequential equilibrium consists of behavioral strategies and beliefs systems satisfying following conditions 1-2

1. Sequential rationality - Each players' strategy is optimal whenever she has to move, given her belief and the other players' strategies.
2. Consistency of beliefs with strategies – Each player's belief is consistent with strategy profile (behavioral strategies of all players)

Weak Sequential Equilibrium

Definition: A weak sequential equilibrium consists of behavioral strategies and beliefs systems satisfying following conditions 1-2

1. Sequential rationality - Each players’ strategy is optimal in the part of the game that follows each of her information sets, given the strategy profile and her belief about the history in the information set that has occurred. In other words – for each player i and each information set of player i, according to player’s i beliefs, her behavioral strategy gives her the highest possible expected utility so that she has no incentive to deviate from her behavioral strategy in any information set.

Weak Sequential Equilibrium

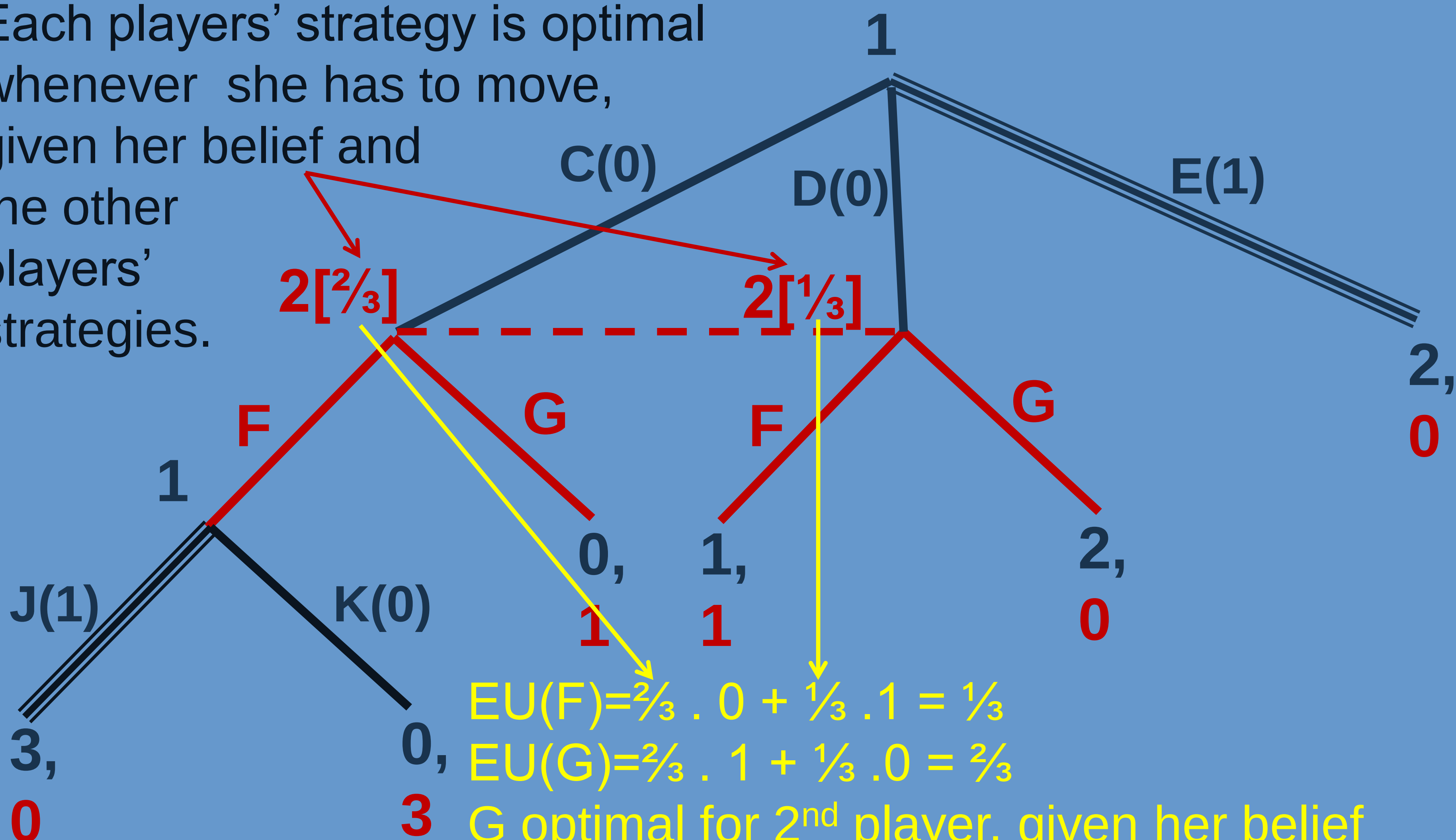
Definition: A weak sequential equilibrium consists of behavioral strategies and beliefs systems satisfying following conditions 1-2

2. Weak consistency of beliefs with strategies – For every information set I_i reached with positive probability given the players strategies, the probability assigned by the belief system to each history h* in I_i is given by

$$\frac{P(h^{*}\text{according to strategy profile})}{\sum_{h\in I_i} P(h\text{ according to strategy profile})}$$

Sequential rationality

Each players' strategy is optimal whenever she has to move, given her belief and the other players' strategies.



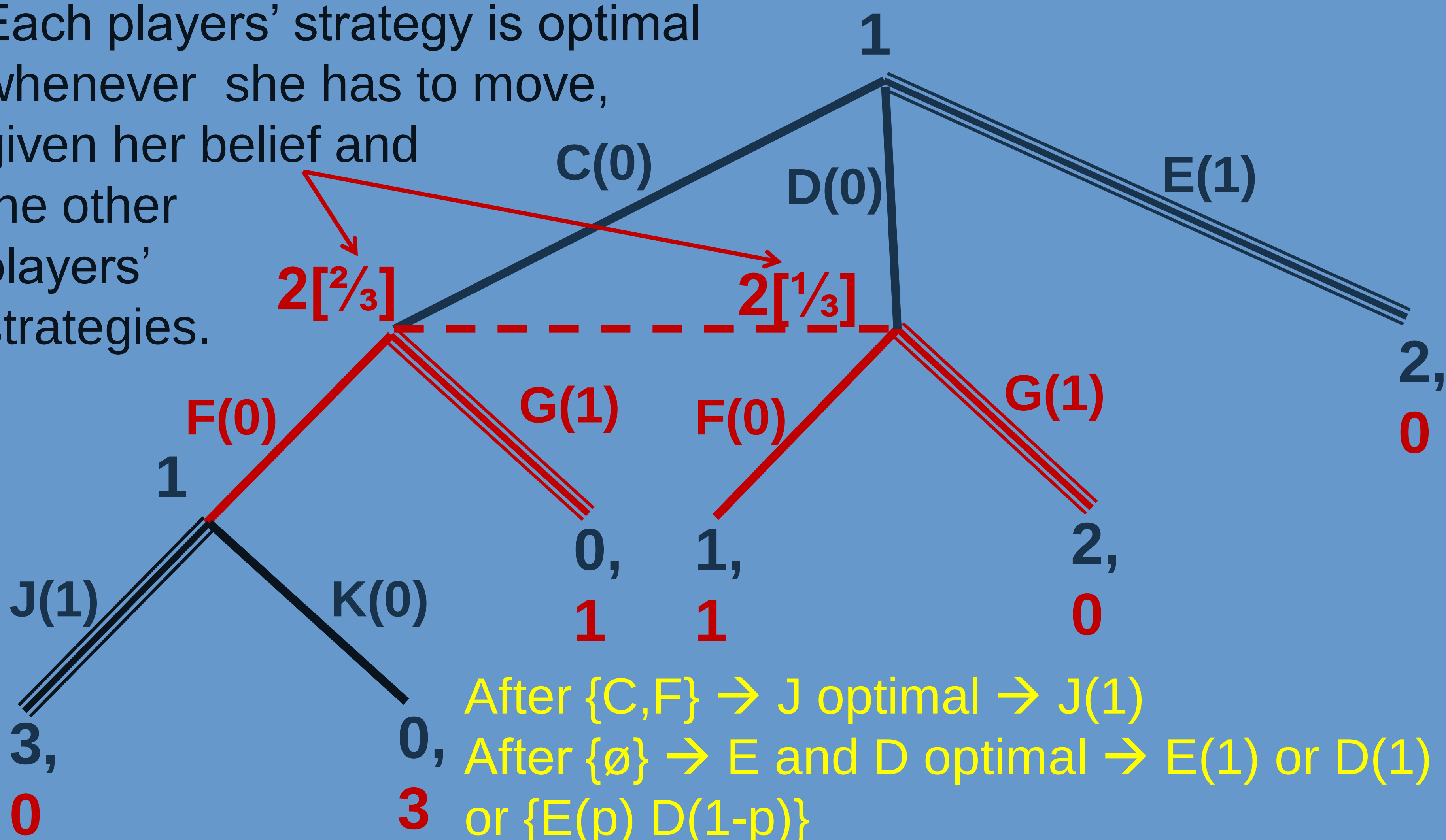
EU(F)=⅔ . 0 + ⅓ .1 = ⅓

EU(G)=⅔ . 1 + ⅓ .0 = ⅔

G optimal for 2nd player, given her belief

Sequential rationality

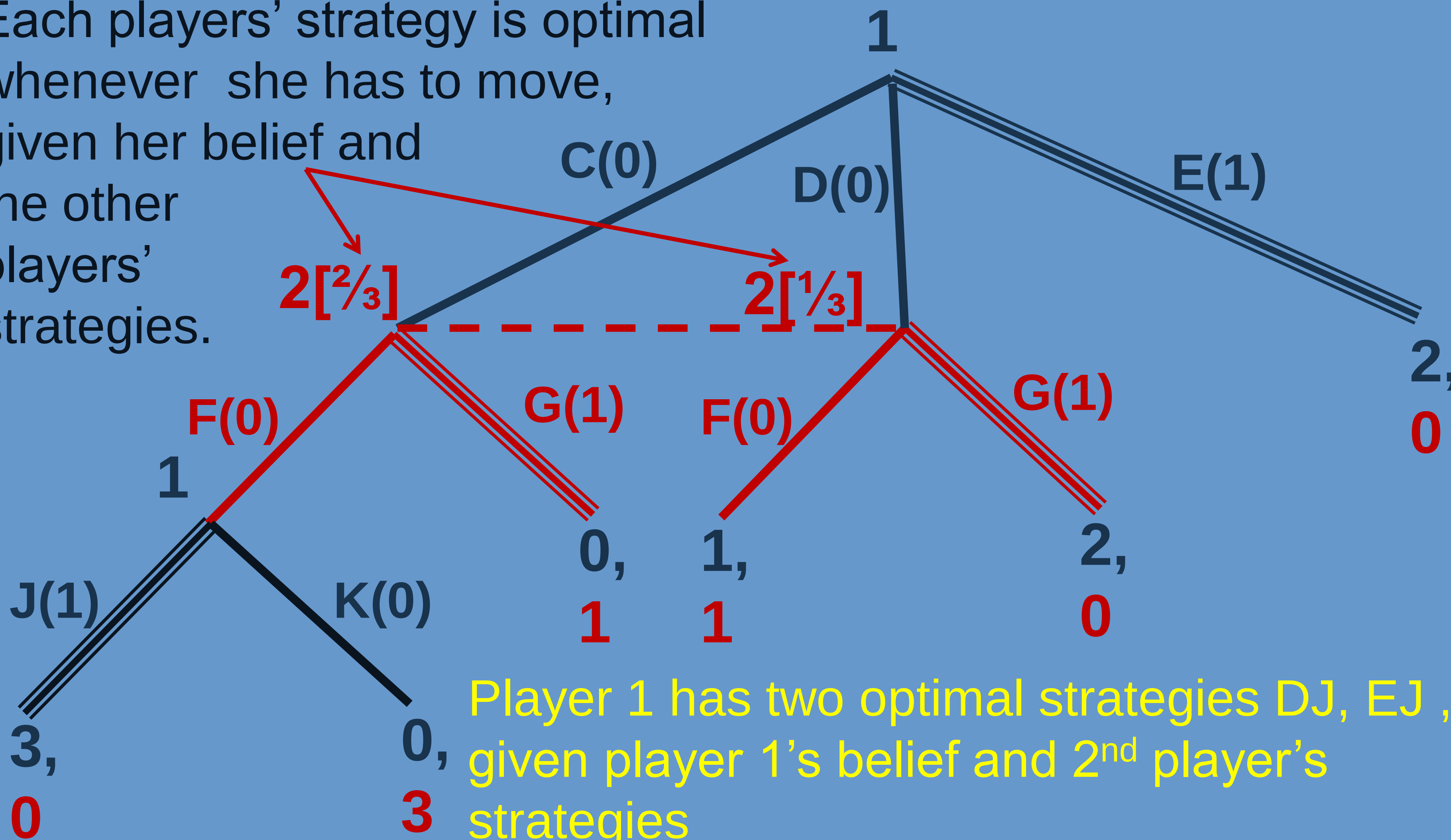
Each players' strategy is optimal whenever she has to move, given her belief and the other players' strategies.



After {C,F} → J optimal → J(1)
After {ø} → E and D optimal → E(1) or D(1) or {E(p) D(1-p)}

Sequential rationality

Each players' strategy is optimal whenever she has to move, given her belief and the other players' strategies.



Player 1 has two optimal strategies DJ, EJ, given player 1's belief and 2nd player's strategies

Consistency of beliefs with strategies

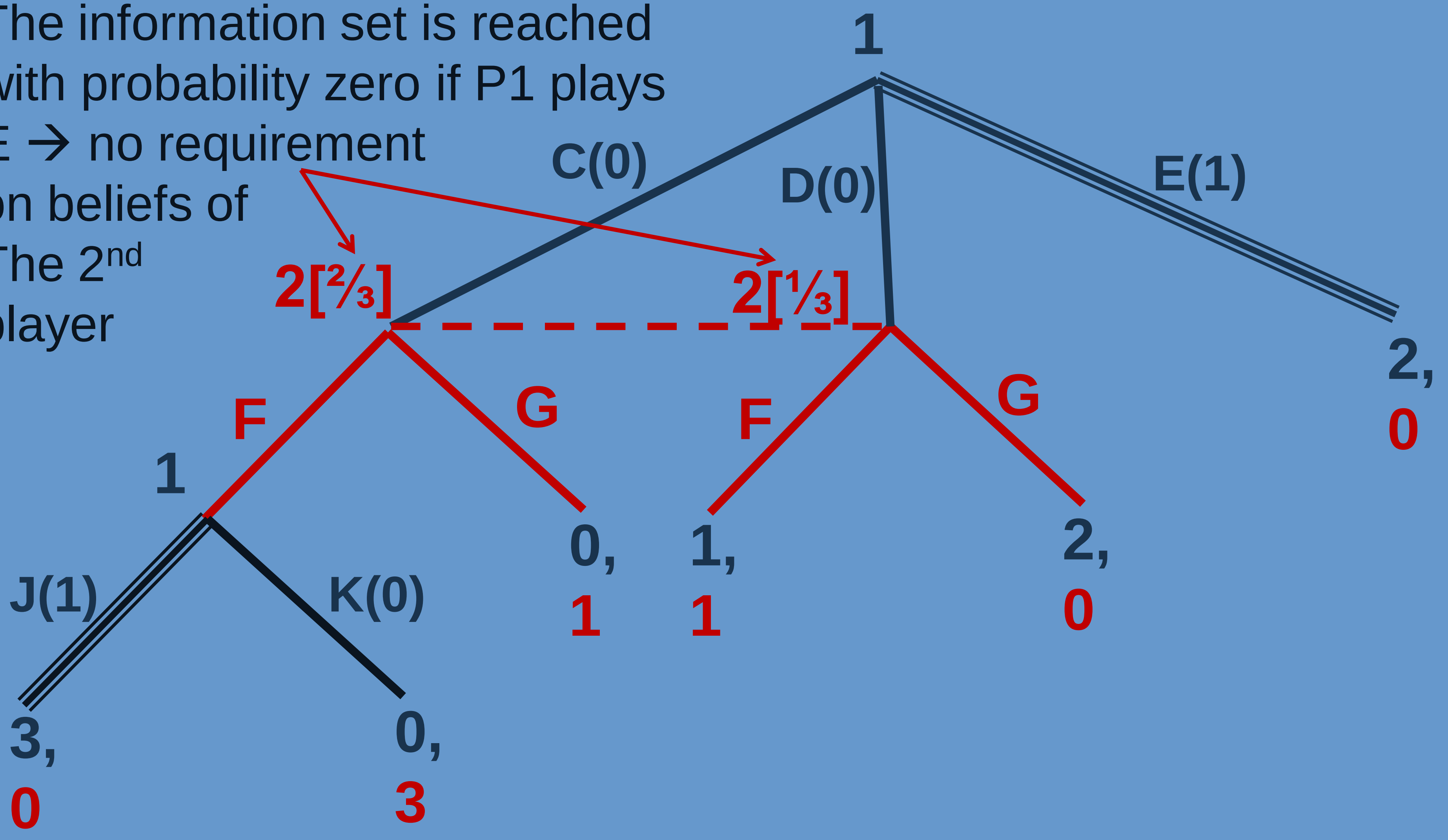
Consistency of beliefs with strategies – Each player's belief is consistent with strategy profile (behavioral strategies of all players)

The idea is that in a steady state, each player's belief must be correct: the probability it assigns to any history must be the probability with which that history occurs if the players adhere to their strategies.

If some information set is reached with probability 0 → player may have any belief at such information set.

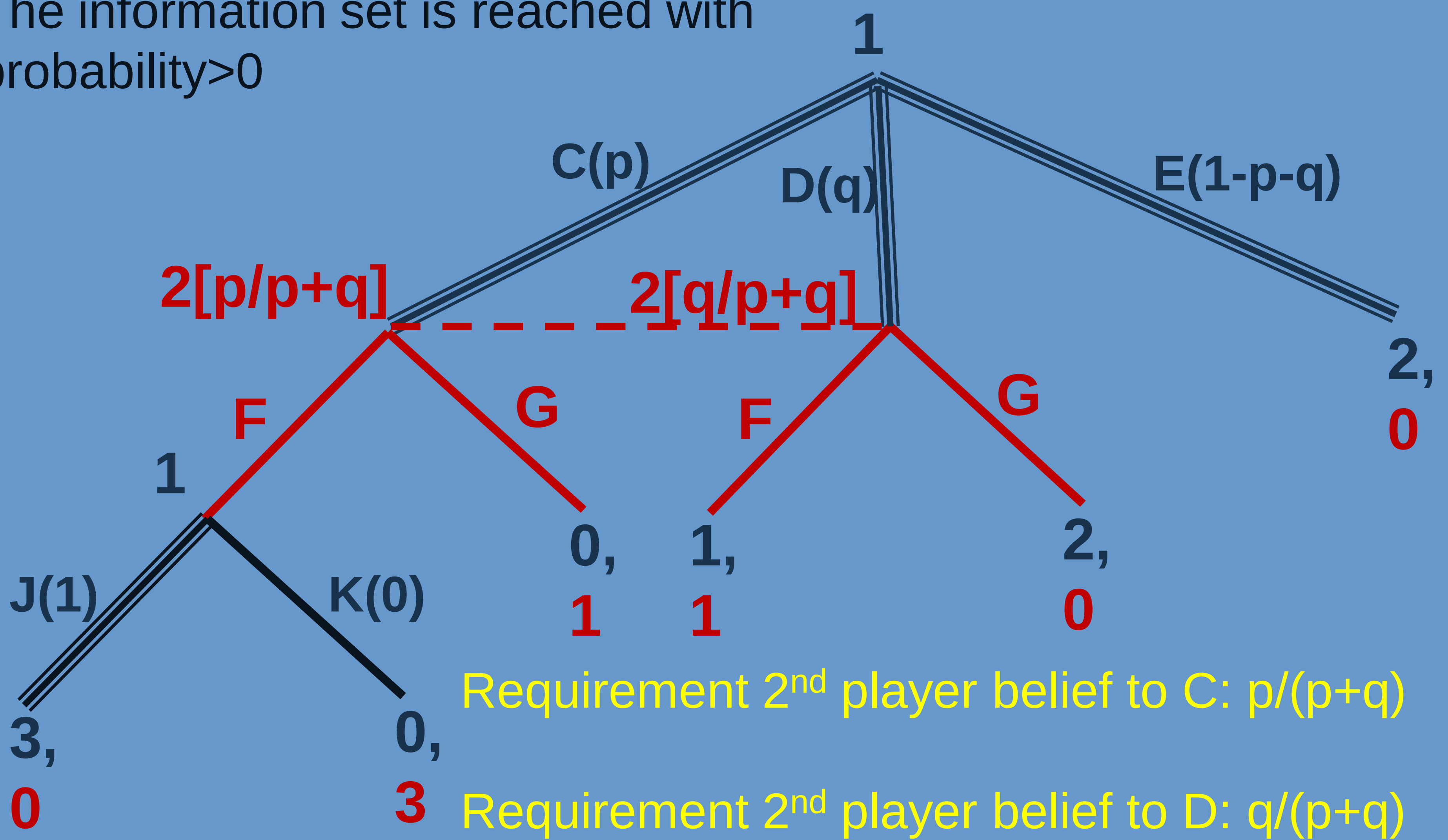
Consistency of beliefs with strategies

The information set is reached with probability zero if P1 plays E → no requirement on beliefs of The 2nd player



