VSE - Introduction to Game Theory
Problem set #1 - Due Wednesday, October 21, 2015 – Suggested solution

Teamwork is an important part of this course. Therefore, please **work in groups of up to 4 students.** Each student can only be in one group. Each group submits **one copy** of problem set with the names of all members.

Homework can be delivered: (1) by email or (2) personally during the lecture or office hours.

No late submissions will be accepted.

Problem 1 [22 points]: Consider Pat and Mat playing the paintball. Pat runs out of paint balls and Mat is aiming at him to shoot. Mat knows that Pat will try to escape and so he is deciding to shoot a little bit to the right or to the left (he knows that Pat will not stay still for sure). Pat can try to move to the left or to the right to escape the ball. Given the close distance and the speed of the ball, they have to choose an action simultaneously. If Pat guesses where Mat shoots correctly and avoids the ball, Pat wins the game. If Pat guesses incorrectly and Mat is shooting to the right, Mat has chance of 80% to hit Pat and win the game. If Pat guesses incorrectly and Mat is shooting to the left, Mat has chance of 40% to hit Pat and win the game.

(a) Find normal form of this game.
(b) Find all pure strategy Nash equilibria of this game.
(c) Find mixed strategy Nash equilibrium of this game.

Solution:

(a) The normal for of the game is as follows:

M \ P	L	R
R	0,10	8,2
L	4,6	0,10

(b) There are no pure strategy NE in this game.

(c) There is one mixed strategy NE in this game: {(1/3,2/3),(2/3,1/3)}

Mat mixes between R and L with probabilities p and 1-p in such a way that Pat is indifferent between playing L and R:

$10p+6(1-p) = 2p+10(1-p) \Rightarrow p=1/3$

Pat mixes between L and R with probabilities q and 1-q in such a way that Mat is indifferent between playing R and L:

$8q = 4(1-q) \Rightarrow q=2/3$

Problem 2 [30 points]: Find all the mixed strategy Nash equilibria of the strategic games below. Sketch best response functions of both players and NE on the graph.

[a]

1 \ 2	L	R
T	6,0	0,6
B	3,2	6,0

[b]

1 \ 2	L	R
T	0,2	0,4
B	4,4	0,2

Solution:

(a) There are no Nash equilibria in pure strategies in this game. To find mixed strategy Nash equilibrium we use the following notation:

- player 1 plays Top with probability p and Bottom with probability $1 - p$
- player 2 plays Left with probability q and Right with probability $1 - q$

For player 1 to mix his strategy, the second player has to choose his mixing strategy such that the first player's payoff from playing Top $(q \cdot 6 + (1 - q) \cdot 0)$ is the same as payoff from playing Bottom $(q \cdot 3+(1-q) \cdot 6)$. If one of these payoffs was higher, there wouldn't be any reason for mixing:

$$q \cdot 6 + (1 - q) \cdot 0 = q \cdot 3 + (1 - q) \cdot 6$$

$$9q = 6$$

$$q = 6/9 = 2/3$$

Similarly, for player 2 to mix his strategy, the payoff from playing Left $(p \cdot 0+(1-p) \cdot 6)$ has to be the same as payoff from playing Right $(p \cdot 2+(1-p) \cdot 0)$. If one of these payoffs was higher, there wouldn't be any reason for mixing:

$$p \cdot 0 + (1 - p) \cdot 2 = p \cdot 6 + (1 - p) \cdot 0$$

$$8p = 2$$

$$p = 1/4$$

Hence, the only mixed strategy Nash equilibrium of this game is: {(3/4,1/4),(2/3,1/3)}.

(b) This game has two Nash equilibria in pure strategies (T,R) and (B,L). Now, we look for mixed strategies equilibria: For player 1 to mix his strategy, the payoff from playing Top $(q \cdot 0+(1-q) \cdot 0)$ has to be the same as payoff from playing Bottom $(q \cdot 4+(1-q) \cdot 0)$. If one of these payoffs was higher, there wouldn't be any reason for mixing:

