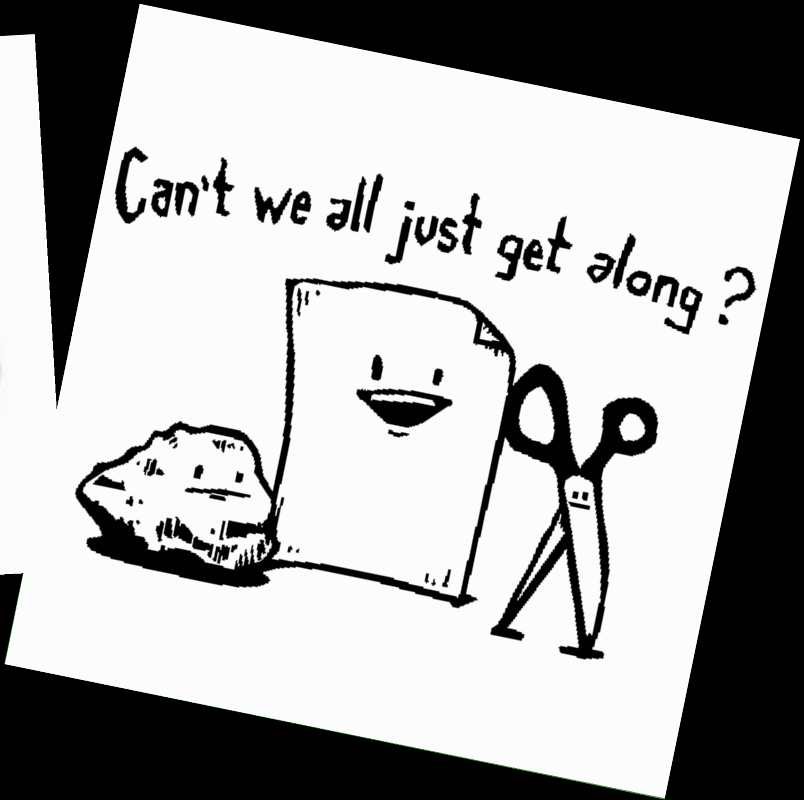


Introduction to Game Theory
Lecture 3

Disclaimer: this presentation is only a supporting material and is not sufficient to master the topics covered during the lecture. Study of relevant books is strongly recommended.

Syllabus

- Contact: kalk00@vse.cz
  home.cerge-ei.cz/kalovcova/teaching.html
- Office hours: Wed 7.30pm – 8.00pm, NB339
  or by email appointment
- Osborne, M. J. – An Introduction to Game Theory
  Gibbons, R. – A Primer in Game Theory
  Suggested articles
- Important information on webpage
- Grading: Midterm 30%, Final 60%,
  Homework 10%, Experiments up to 5%

NE - Review

- Nash Equilibrium is a concept of a *steady state* in given situation
- No one can *unilaterally* improve their payoff, therefore no one has incentive to deviate from equilibrium action
- Nash Equilibrium is an action profile in which every player's action is best response to every other player's action

NE - Review

- No player wishes to change her behavior, knowing the other players' behavior => there are no regrets
- Equilibrium behavior is based on general knowledge and experience with similar players and situations; not on particular circumstances