Consistency of beliefs with strategies

The information set is reached with probability>0

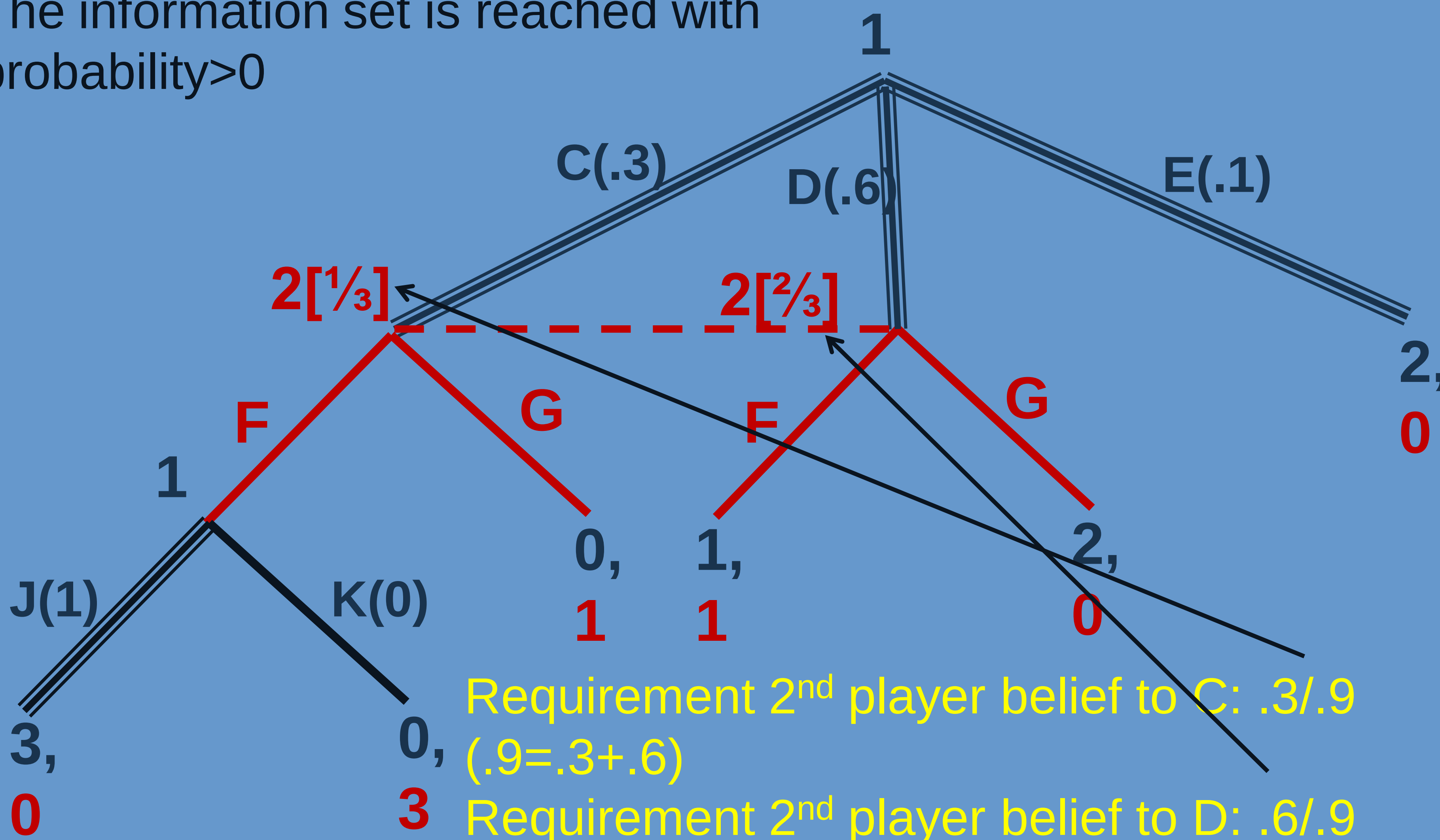


Requirement 2nd player belief to C: p/(p+q)

Requirement 2nd player belief to D: q/(p+q)

Consistency of beliefs with strategies

The information set is reached with probability>0



Requirement 2nd player belief to C: .3/.9 (.9=.3+.6)

Requirement 2nd player belief to D: .6/.9

Finding weak sequential equilibria

1) If the game has any subgame → find at first the weak sequential equilibria of the subgame

2) Identify all possible strategies of all players

First way:

3) Find all NE of the game – in similar way as in the case of dynamic games – table with all possible strategies
4) Compute the beliefs – consistent with the strategies
5) Check all NE whether they satisfies the 2 conditions for the weak sequential equilibrium

Finding weak sequential equilibria

1) If the game has any subgame → find at first the weak sequential equilibria of the subgame

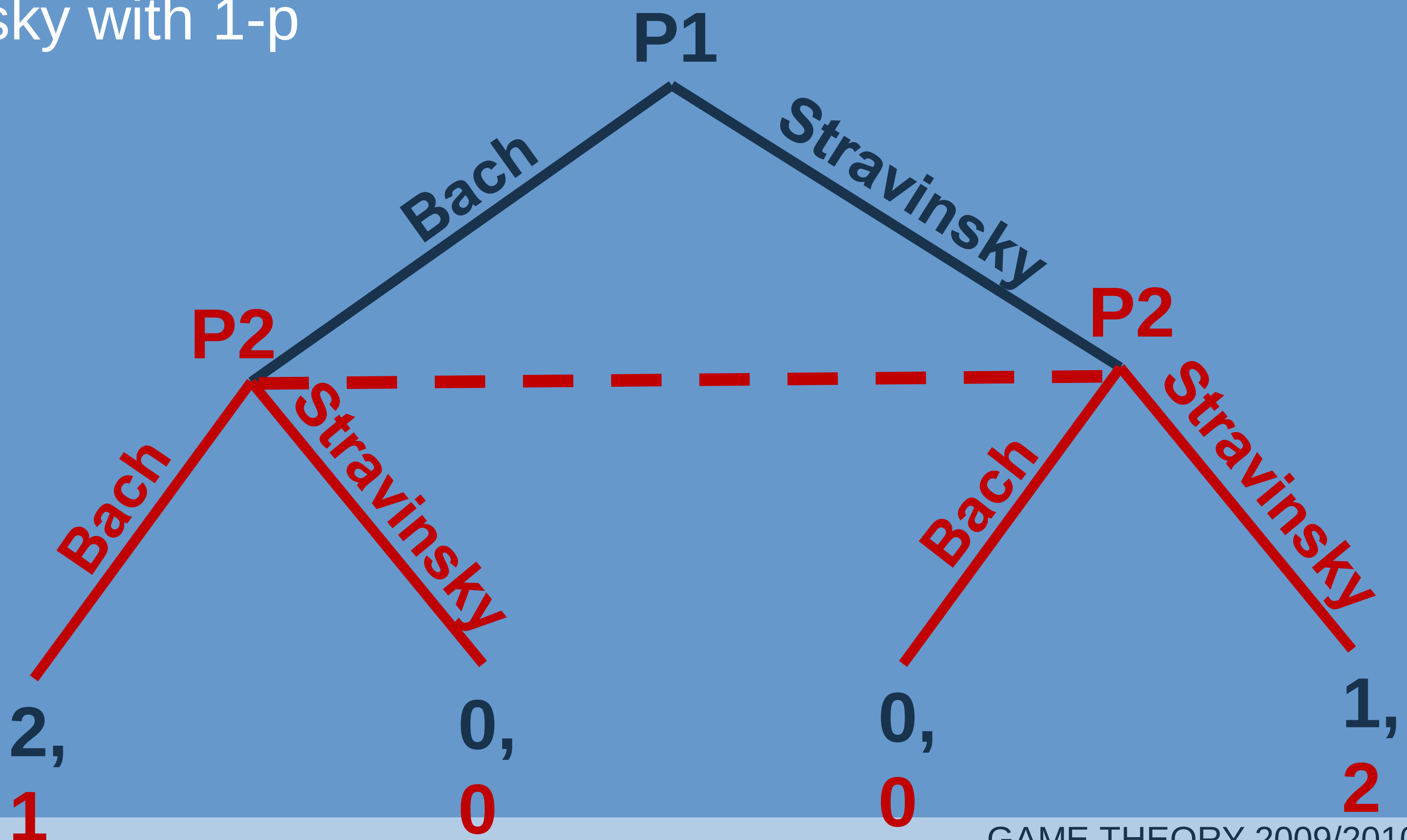
2) Identify all possible strategies of all players

SIMPLE GAMES: (2 players, finite number of actions)

3) In our quite simple games - start from the beginning by analyzing one after each other strategies of the first player and compute the respective beliefs of the other players, given the strategy of first player
4) Continue by finding the optimal strategies of further players, given their beliefs and strategies of the other players.
5) Check for the equilibrium

B or S: imperfect information

1) No Subgame
2) Strategies of P1: B(1) - bach with probability 1, S(1) – stravinsky with probability 1, B(p)S(1-p) – bach with probability p, stravinsky with 1-p

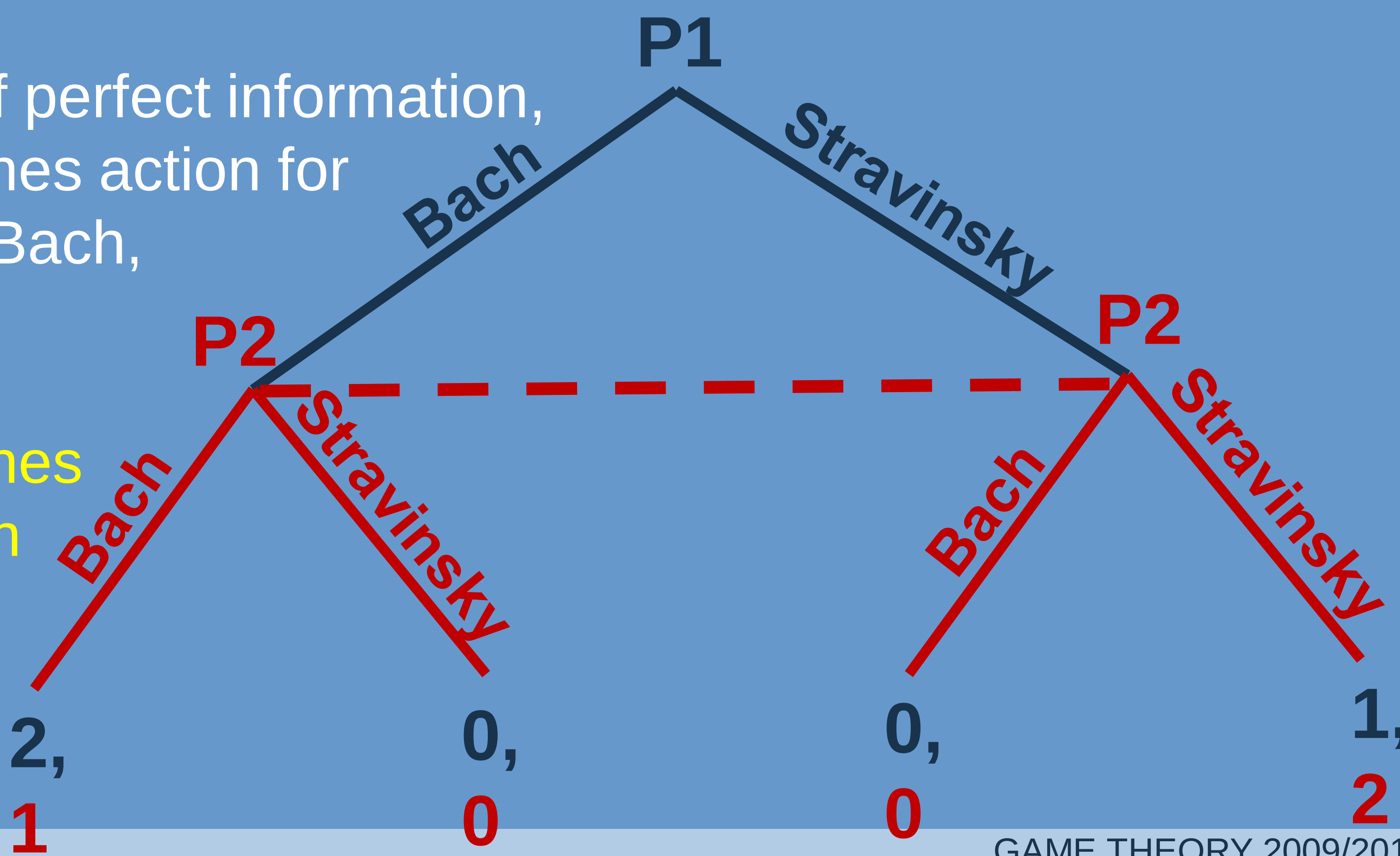


B or S: imperfect information

2) Strategies of P2: B(1) - bach with probability 1, S(1) – stravinsky with probability 1, B(p)S(1-p) – bach with probability p, stravinsky with 1-p

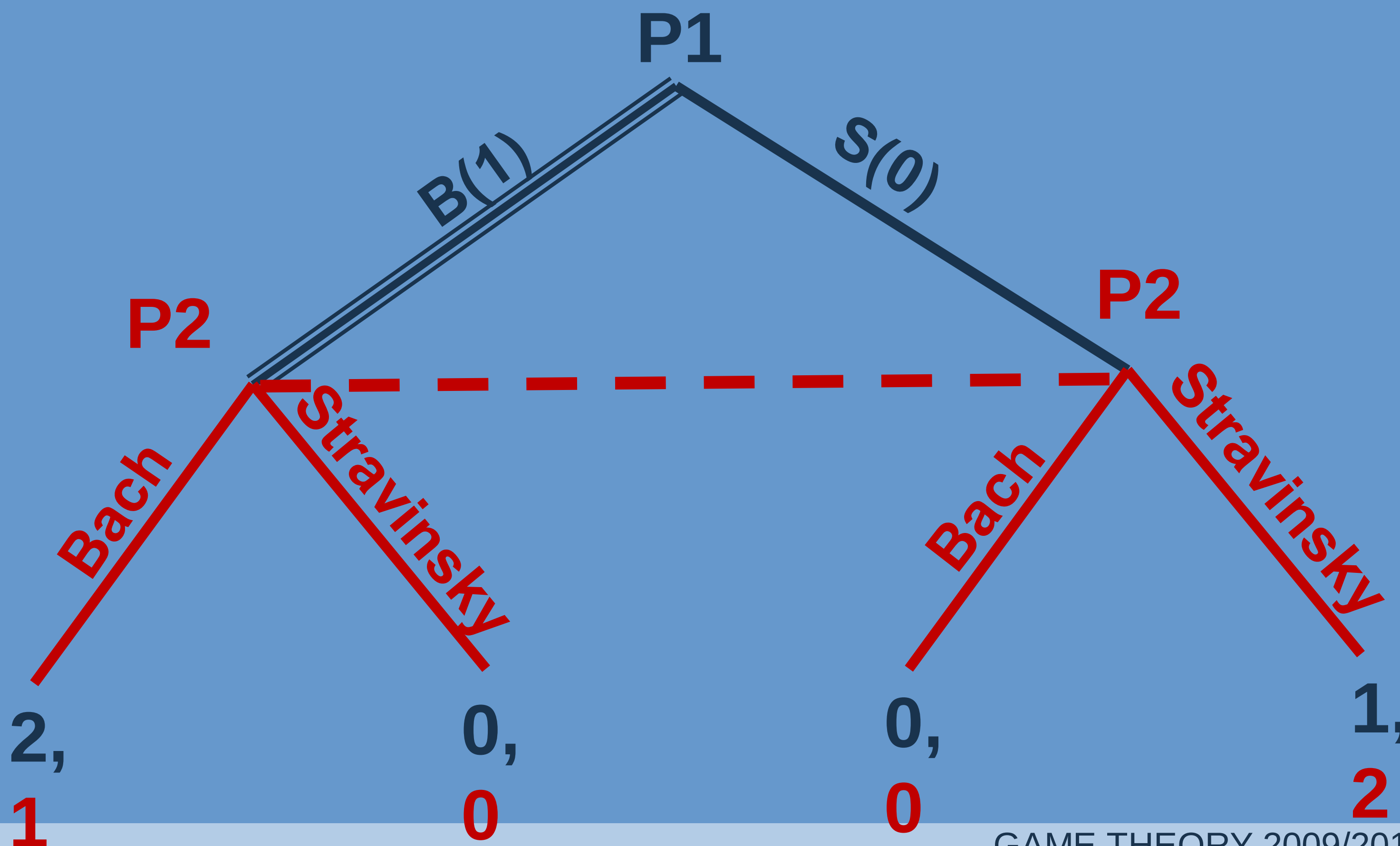
In the case of perfect information, Strategy defines action for each history Bach, Stravinsky