$$q \cdot 0 + (1 - q) \cdot 0 = q \cdot 4 + (1 - q) \cdot 0$$

$$4q = 0$$

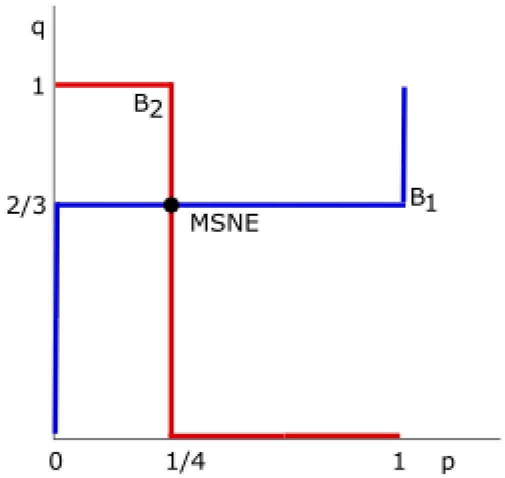
$$q = 0$$

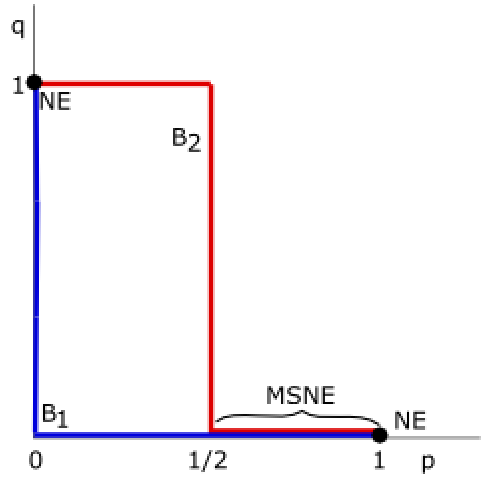
So the second player always chooses Right (if he played Left with positive probability, player 1 would never mix, only play Bottom). Now the first player has to mix in such a way that he makes Right more attractive for the second player than Left.

Therefore, the payoff from playing Left ($p \cdot 2 + (1-p) \cdot 4$) has to be smaller or equal than payoff from playing Right ($p \cdot 4 + (1-p) \cdot 2$).

$$p \cdot 2 + (1-p) \cdot 4 \le p \cdot 4 + (1-p) \cdot 2$$

$$4p \ge 2 \Rightarrow p > 1/2$$





Problem 3 [24 points]: Go to the link below and watch the movie extract. Model this situation with help of game theory, i.e. describe players, their actions, action profiles and payoffs. Find all pure strategy Nash equilibria in this game.

https://www.youtube.com/watch?v=U_eZmEiyTo0

Solution: This situation can be described as a two player static game (you can also use dynamic game concept):

Players: Westley, Vizzini
Actions:

- Westley: put poison to the cup close to Vizzini (C) or far from him (F).
- Vizzini: drink from the cup which is closer to him (C) or from the cup which is far from him (F).

Payoffs:

- Player to drink the poison loses the game and dies, e.g. payoff is -100
- Player to drink a good wine wins the game, e.g. 50

W \ V	C	F
C	50,-100	-100,50
F	-100,50	50,-100

There are no pure strategy Nash equilibria in this game.

Problem 4 [24 points]: Find strictly dominated strategies using also mixed strategies and use iterative elimination of strictly dominated strategies to simplify the following game as much as possible.

1 \ 2	A	B	C
D	1,2	2,4	4,1
E	3,3	2,2	1.5,1
F	4,3	4,2	1,4

Solution: We start with best responses and NE in pure strategies. Strategy that is best response to some of opponent's actions can never be eliminated.

1 \ 2	A	B	C
D	1,2	2,$\underline{4}$	$\underline{4}$,1
E	3,$\underline{3}$	2,2	1.5,1
F	$\underline{4}$,3	$\underline{4}$,2	1,$\underline{4}$

For the second player, all strategies (A, B, and C) are best responses to some of opponent's action. Only strategy that is never best response to other player's actions is E of the first player. This is our candidate for elimination. E is not strictly dominated by D or F only. So we are looking for such mixing of D and F that this mixed strategy strictly dominates E.

$$pD + (1 - p)F > E$$

Player 2: A: $p + 4(1 - p) > 3 \Rightarrow p < 1/3$

Player 2: B: $2p + 4(1 - p) > 2 \Rightarrow p < 1$

Player 2: C: $4p + 1(1 - p) > 1.5 \Rightarrow p > 1/6$

If we pick a random number p from interval (1/6,1/3) then mixed strategy $pD+(1-p)F$ gives a higher payoff than strategy *E*. Hence, *E* is strictly dominated and we can eliminate this strategy. For example, for p = 1/4 we have: 1/4D+3/4F = (3.25,3.5,1.75) while E yields payoffs (3,2,1.5). After the elimination of E our table representing the game is as follows.

1 \ 2	A	B	C
D	1,2	2,$\underline{4}$	$\underline{4}$,1
F	$\underline{4}$,3	$\underline{4}$,2	1,$\underline{4}$

Now, the only strategy that is never best response to any opponent's actions is A of the second player. No pure strategy strictly dominates *A*, we are looking for such value of p that $pB + (1 - p)C > A$:

$$pB + (1 - p)C > A$$

Player 1: D: $4p + 1(1 - p) > 2 \Rightarrow p > 1/3$

Player 1: F: $2p + 4(1 - p) > 3 \Rightarrow p < 1/2$

If we pick a random number p from interval (1/3,1/2) then mixed strategy pB +(1−p)C gives a higher payoff than strategy A. Hence, A is strictly dominated and we can eliminate this strategy. For example, for p = 5/12 we have: 5/12B+7/12C = (27/12,38/12) while A yields payoffs (2,3).

1 \ 2	B	C
D	2,$\underline{4}$	$\underline{4}$,1
F	$\underline{4}$,2	1,$\underline{4}$

Now, each action of each player is sometimes a best response and hence no more strategies can be eliminated.