Introduction to Game Theory
Lecture 9

Disclaimer: this presentation is only a supporting material and is not sufficient to master the topics covered during the lecture. Study of relevant books is strongly recommended.

Today

- Applications of NE and SPNE
  - Auctions
    - English Auction
    - Second-Price Sealed-Bid Auction
    - First-Price Sealed-Bid Auction

Auctions

- Used to allocate:
  - Art
  - Government bonds
  - Radio spectrum
- Forms:
  - Sequential bidding
  - Bid placed in sealed envelopes
- Application of Game Theoretic Approach:
  - Find most effective design

Auction 1

- Rules:
  - bidders sequentially submit increasing bids
  - person that made current bid wins if no one wishes to submit a higher bid
  - everybody knows their personal value of an object

    => before the bidding starts, every bidder knows their "maximal bid"

- this type of auction is called English Auction

Auction 1

- Bidder with the highest maximal bid wins
  - he pays the second maximal bid
  - only two bidders matter for the outcome
    - bidder with the highest maximal bid – B1
    - bidder with the 2nd highest maximal bid – B2
  - to win, B1 has to bid only "slightly" more than maximal bid of B2
  - if the bidding increment is small -> we take the winning price to be equal to the 2nd highest maximal bid

Auction 2

- Rules:
  - all bidders submit their bids simultaneously
  - all bidders place their bids in sealed envelopes
  - bidder with the highest bid wins
  - winning bidder pays the 2^{nd} highest bid

  - this type of auction is called Second Price Sealed Bid Auction

Auction 2

- Bidder with the highest maximal bid wins
  - he pays the second maximal bid
  - only two bidders matter for the outcome
    - bidder with the highest maximal bid – B1
    - bidder with the 2nd highest maximal bid – B2
  - to win, B1 has to bid only “slightly” more than maximal bid of B2
  - price is equal to the 2nd highest maximal bid

Auction 1 and Auction 2

- Auction 1 and Auction 2 lead to the same outcome:
  - winner is the same
  - winner pays the same price

Second Price Sealed Bid

Auction as a strategic game

- Players: n bidders
- Actions: all possible bids (non negative numbers)
- Payoffs: difference between value and second highest bid if win, zero otherwise

Second Price Sealed Bid

Notation:

- n players are ordered according to their valuations: $v_1 > v_2 > v_3 > \ldots > v_n$
- each player i submits a bid b_i
- if b_i is highest and b_j second highest bid:
  - bidder i gets $v_i - b_j$
  - all other bidders get zero

Second Price Sealed Bid

Nash Equilibrium 1:

- Every bidder bids their value:

  $(b_1, b_2, b_3, \ldots, b_n) = (v_1, v_2, v_3, \ldots, v_n)$

  - bidder 1, with value v_1, wins and pays b_2
  - bidder 1 has payoff $(v_1 - b_2)$
  - all the other bidders get zero

Second Price Sealed Bid

Every bidder bids their value is NE:
$(b_1, b_2, b_3, \ldots, b_n) = (v_1, v_2, v_3, \ldots, v_n)$

Winner – bidder 1:

- bid more – nothing changes
- bid less – lose and get nothing

=> winner has no incentive to deviate

Loser k:

- bid less – nothing changes
- bid more (less than v1) – nothing changes
- bid more ($b_k \geq v_1$) – win, but earn $v_k - b_k < 0$

=> losers have no incentive to deviate

Second Price Sealed Bid

Nash Equilibrium 2:

- First bidder bids his value, others bid zero: $(b_1,b_2,b_3,\ldots,b_n) = (v_1,0,0,\ldots,0)$
  - bidder 1, with value v_1, wins and pays 0
  - bidder 1 has payoff v_1
  - all the other bidders get zero

Second Price Sealed Bid

First bidder bids v_1, others bid zero is NE:
$(b_1,b_2,b_3,\ldots,b_n) = (v_1,0,0,\ldots,0)$

Winner – bidder 1:

- bid more – nothing changes
- bid less – nothing changes

=> winner has no incentive to deviate

Loser k:

- bid less – not possible
- bid more (less than v_1) – nothing changes
- bid more ($b_k \geq v_1$) – win, but earn $v_k - b_k < 0$