NOW!! Strategy defines action at each Information set



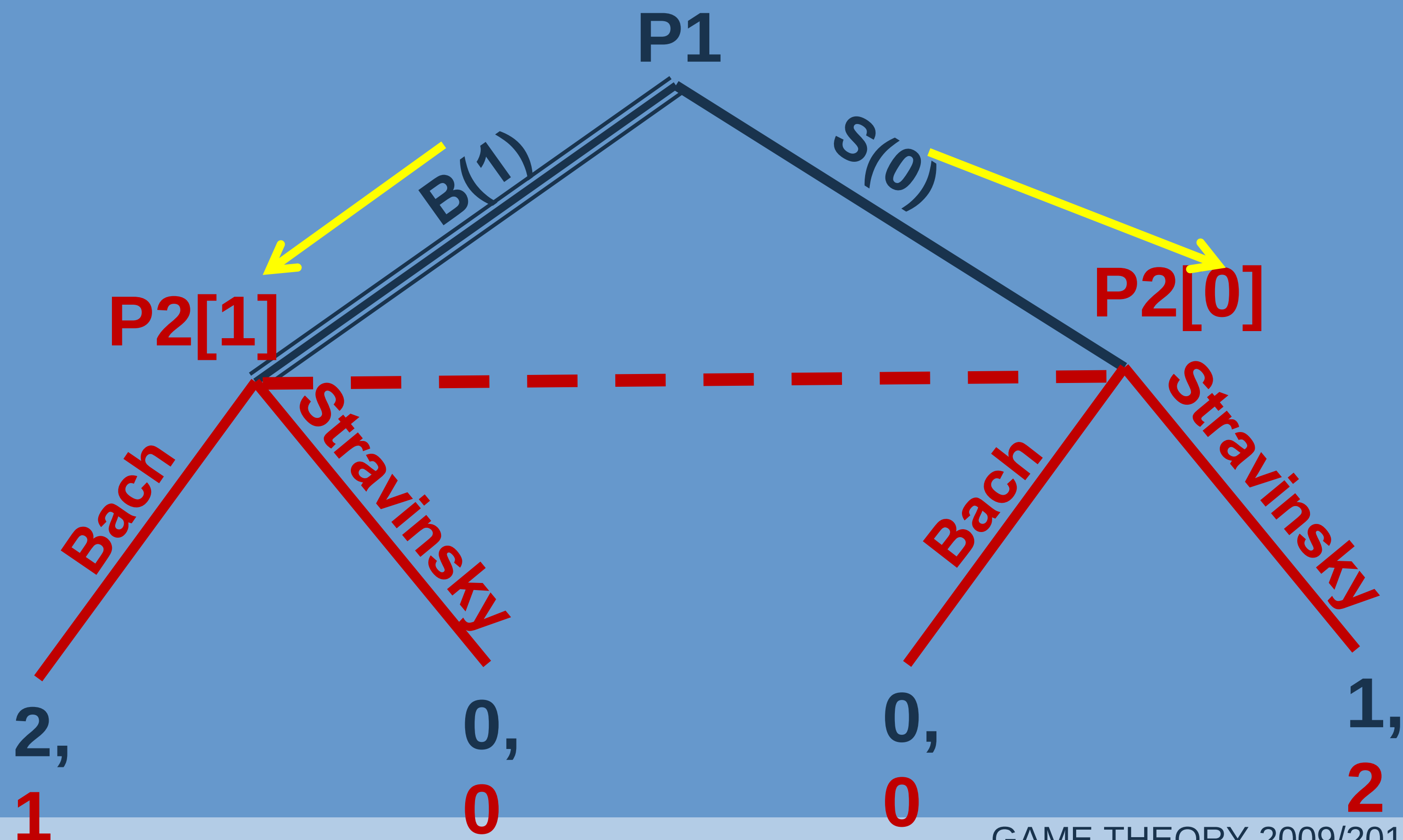
B or S: imperfect information

2) Strategies of P1: B(1) - bach with probability 1
3) Beliefs of P2



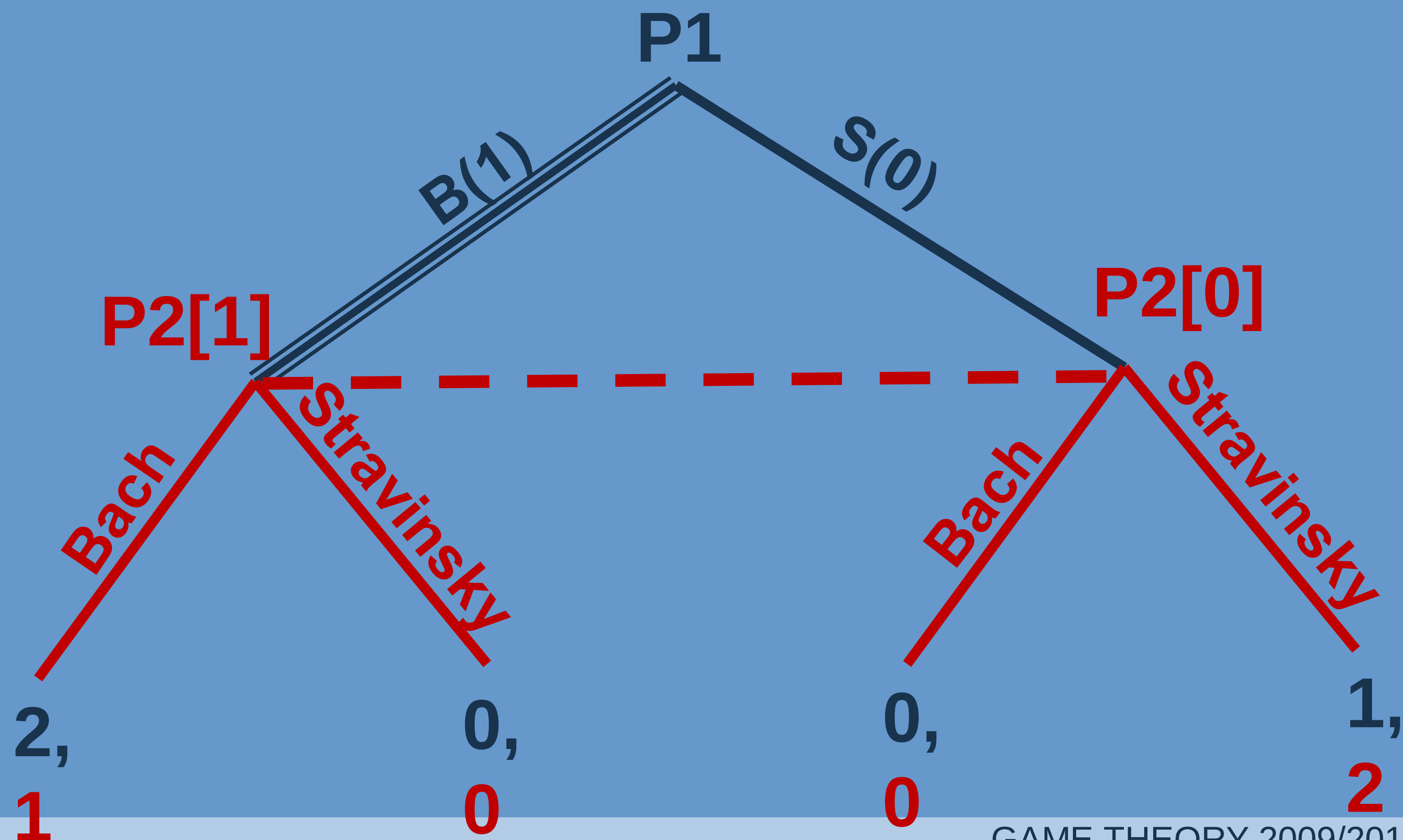
B or S: imperfect information

2) Strategies of P1: B(1) - bach with probability 1
3) Beliefs of P2 - belief is consistent with strategy profile



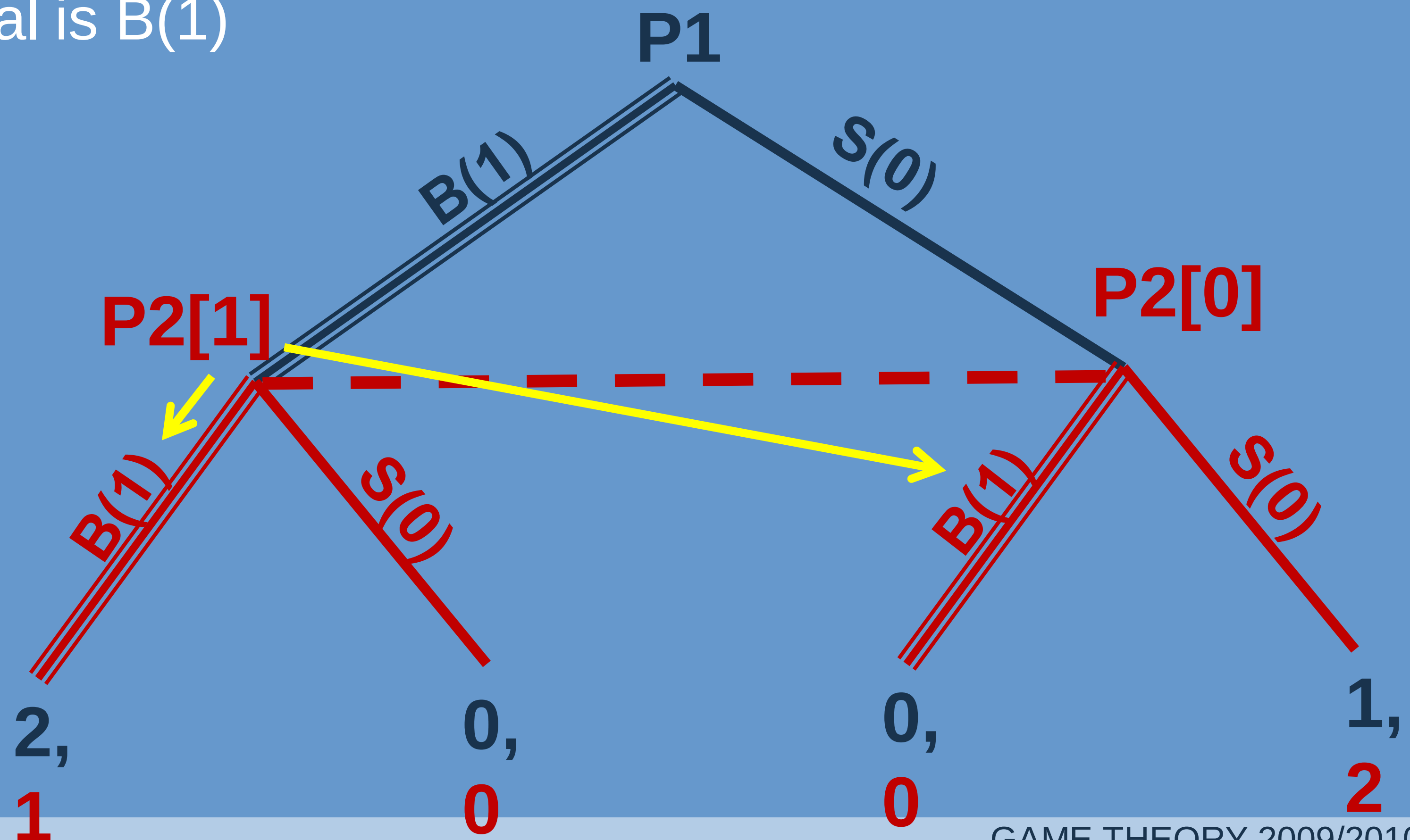
B or S: imperfect information

2) Strategies of P1: B(1) - bach with probability 1
3) Beliefs of P2
4) Optimal strategy of P2, given her belief and P1 plays B(1)



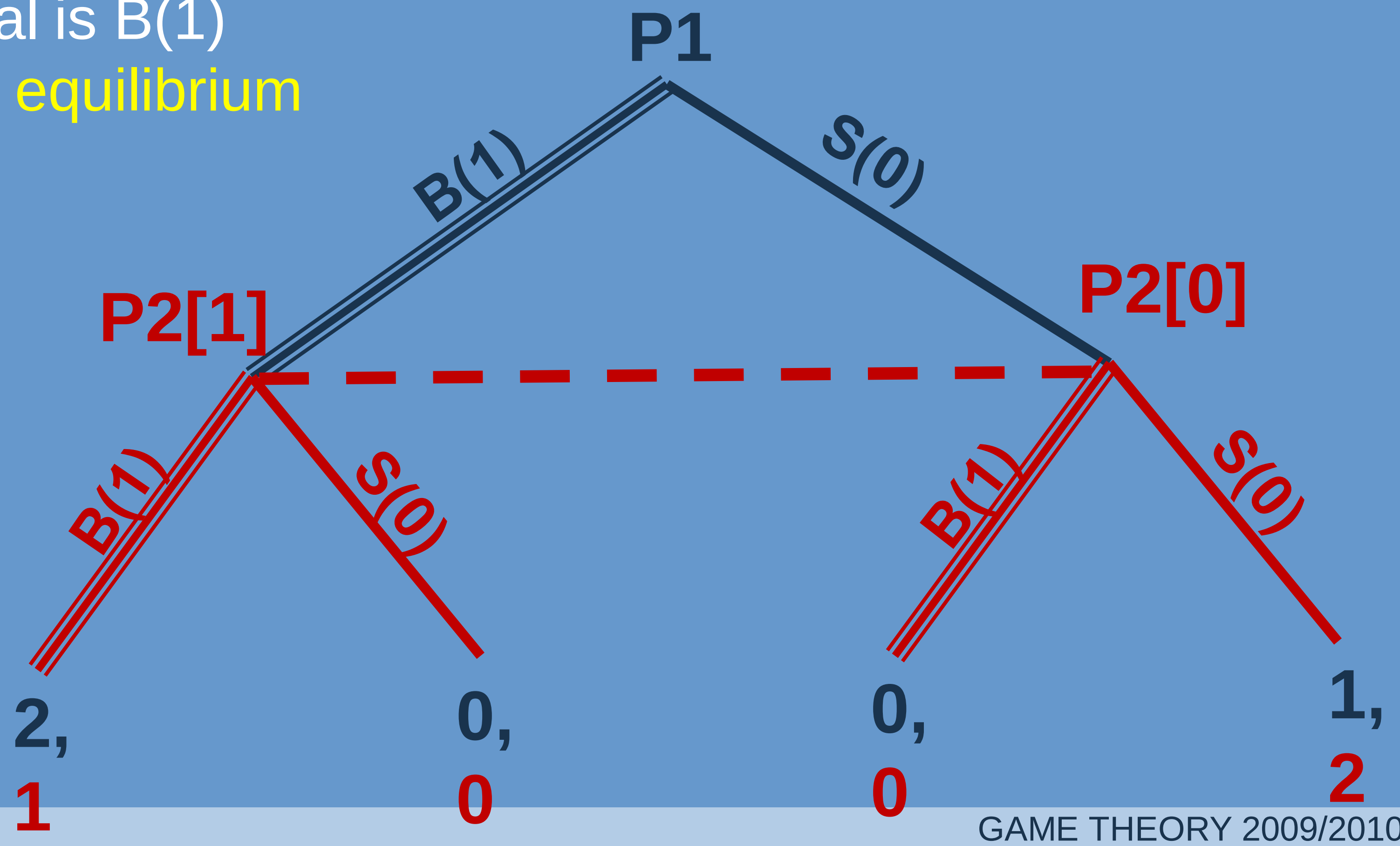
B or S: imperfect information

2) Strategies of P1: B(1) - bach with probability 1
3) Beliefs of P2
4) Optimal strategy of P2, given her belief and P1 plays B(1) –
   Optimal is B(1)



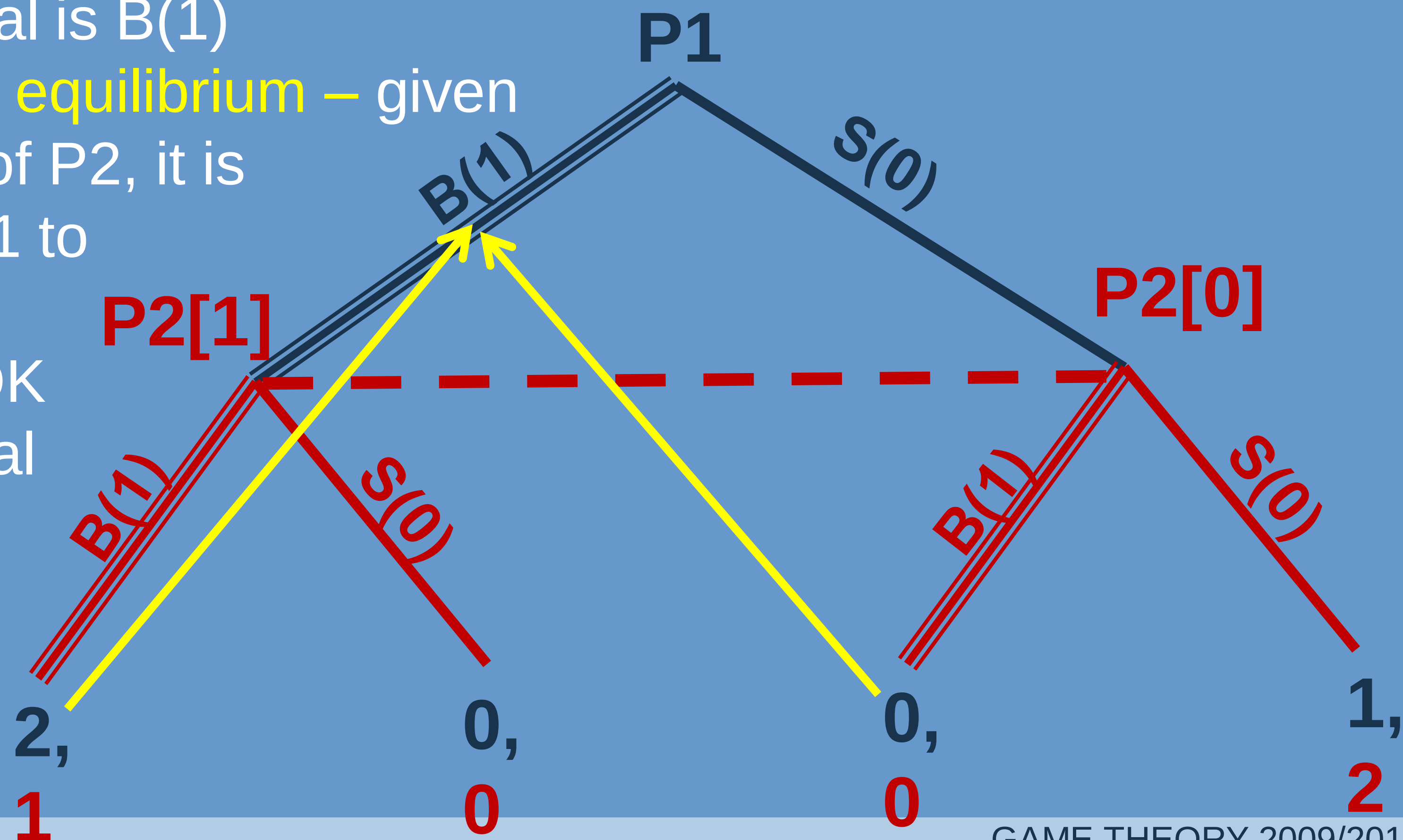
B or S: imperfect information

2) Strategies of P1: B(1) - bach with probability 1
3) Beliefs of P2
4) Optimal strategy of P2, given her belief and P1 plays B(1) –
   Optimal is B(1)
5) Check for equilibrium



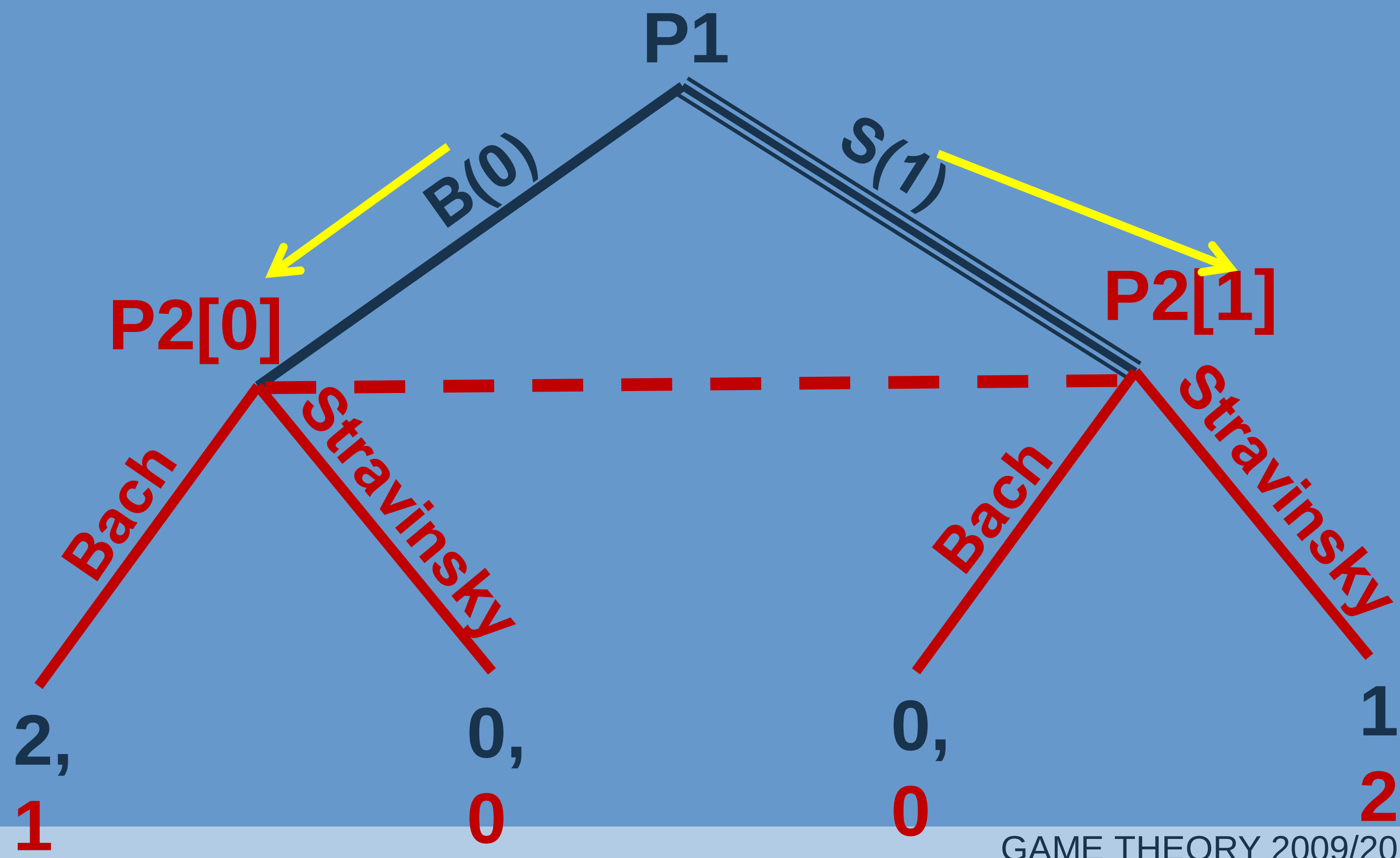
B or S: imperfect information

2) Strategies of P1: B(1) - bach with probability 1
3) Beliefs of P2
4) Optimal strategy of P2, given her belief and P1 plays B(1) –
   Optimal is B(1)
5) Check for equilibrium – given the strategy of P2, it is optimal for P1 to play bach→ equilibrium OK B(1) is optimal



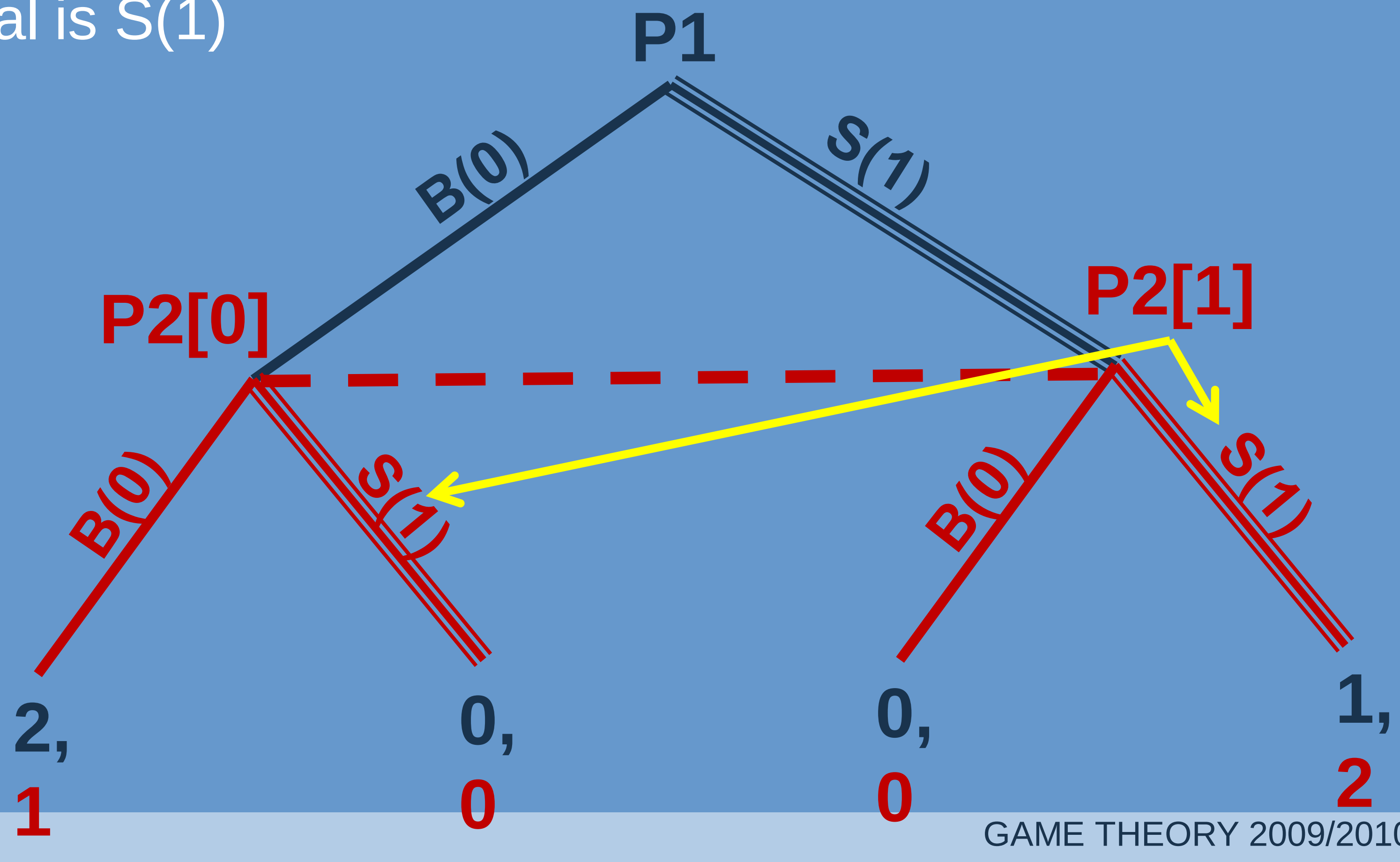
B or S: imperfect information

2) Strategies of P1: B(1) - stravinsky with probability 1
3) Beliefs of P2 - belief is consistent with strategy profile



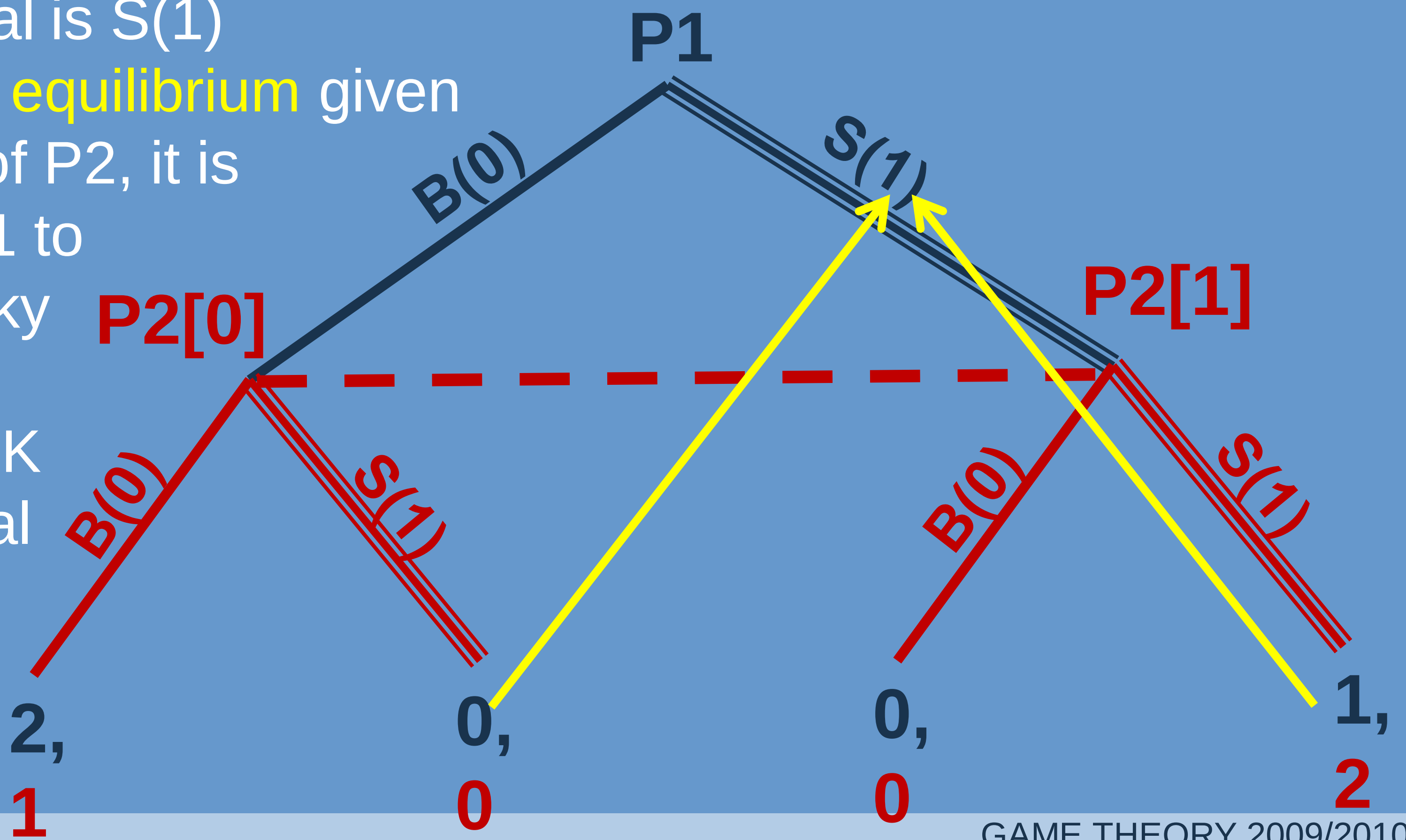
B or S: imperfect information

2) Strategies of P1: B(1) - bach with probability 1
3) Beliefs of P2
4) Optimal strategy of P2, given her belief and P1 plays S(1) –
   Optimal is S(1)



B or S: imperfect information

2) Strategies of P1: B(1) - bach with probability 1
3) Beliefs of P2
4) Optimal strategy of P2, given her belief and P1 plays S(1) –
   Optimal is S(1)
5) Check for equilibrium given the strategy of P2, it is optimal for P1 to play Stravinsky
→
equilibrium OK
S(1) is optimal



B or S: imperfect information

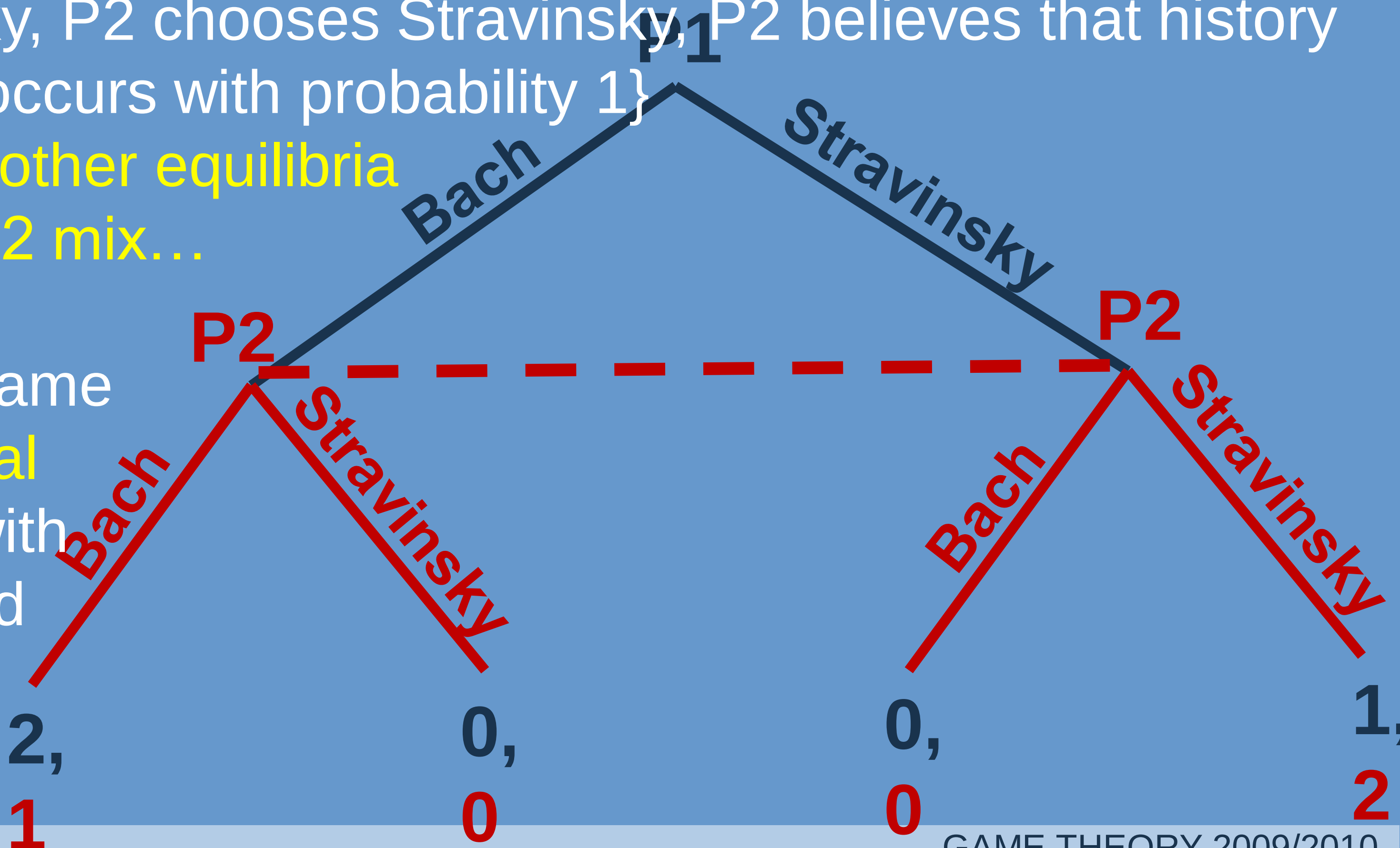
The game has weak sequential equilibria:

{P1 – Bach, P2 chooses Bach, P2 believes that history Bach occurs with probability 1}

{P1 – Stravinsky, P2 chooses Stravinsky, P2 believes that history Stravinsky occurs with probability 1}

There are also other equilibria when P1 and P2 mix…

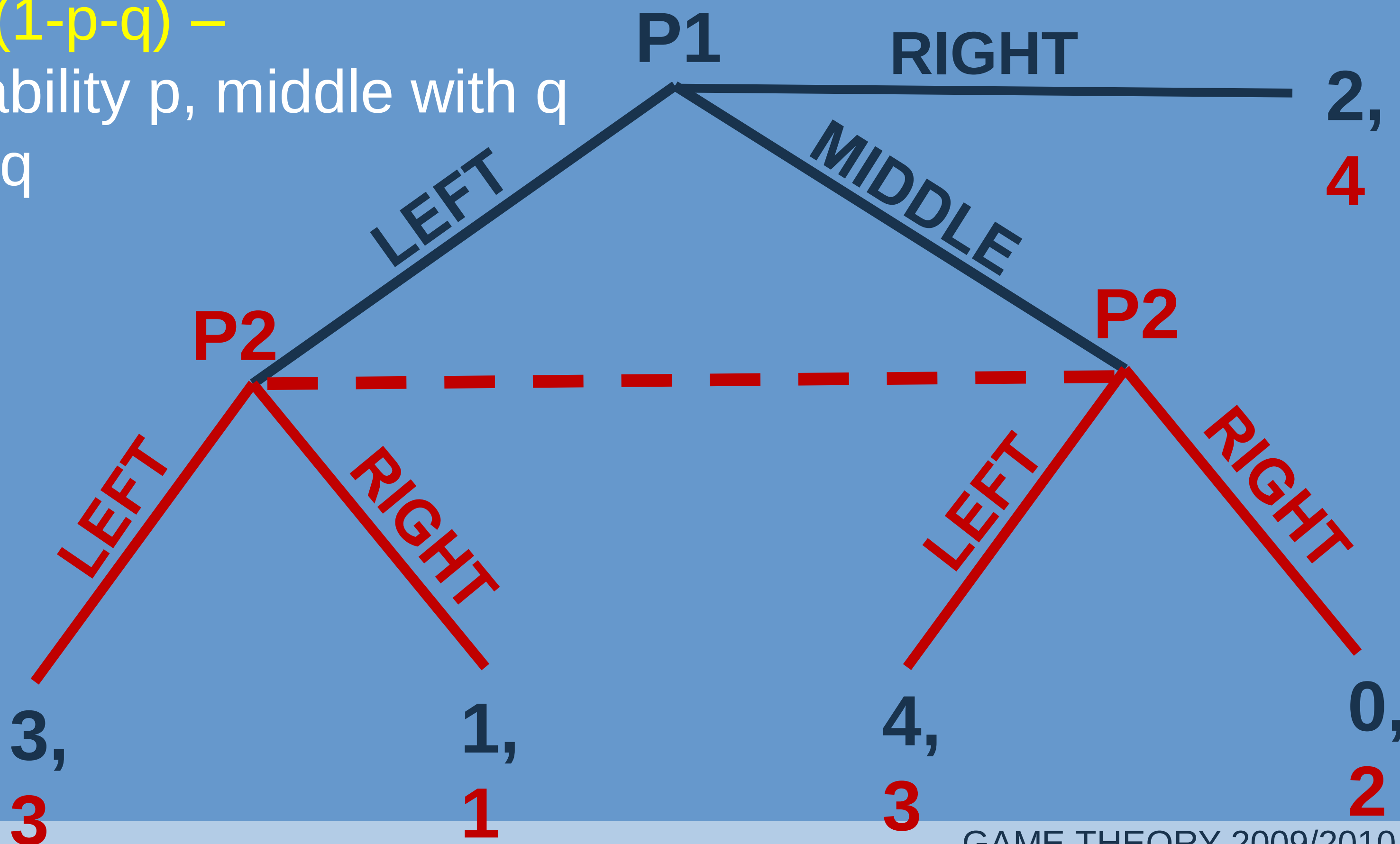
In this simple game Weak sequential eq. coincides with both, SBNE and NE



Example 1

1) No Subgame
2) Strategies of P1: L(1) - left with probability 1, M(1) – middle with probability 1, R(1) – right with probability 1, L(p)M(q)R(1-p-q) –

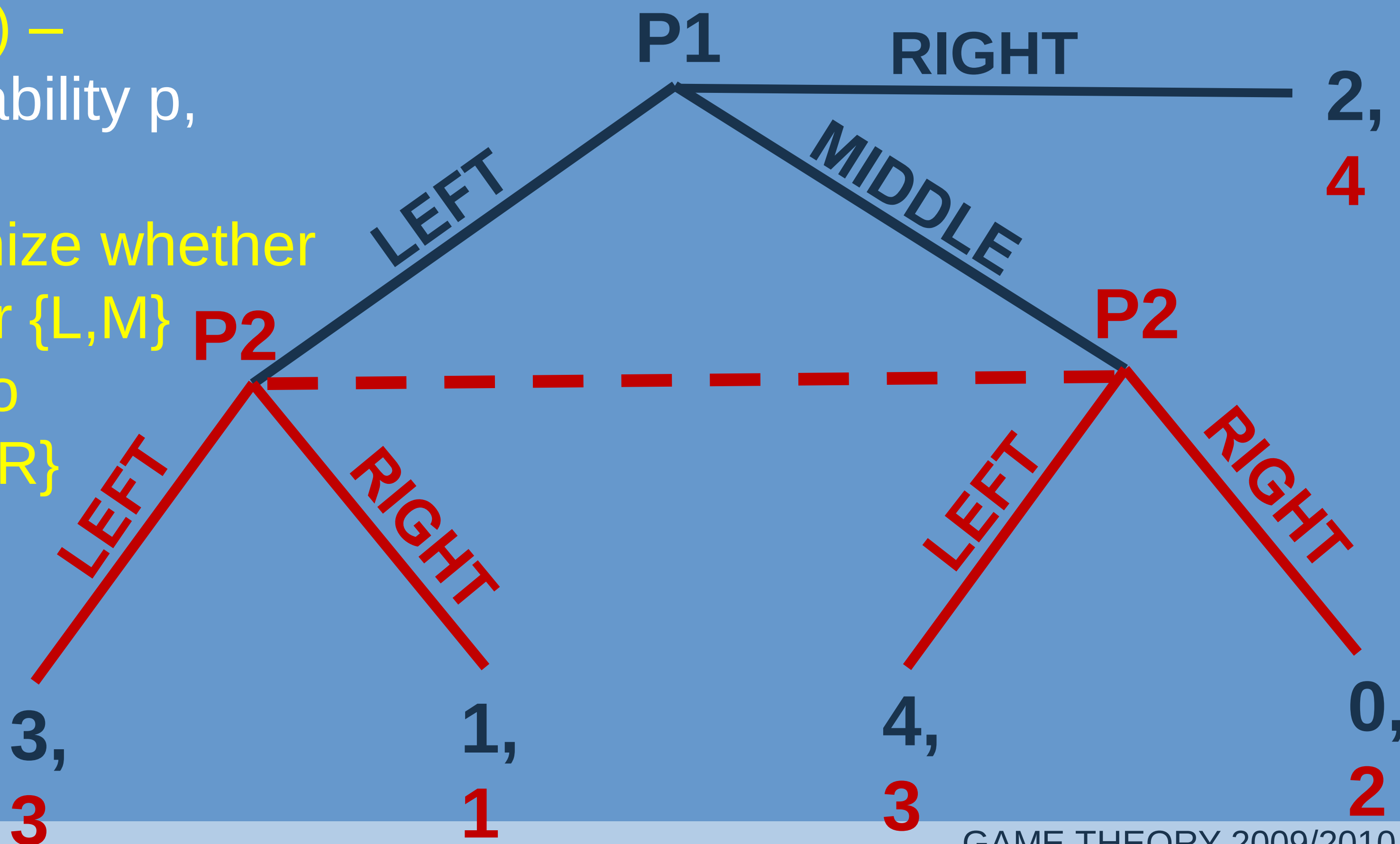
left with probability p, middle with q right with 1-p-q



Example 1

1) No Subgame
2) Strategies of P2: after {L,M} - L(1) - left with probability 1, R(1) – middle with probability 1, L(p) R(1-p) – left with probability p, right with 1-p