=> losers have no incentive to deviate

Second Price Sealed Bid

Nash Equilibrium 3:

- Bidders bid in the following way: $(b_1,b_2,b_3,\ldots,b_n) = (v_2,v_1,0,\ldots,0)$
  - bidder 2, with value v_2, wins and pays v_2
  - bidder 2 has payoff 0
  - all the other bidders get zero

Second Price Sealed Bid

Bidders bidding in the following way is NE:

$(b_1, b_2, b_3, \ldots, b_n) = (v_2, v_1, 0, \ldots, 0)$

Winner – bidder 2:

- bid more – nothing changes
- bid less (still more than v_2) – nothing changes
- bid less (less than v_2) – lose and get nothing

=> winner has no incentive to deviate

Loser k:

- bid less – nothing changes
- bid more (less than v_1) – nothing changes
- bid more ($b_k \geq v_1$) – win, but earn $v_k - b_k \leq 0$

=> losers have no incentive to deviate

Second Price Sealed Bid

NE 3 - $(b_1,b_2,b_3,\ldots,b_n) = (v_2,v_1,0,\ldots,0)$:

- bidder 1 has to believe that bidder 2 will continue bidding up to v_1, then bidding v_2 is best response
- bidder 2 is taking risk of negative payoff if bidder 1 bids more than v_2
- still, given that all bidders bid according to NE 3, everybody is playing the best response
- for bidder two, bidding v_1 is weakly dominated by bidding v_2

In general: in a second-price sealed-bid auction, a player's bid equal to her valuation weakly dominates all her other bids

Second Price Sealed Bid

Many Nash Equilibria, but one is special:

- NE where every bidder bids her value $(b_1,b_2,b_3,\ldots,b_n) = (v_1,v_2,v_3,\ldots,v_n)$ is the only one where every player's action weakly dominates all her other actions

Auction 3

- Two players participate in the English auction (bidders sequentially submit increasing bids) for 100CZK banknote
- Person that made current bid wins if no one wishes to submit a higher bid
- <u>BOTH bidders must pay the highest amount they bid</u>

- NE: no Nash equilibria in pure strategies in static form of this auction

Auction 4

- all bidders submit their bids simultaneously
- all bidders place their bids in sealed envelopes
- bidder with the highest bid wins
- winning bidder pays her own bid

- this type of auction is called
  First Price Sealed Bid Auction

First Price Sealed Bid

- Winner pays the price she bids, not the second highest price
- We assume games with perfect information, i.e. everybody knows value of all bidders

- Players: n bidders
- Actions: all possible bids (non negative numbers)
- Payoffs: difference between value and bid if win, zero otherwise

First Price Sealed Bid

Nash Equilibrium 1:

- Bidders bid in the following way: $(b_1,b_2,b_3,\ldots,b_n) = (v_2,v_2,v_3,\ldots,v_n)$
  - bidder 1, with value v1, wins and pays v_2
  - bidder 1 has payoff (v_1-v_2)
  - all the other bidders get zero

First Price Sealed Bid

Bidders bid in the following way is NE:
$(b_1,b_2,b_3,\ldots,b_n) = (v_2,v_2,v_3,\ldots,v_n)$

Winner – bidder 1:

- bid more – still win, pay more
- bid less – lose and get nothing

=> winner has no incentive to deviate

Loser k:

- bid less – nothing changes
- bid more (less than v_2) – nothing changes
- bid more ($b_k \geq v_2$) – win, but earn $v_k - b_k < 0$

=> losers have no incentive to deviate

First Price Sealed Bid

- First-price sealed-bid auction has many NE
- In all of them, bidder 1 wins the auction

- First-price sealed-bid auction where bidders bid $(b_1,b_2,b_3,\ldots,b_n) = (v_2,v_2,v_3,\ldots,v_n)$ yields the same outcome as Second-price sealed-bid auction

Note: usually we do not know the value of other bidders -> we use expected value

Summary

- Game theoretic approach and concept of Nash equilibrium has many useful applications
- Auctions – Nash equilibrium concept helps to determine the winner and the return for the owner of the object being sold
- This allow us to compare different types of auctions in terms of price paid
- In case of perfect information – First- and Second-price sealed-bid auction yields the same results

Midterm Exam

Histogram – points on horizontal and number of students on vertical axis