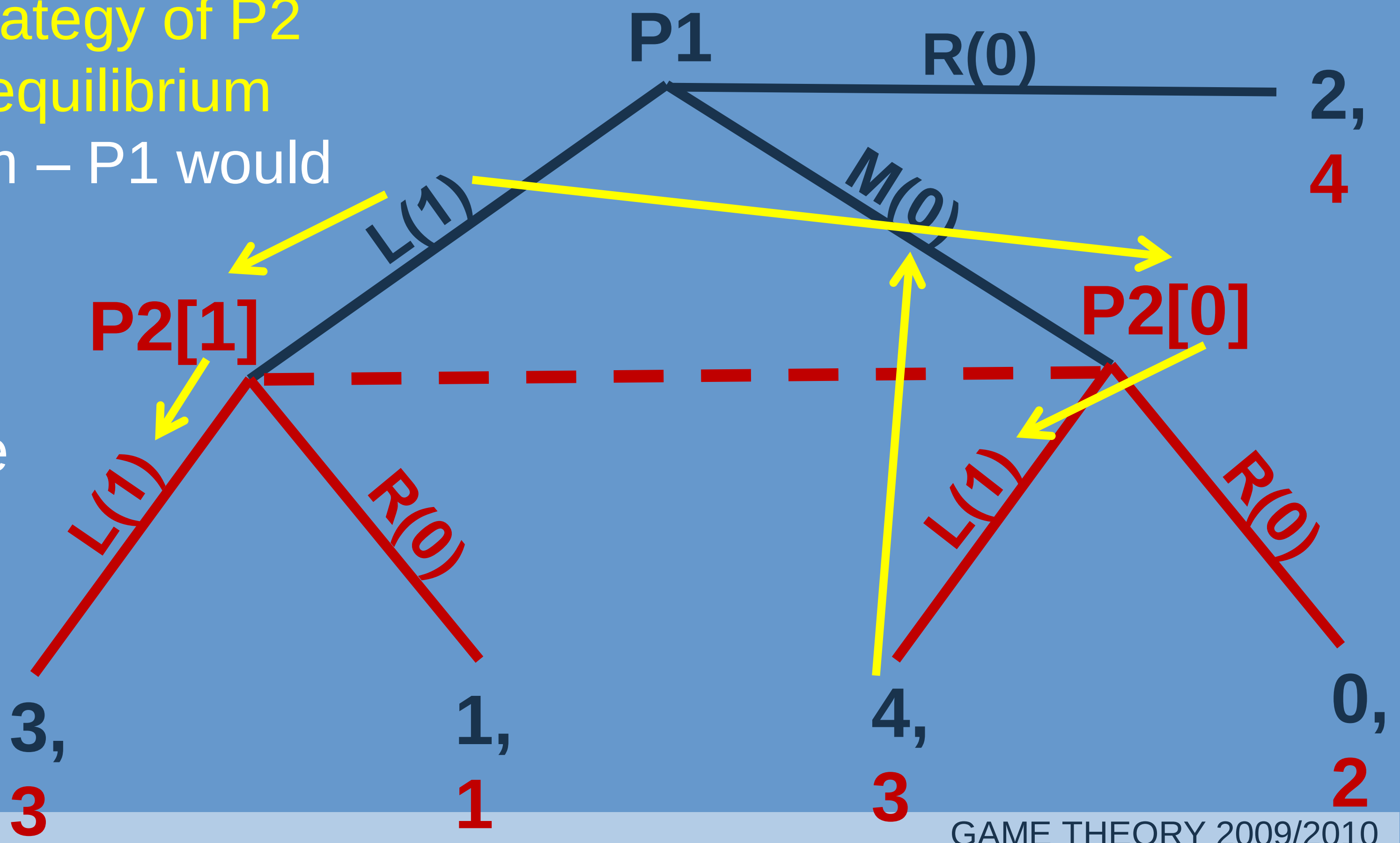
P2 can recognize whether she is at {R} or {L,M}
But she has no choices after {R}



Example 1

1) No Subgame
2) Strategies of P1: L(1) - left with probability 1
3) Beliefs of P2
4) Optimal strategy of P2
5) Check for equilibrium

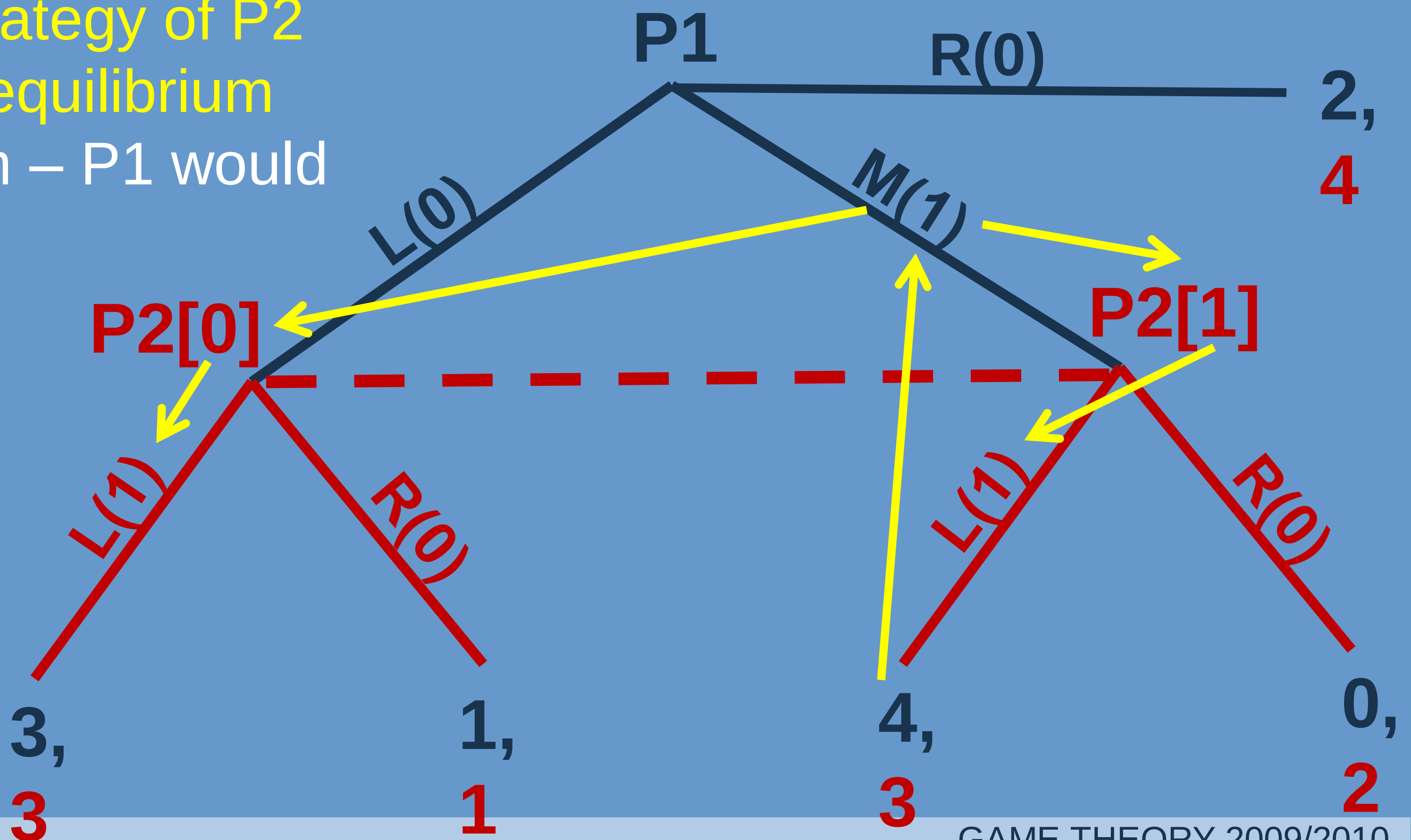
Not equilibrium – P1 would choose M(1) given the P2 strategy.
So not the one we assumed in step 2



Example 1

1) No Subgame
2) Strategies of P1: M(1) - middle with probability 1
3) Beliefs of P2
4) Optimal strategy of P2
5) Check for equilibrium

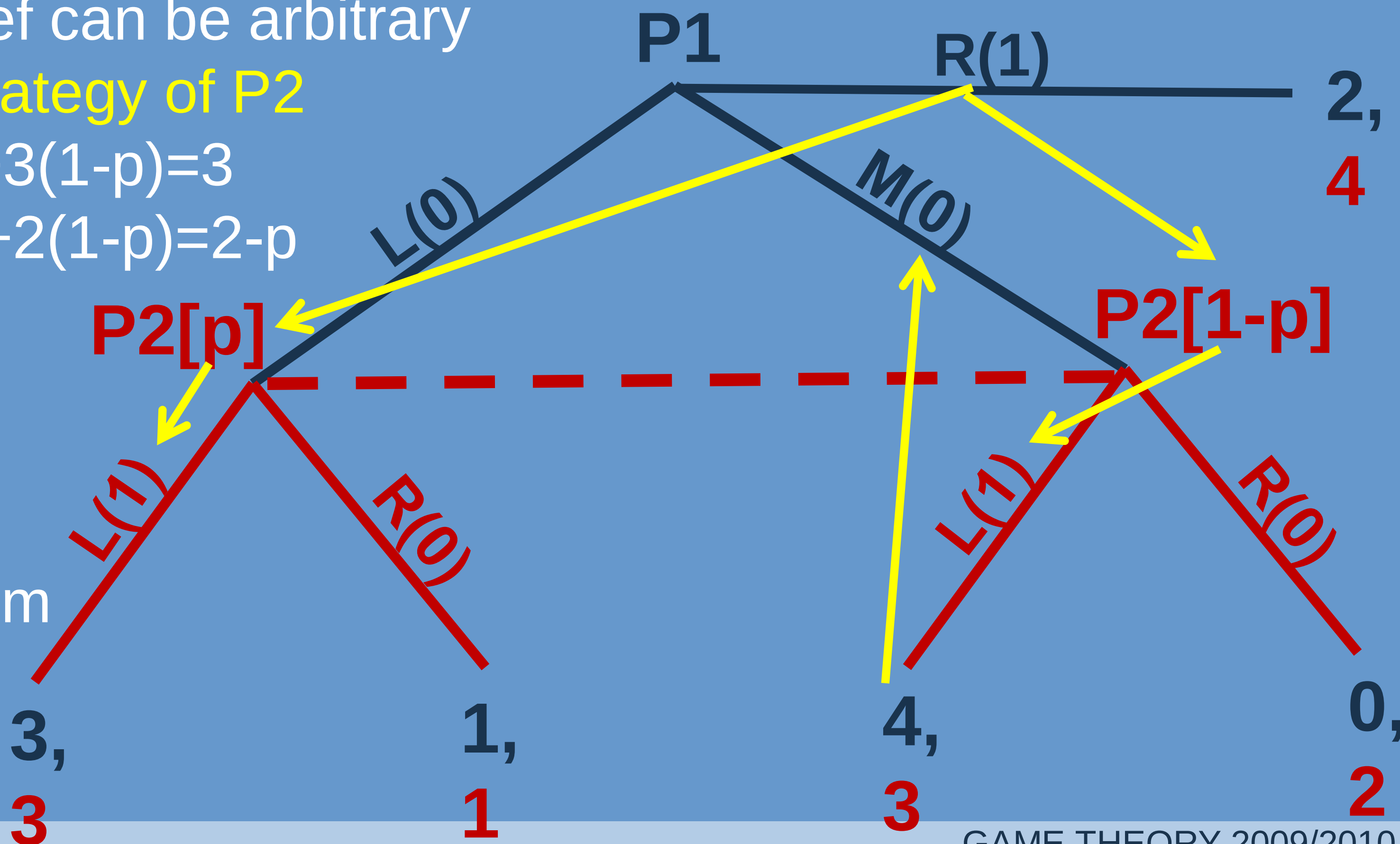
OK equilibrium – P1 would choose M(1) given the P2 strategy



Example 1

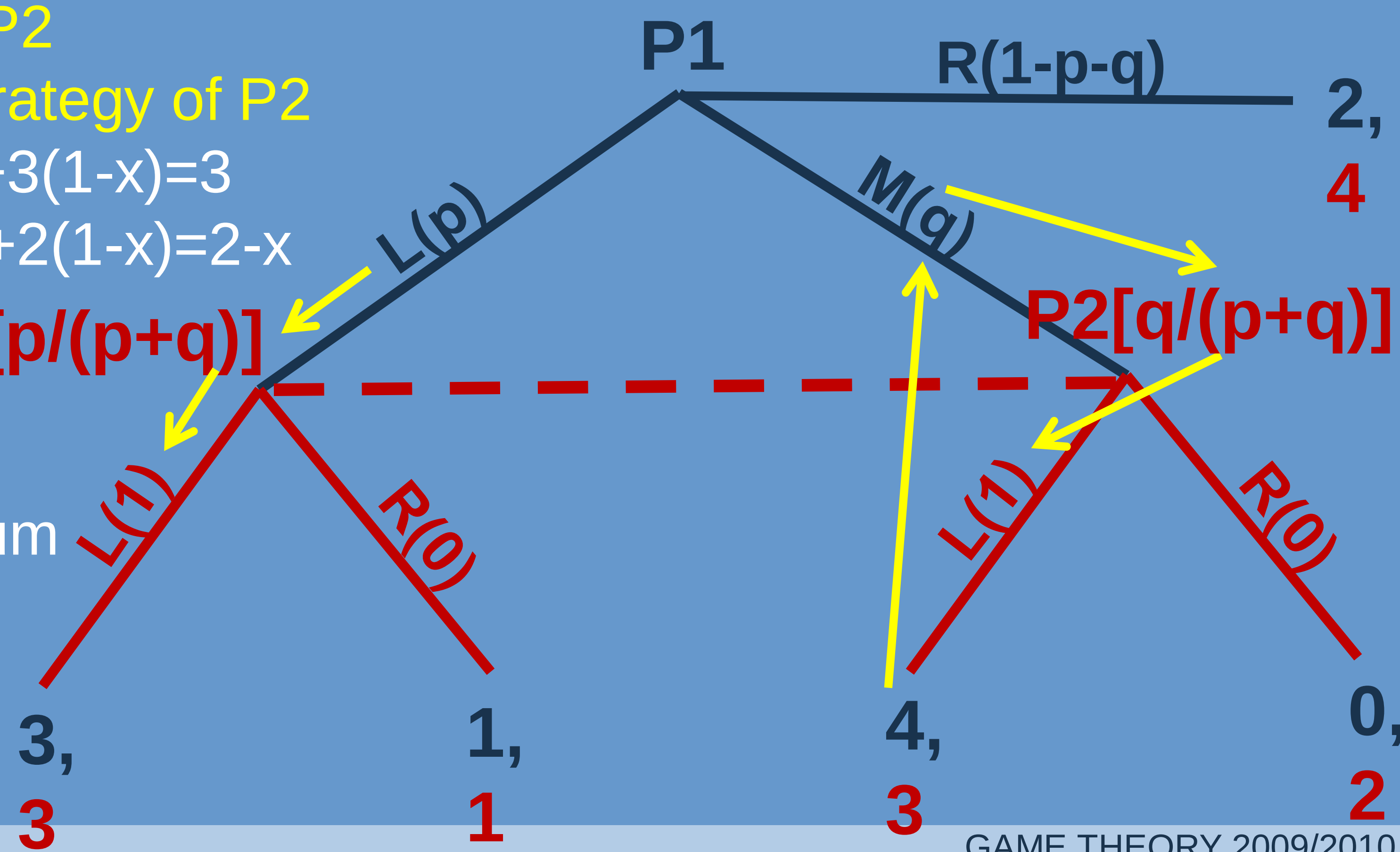
1) No Subgame
2) Strategies of P1: R(1) - right with probability 1
3) Beliefs of P2 – information set is not reached, so the belief can be arbitrary
4) Optimal strategy of P2
   EU(L)=3p+3(1-p)=3
   EU(R)=1p+2(1-p)=2-p
   R is never optimal
5) Check for equilibrium

NOT equilibrium
 P1 would choose M(1)



Example 1

1) No Subgame
2) Strategies of P1: L(p)M(q)R(1-p-q) – left with probability p, middle with q right with 1-p-q
3) Beliefs of P2
4) Optimal strategy of P2
   EU(L)=3x+3(1-x)=3
   EU(R)=1x+2(1-x)=2-x

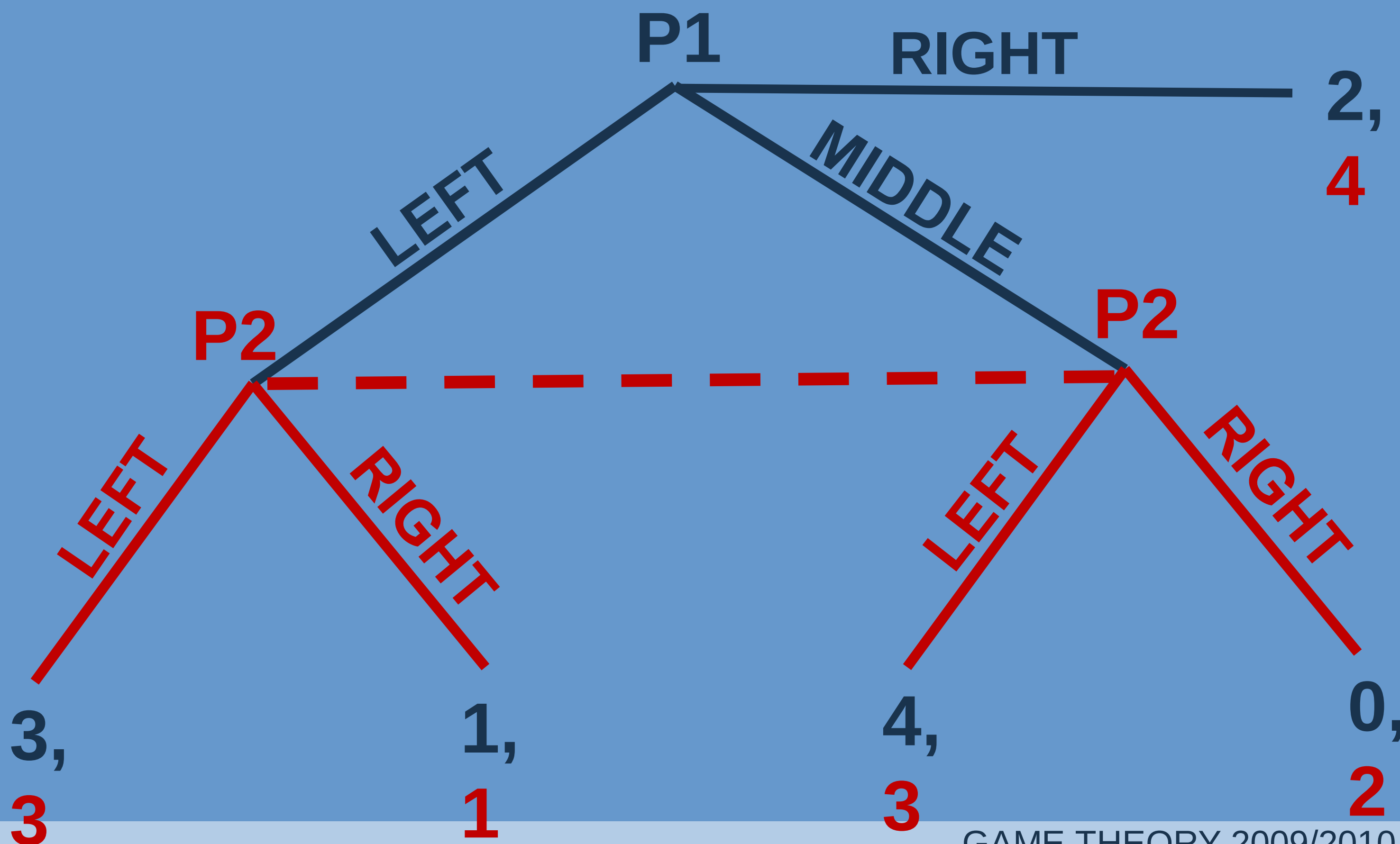


5) Check for equilibrium
NOT equilibrium
P1 would choose M(1) given the P2 strategy

Example 1

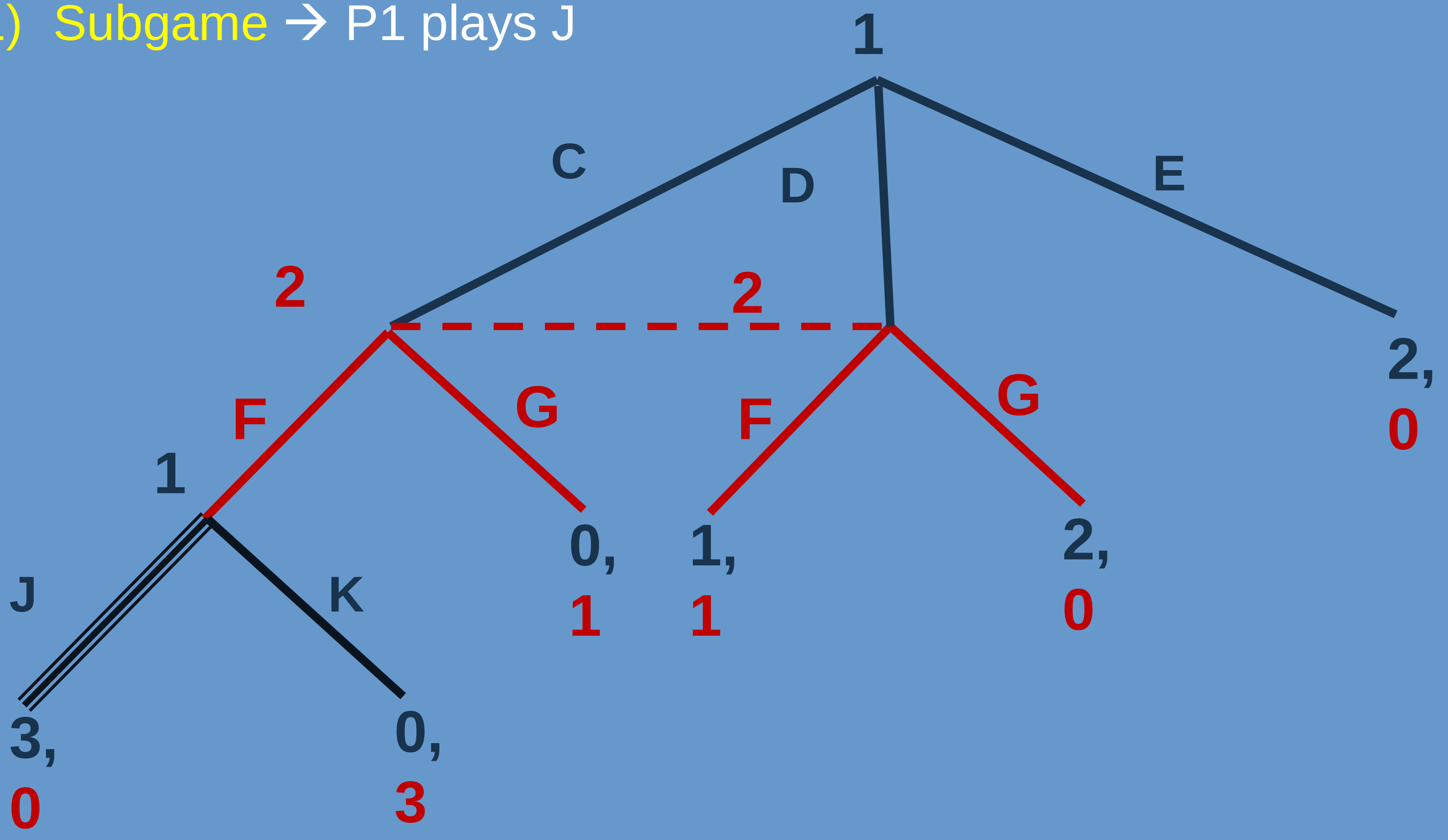
The game has only 1 weak sequential equilibrium:

{P1 – middle, P2 chooses left after {L,M}, P2 believes that history middle occurs with probability 1}



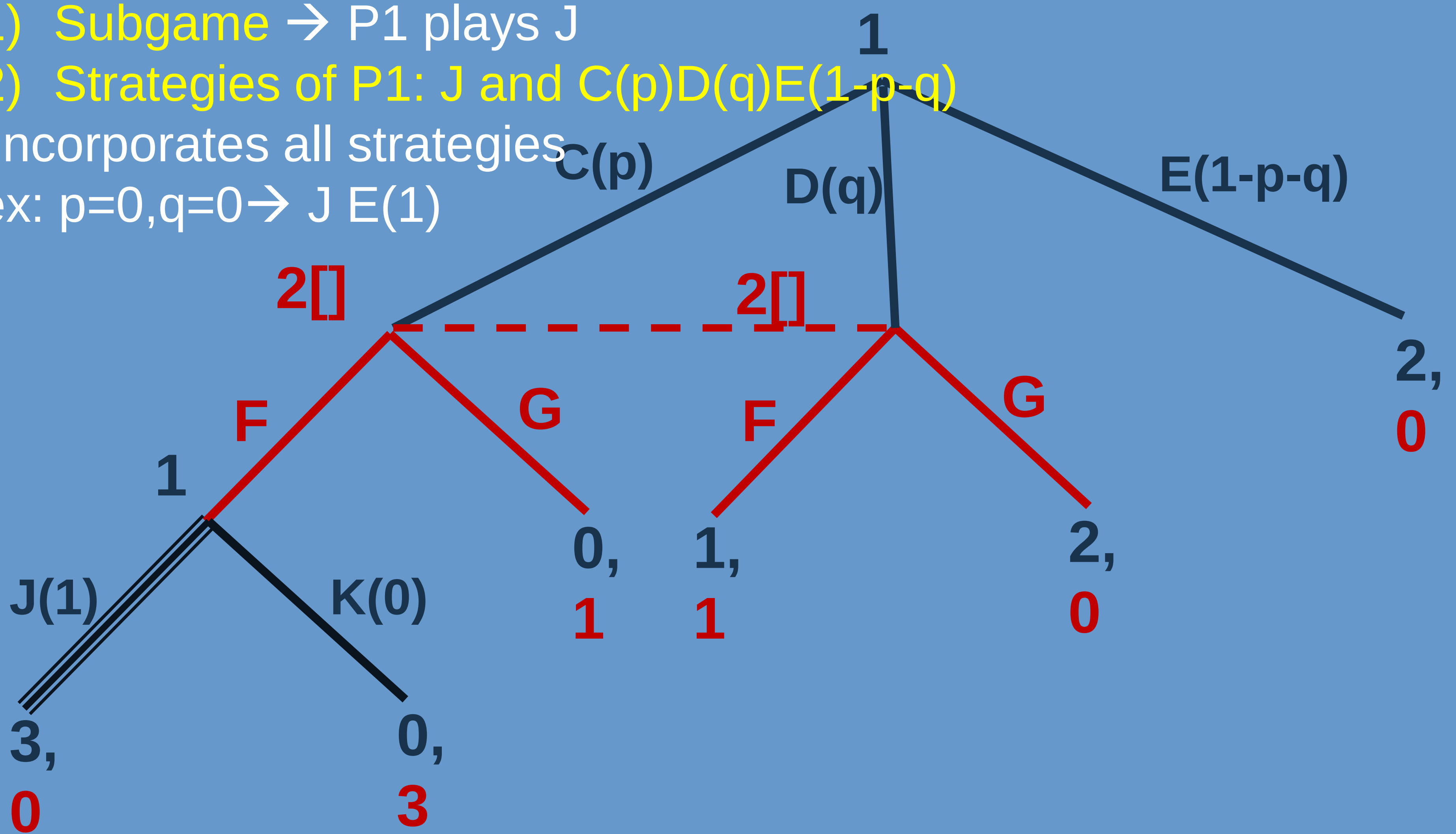
Example

1) Subgame → P1 plays J



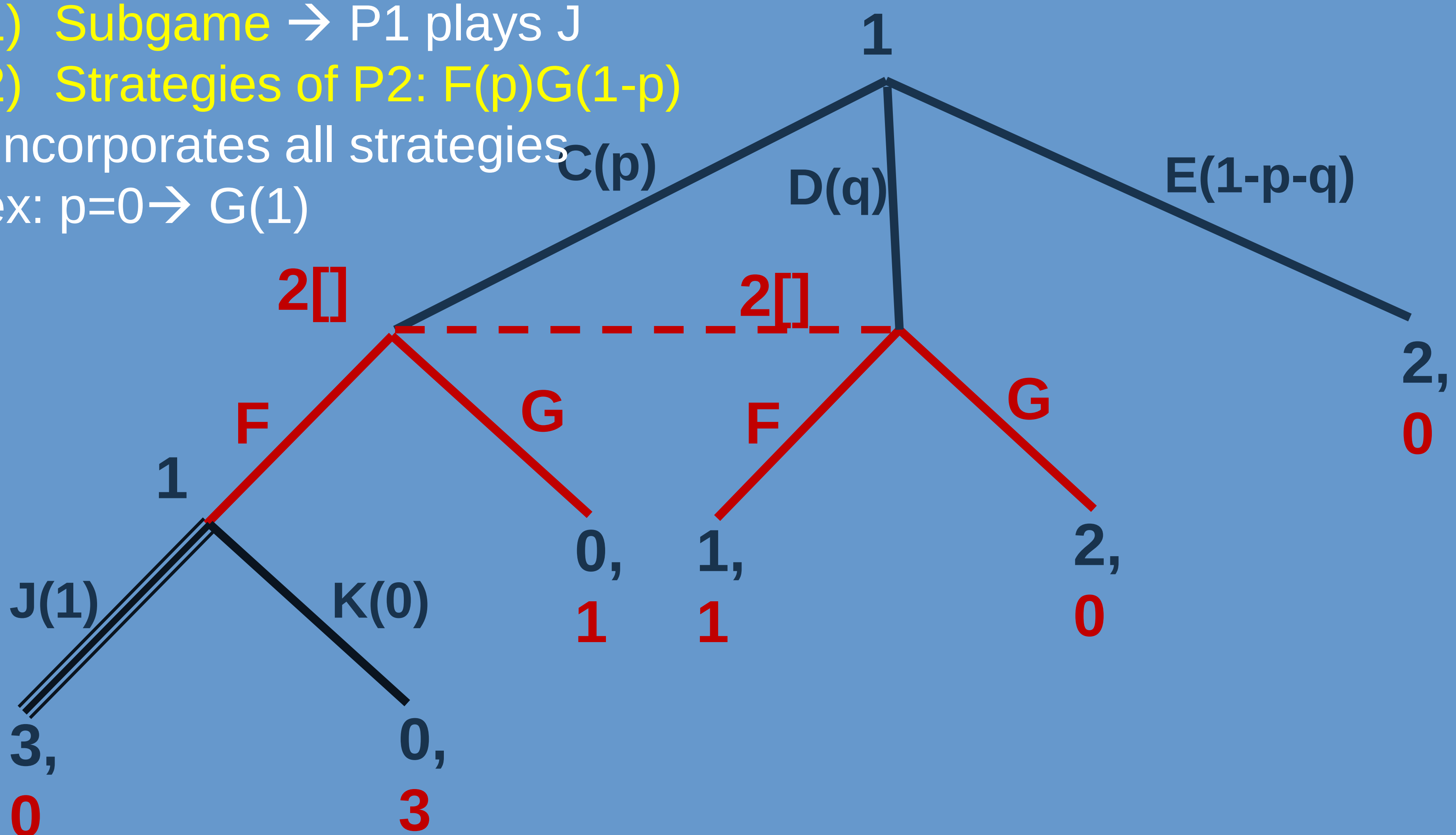
Example

1) Subgame → P1 plays J
2) Strategies of P1: J and C(p)D(q)E(1-p-q)
 incorporates all strategies
ex: p=0,q=0→ J E(1)



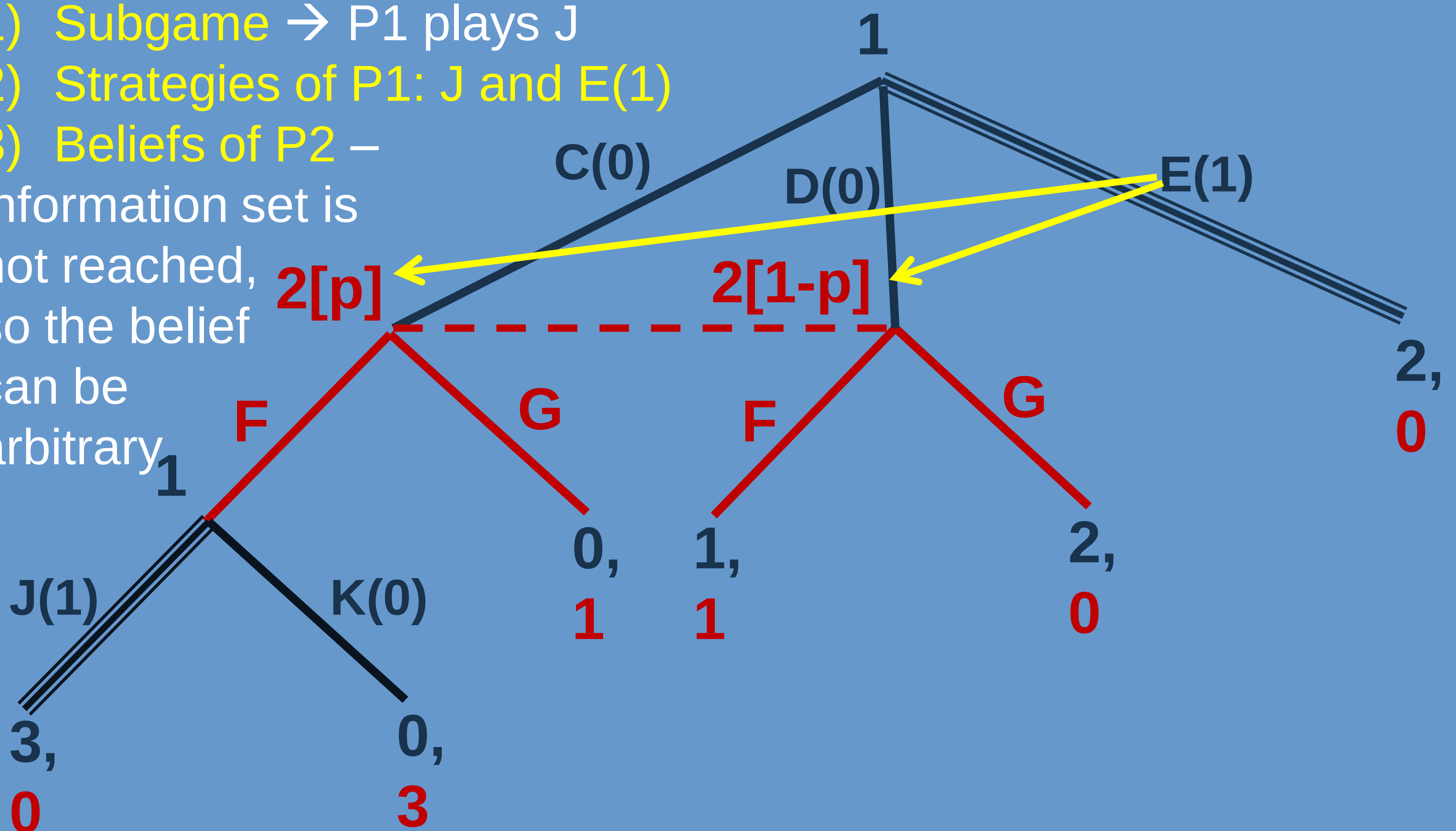
Example

1) Subgame → P1 plays J
2) Strategies of P2: F(p)G(1-p)
 incorporates all strategies
ex: p=0→ G(1)



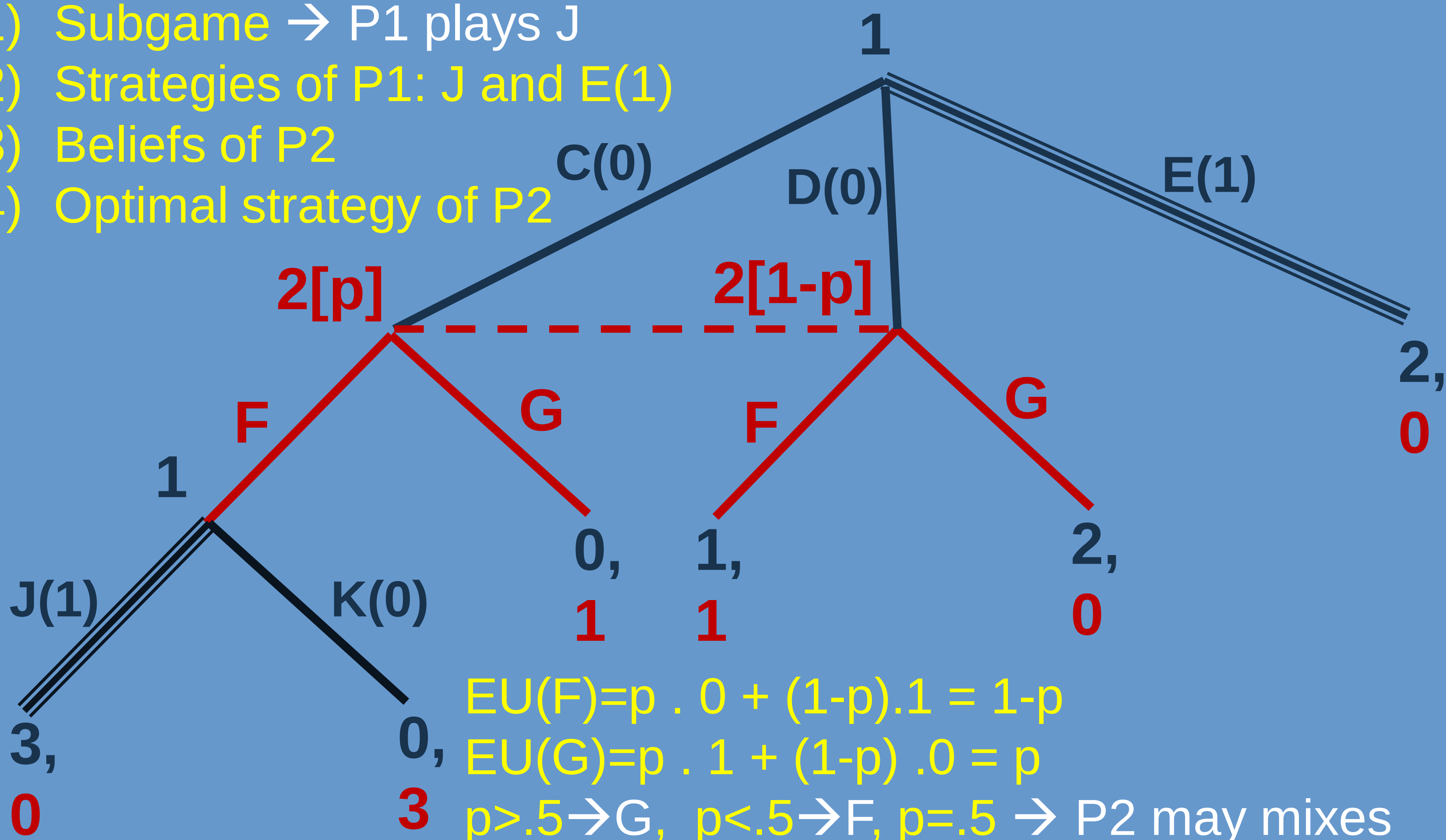
Example

1) Subgame → P1 plays J
2) Strategies of P1: J and E(1)
3) Beliefs of P2 – information set is not reached, so the belief can be arbitrary



Example

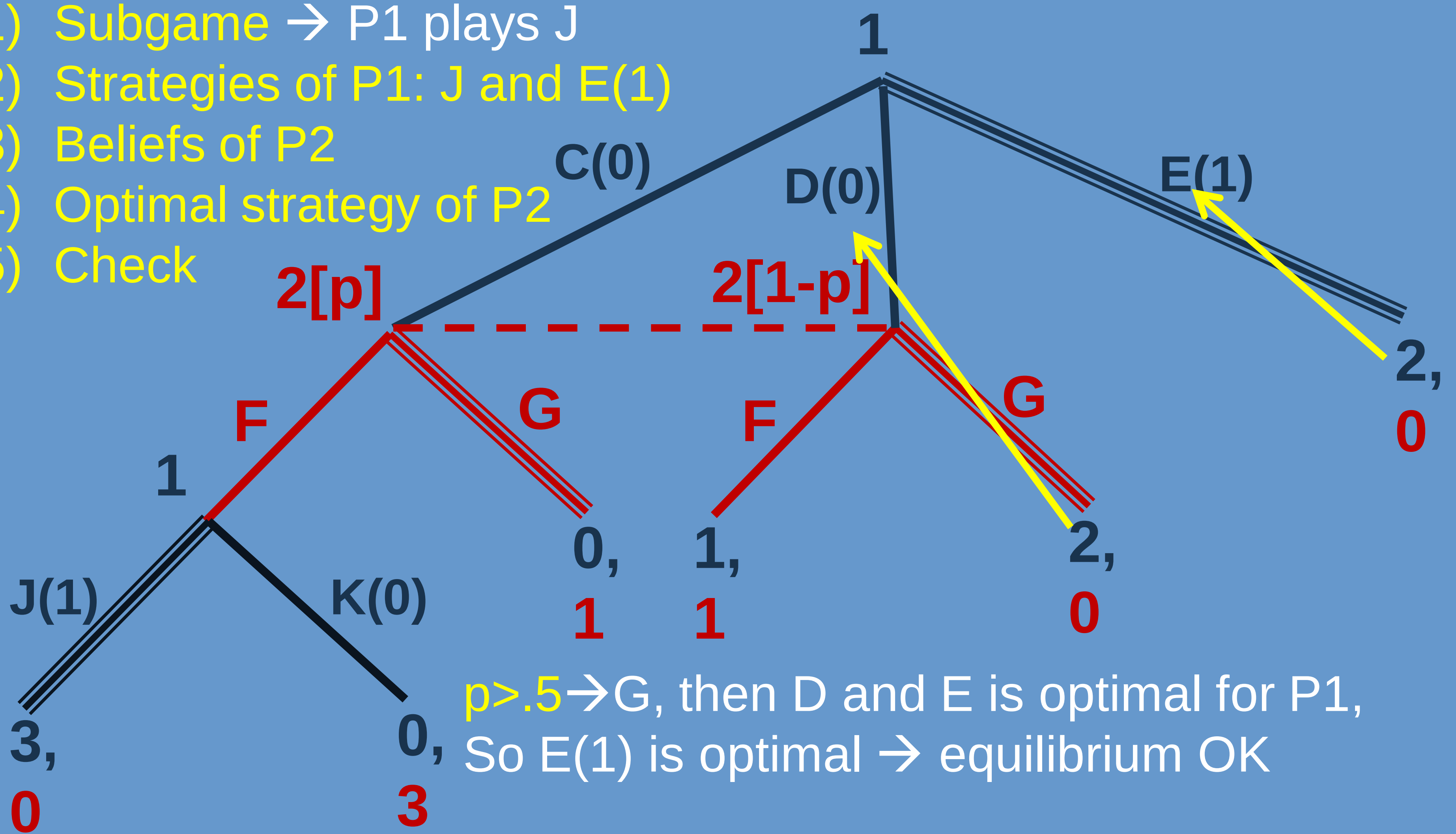
1) Subgame → P1 plays J
2) Strategies of P1: J and E(1)
3) Beliefs of P2
4) Optimal strategy of P2



EU(F)=p . 0 + (1-p).1 = 1-p
EU(G)=p . 1 + (1-p) .0 = p
p>.5→G, p<.5→F, p=.5 → P2 may mixes

Example

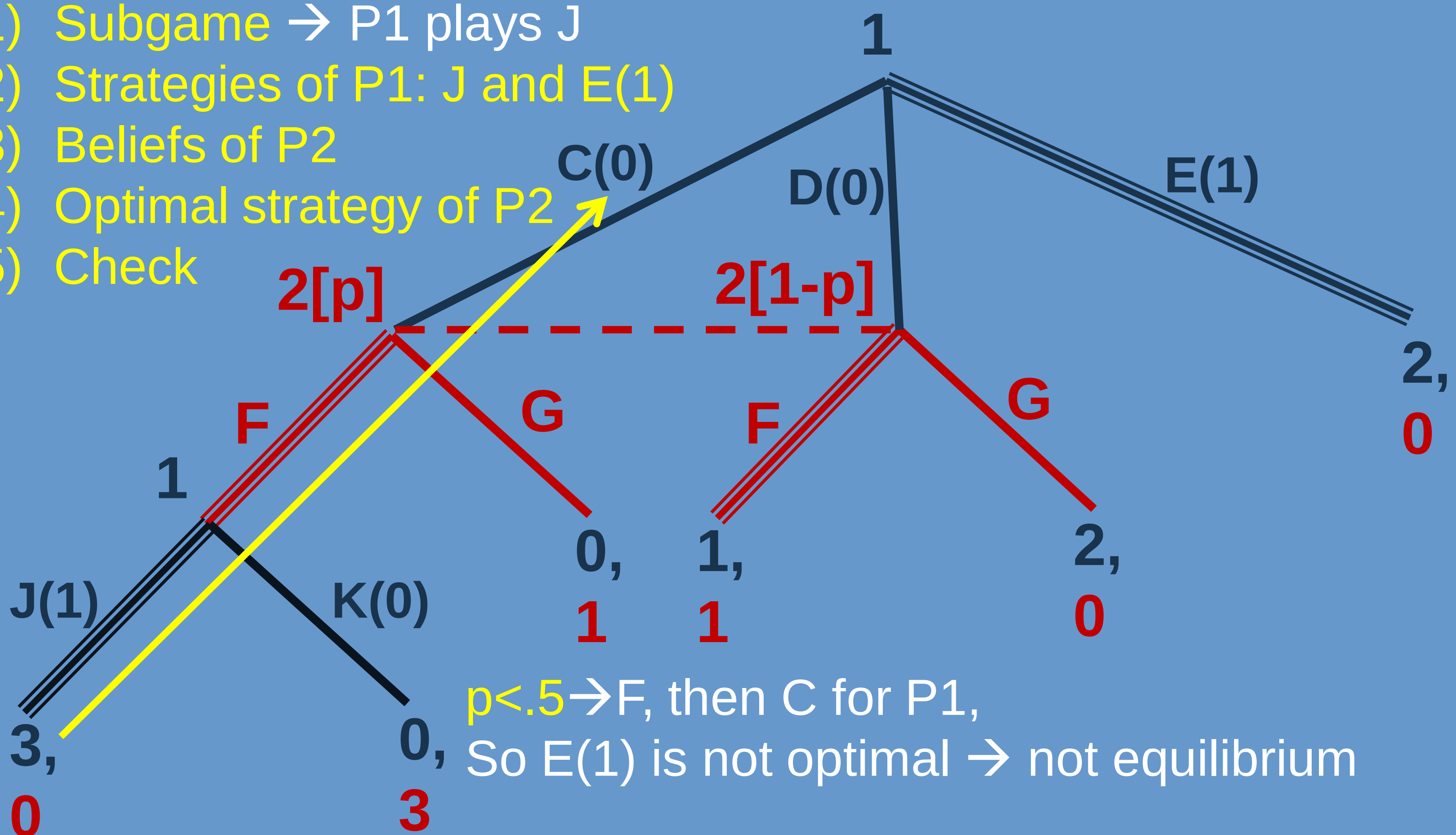
1) Subgame → P1 plays J
2) Strategies of P1: J and E(1)
3) Beliefs of P2
4) Optimal strategy of P2
5) Check



p>.5 → G, then D and E is optimal for P1,
So E(1) is optimal → equilibrium OK

Example

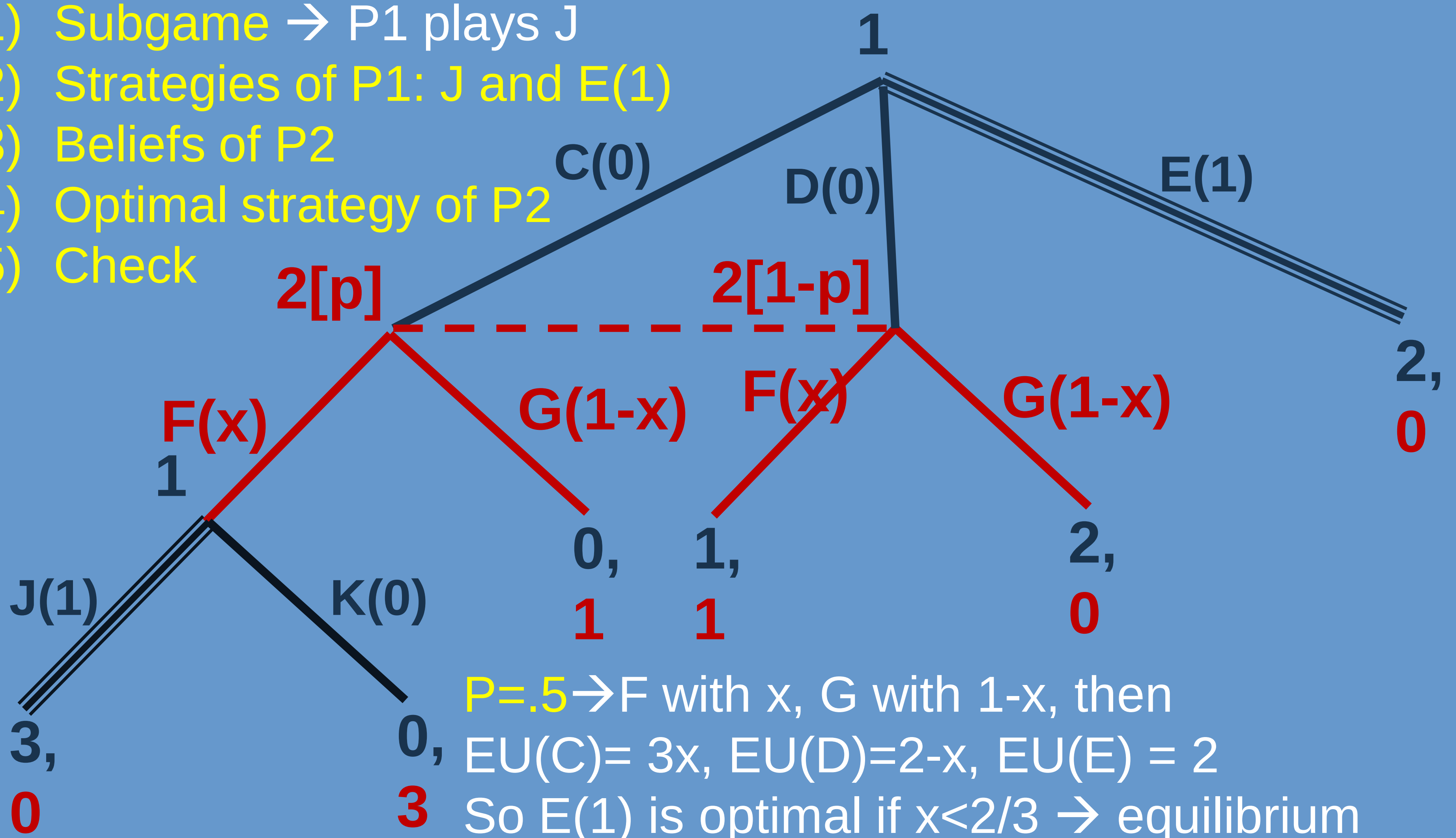
1) Subgame → P1 plays J
2) Strategies of P1: J and E(1)
3) Beliefs of P2
4) Optimal strategy of P2
5) Check



p<.5→F, then C for P1,
So E(1) is not optimal → not equilibrium

Example

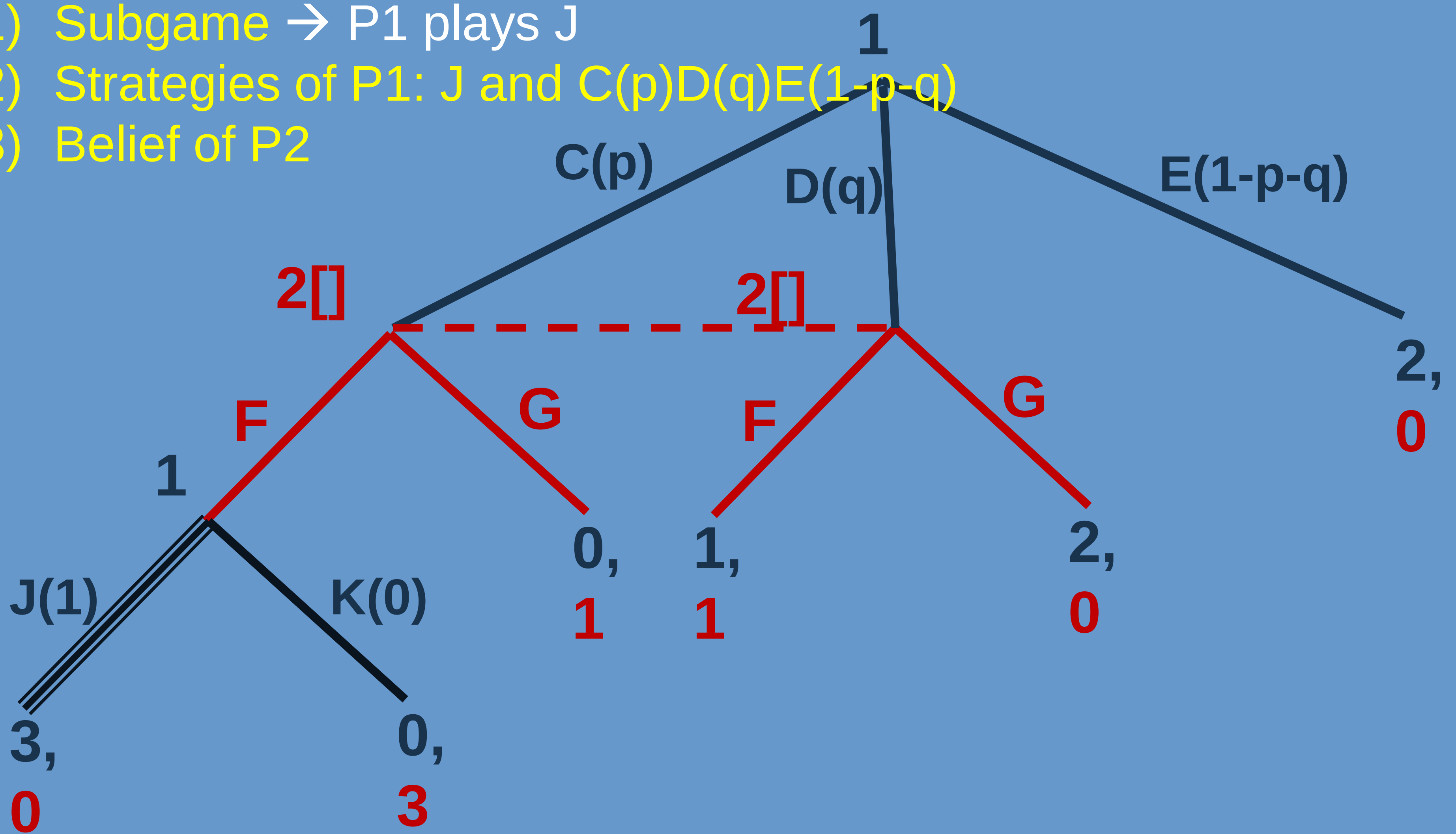
1) Subgame → P1 plays J
2) Strategies of P1: J and E(1)
3) Beliefs of P2
4) Optimal strategy of P2
5) Check



P=.5→F with x, G with 1-x, then
EU(C)= 3x, EU(D)=2-x, EU(E) = 2
So E(1) is optimal if x<2/3 → equilibrium

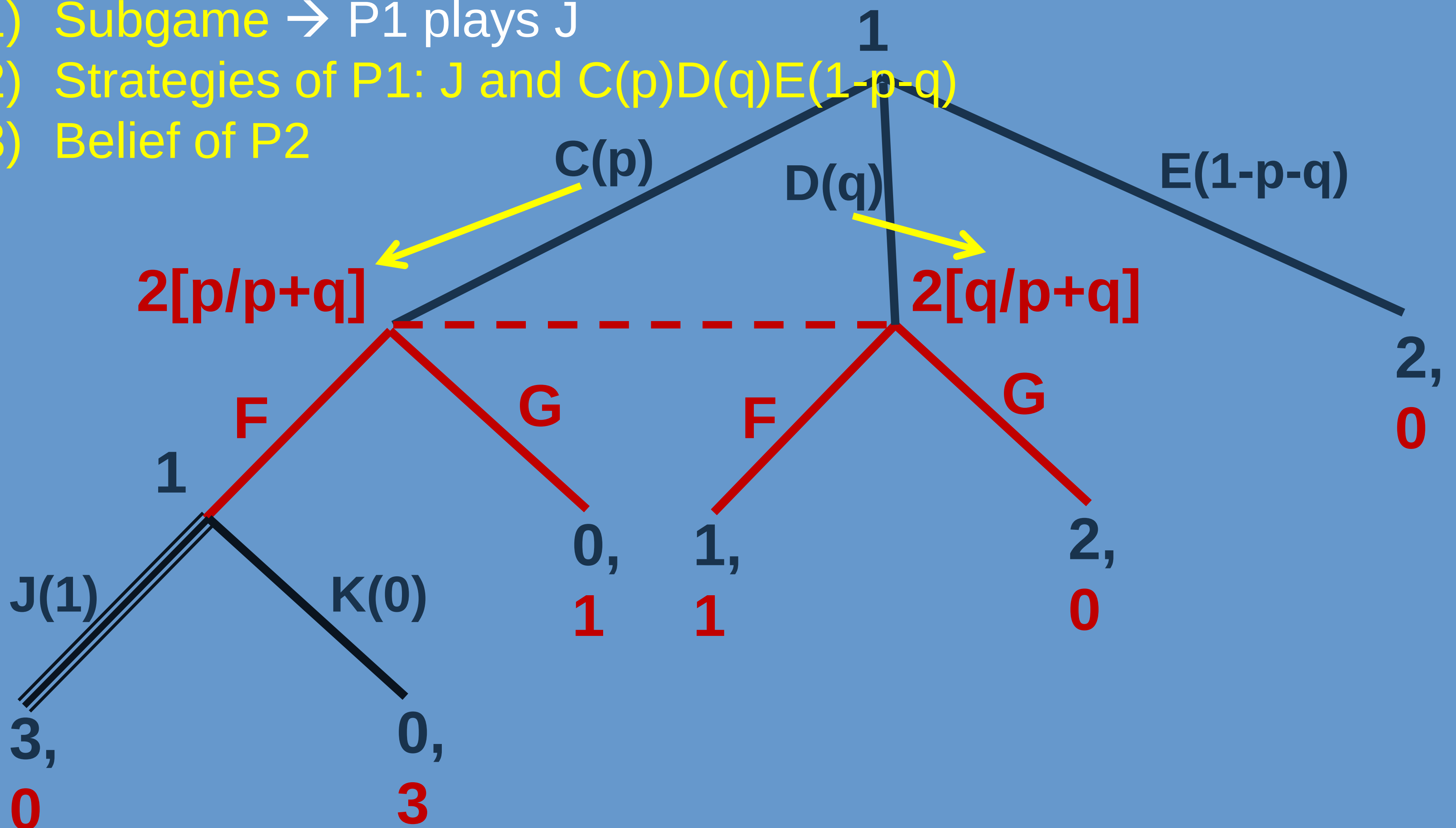
Example

1) Subgame → P1 plays J
2) Strategies of P1: J and C(p)D(q)E(1-p-q)
3) Belief of P2



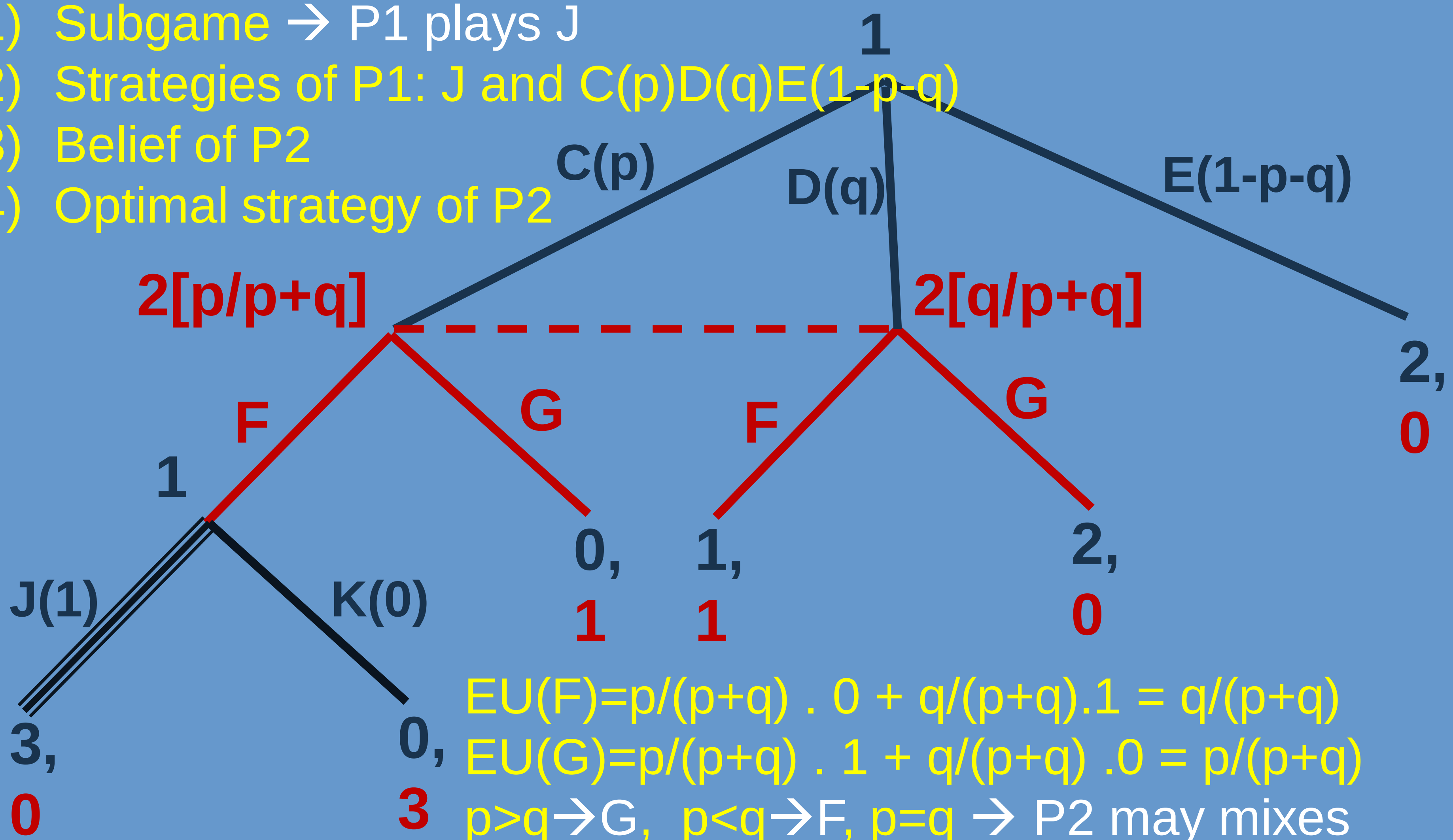
Example

1) Subgame → P1 plays J
2) Strategies of P1: J and C(p)D(q)E(1-p-q)
3) Belief of P2



Example

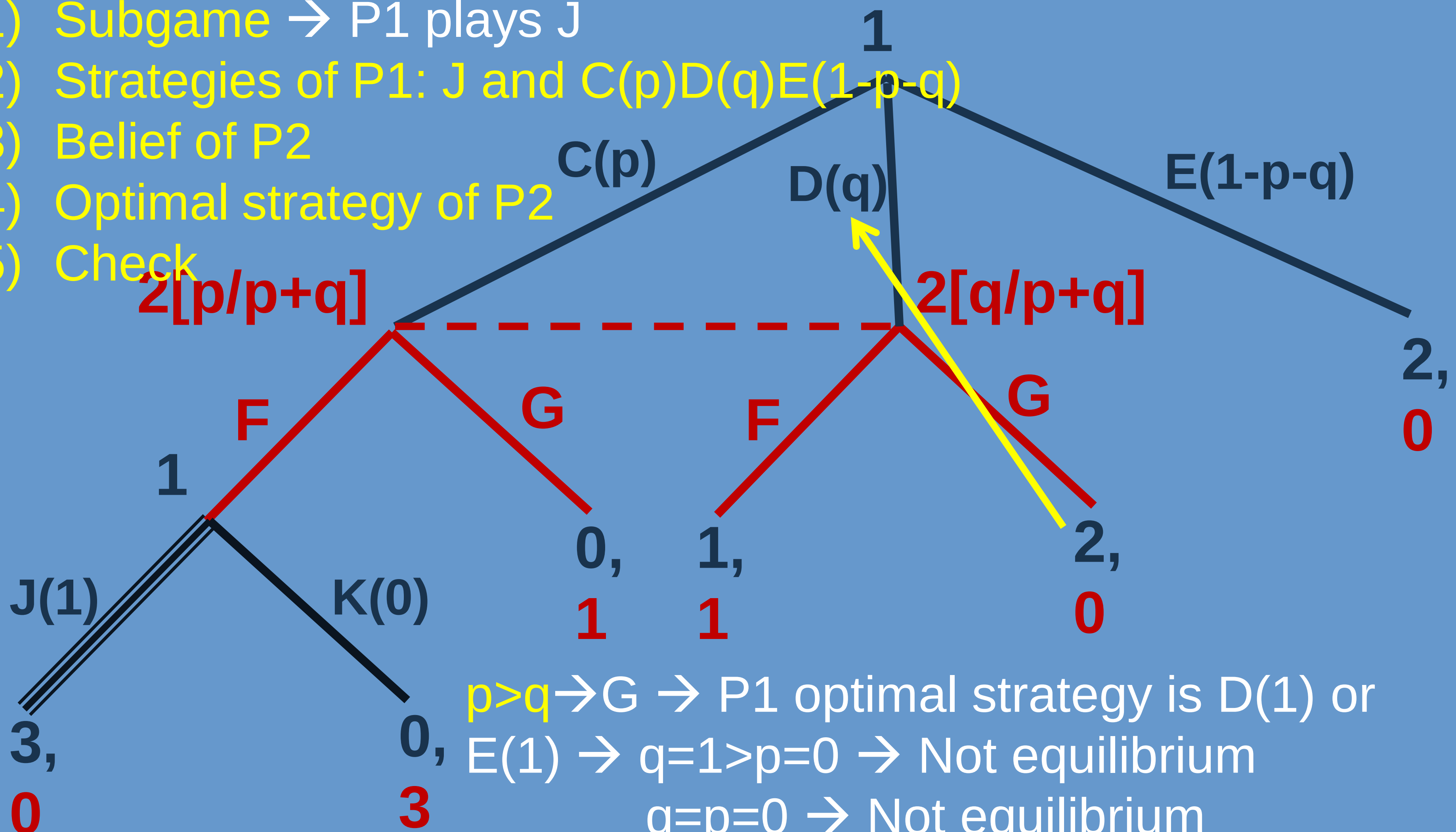
1) Subgame → P1 plays J
2) Strategies of P1: J and C(p)D(q)E(1-p-q)
3) Belief of P2
4) Optimal strategy of P2



EU(F)=p/(p+q) . 0 + q/(p+q).1 = q/(p+q)
EU(G)=p/(p+q) . 1 + q/(p+q) .0 = p/(p+q)
p>q→G, p<q→F, p=q → P2 may mixes

Example

1) Subgame → P1 plays J
2) Strategies of P1: J and C(p)D(q)E(1-p-q)
3) Belief of P2
4) Optimal strategy of P2
5) Check

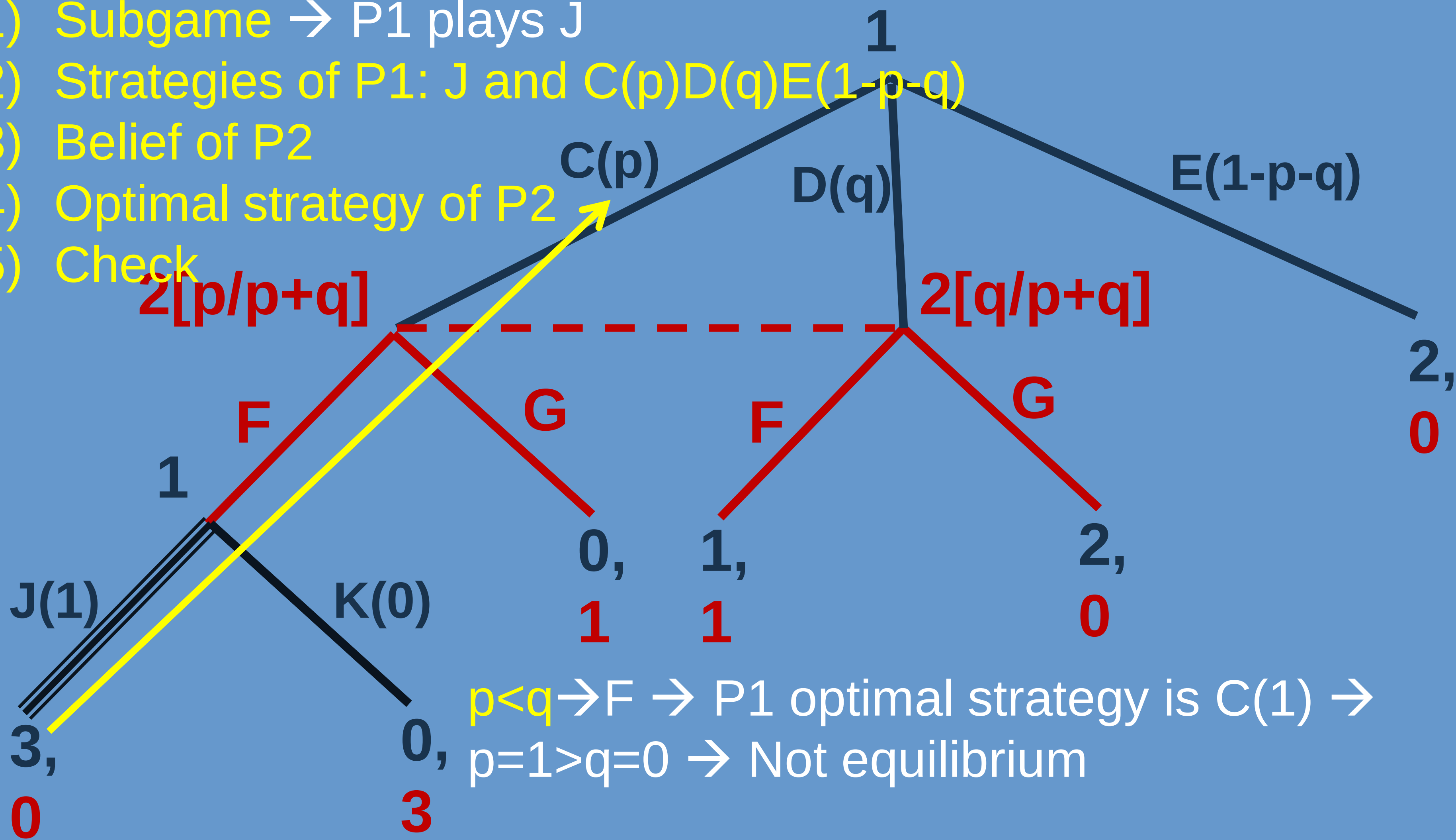


p>q→G → P1 optimal strategy is D(1) or E(1) → q=1>p=0 → Not equilibrium

q=p=0 → Not equilibrium

Example

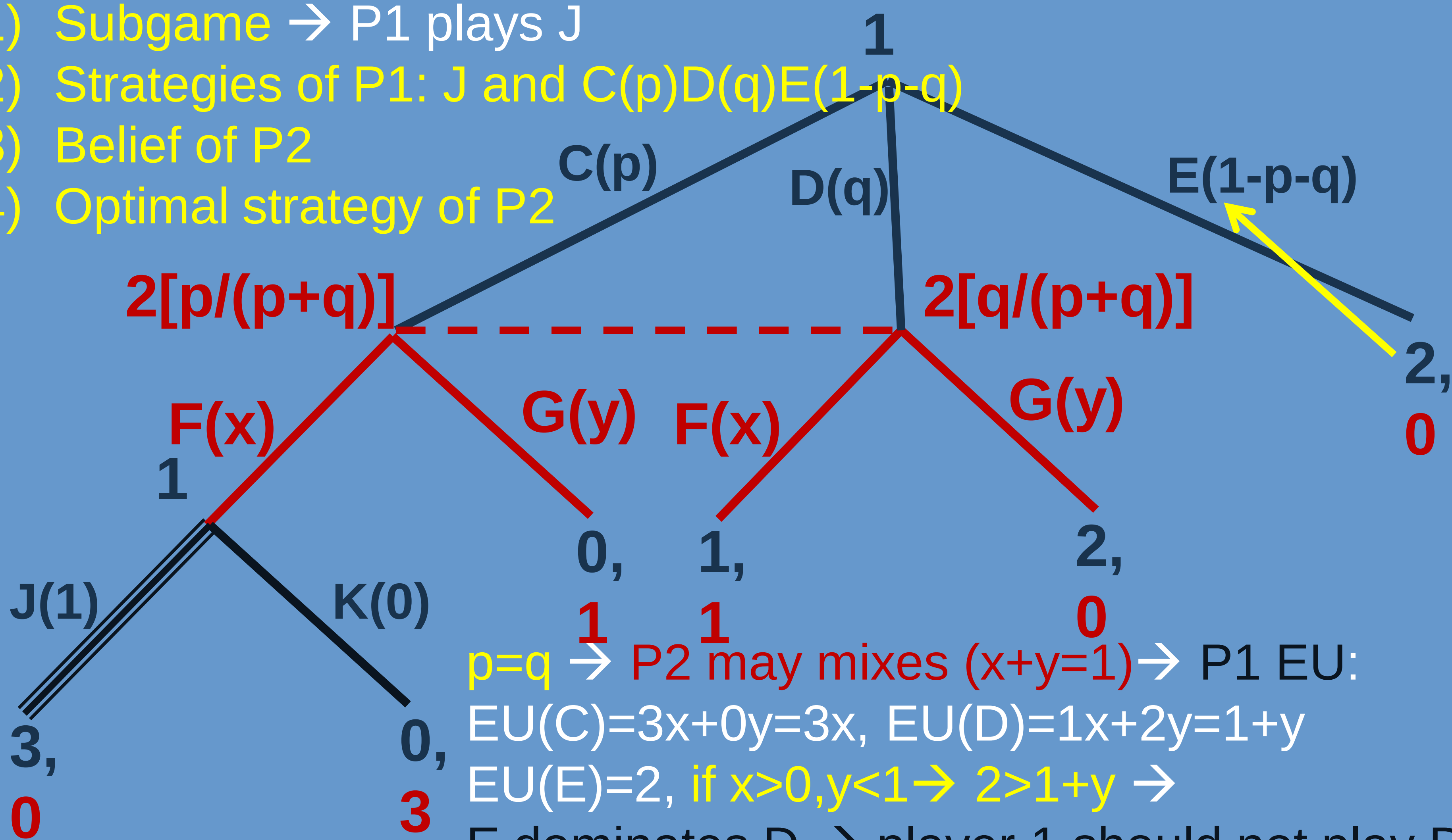
1) Subgame → P1 plays J
2) Strategies of P1: J and C(p)D(q)E(1-p-q)
3) Belief of P2
4) Optimal strategy of P2
5) Check



p<q→F → P1 optimal strategy is C(1) →
p=1>q=0 → Not equilibrium

Example

1) Subgame → P1 plays J
2) Strategies of P1: J and C(p)D(q)E(1-p-q)
3) Belief of P2
4) Optimal strategy of P2



p=q → P2 may mixes (x+y=1)→ P1 EU:

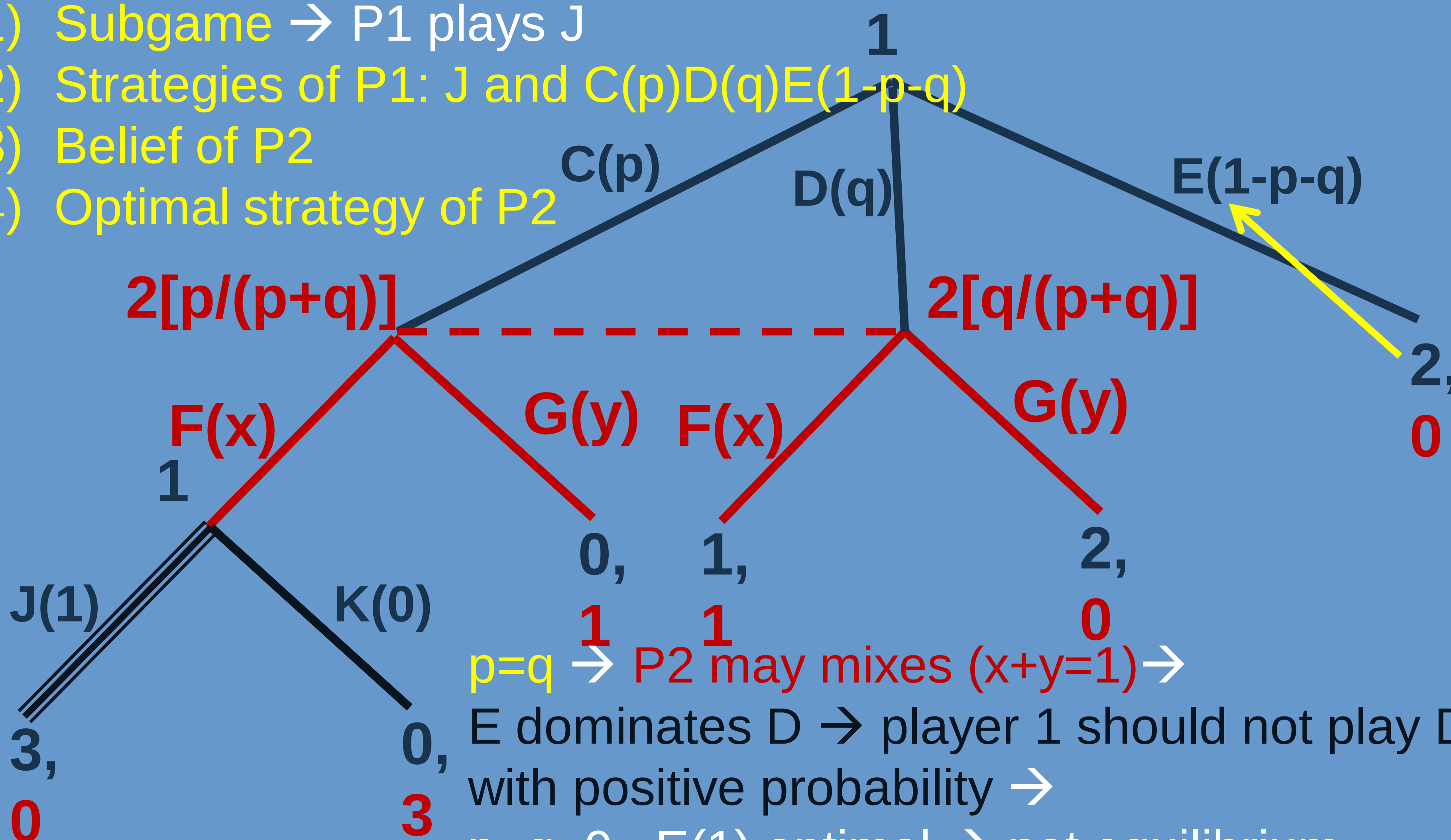
EU(C)=3x+0y=3x, EU(D)=1x+2y=1+y

EU(E)=2, if x>0,y<1→ 2>1+y →

E dominates D → player 1 should not play D

Example

1) Subgame → P1 plays J
2) Strategies of P1: J and C(p)D(q)E(1-p-q)
3) Belief of P2
4) Optimal strategy of P2

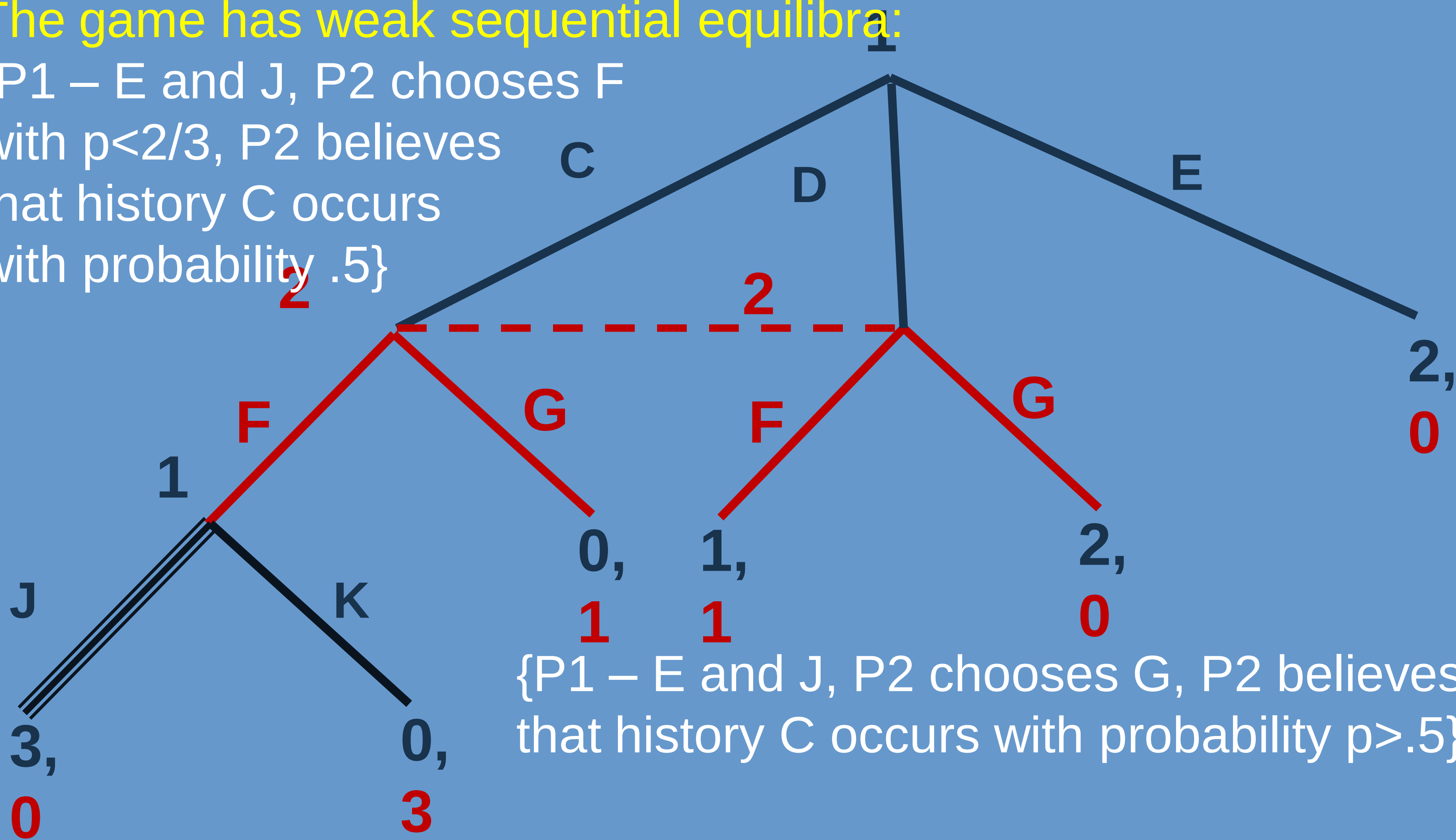


p=q → P2 may mixes (x+y=1)→
E dominates D → player 1 should not play D with positive probability →
p=q=0 , E(1) optimal → not equilibrium

Example

The game has weak sequential equilibra:

{P1 – E and J, P2 chooses F with p<2/3, P2 believes that history C occurs with probability .5}



{P1 – E and J, P2 chooses G, P2 believes that history C occurs with probability p>.5}

Summary

- Dynamic games with incomplete information
- Weak sequential equilibrium

(in our simple cases coincides with perfect Bayesian equilibrium in Gibbons)

- Gibbons 2.4.A, 4; Osborne 10

NEXT WEEK:

weak sequential equilibrium,Signaling games