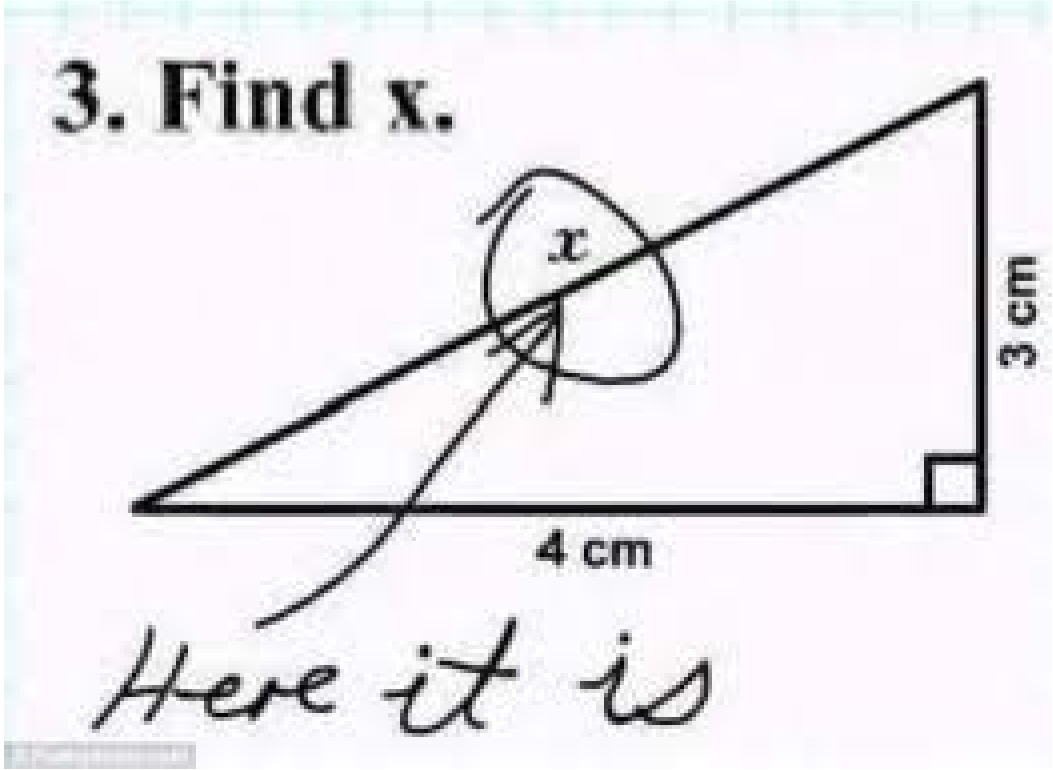
NE - Review

- We can find Nash equilibria by:
  - Elimination of strictly dominated strategies
  - “Circle Method”
- Elimination of weakly dominated strategies leads to:
  - strict Nash equilibria
  - but can eliminate nonstrict Nash equilibria
- That is why we only eliminate strictly dominated strategies
- Elimination method is sometimes imprecise, NE (Circle Method, Best responses) is stronger.

Preview

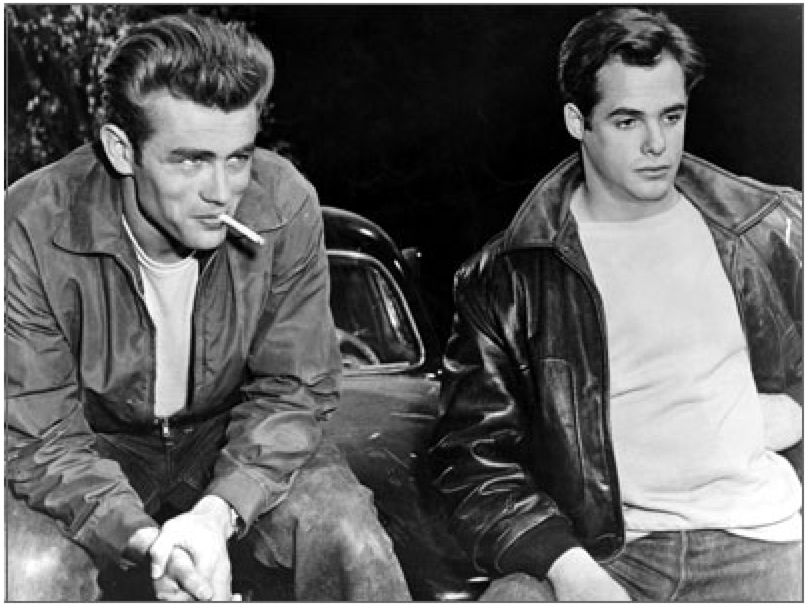
- Best response functions (why circles work)
- Mixed strategies Nash equilibrium

Best Response



Chicken Game

- Jim and Buzz are driving cars towards each other
- Who turns first is a chicken
- If nobody turns the car, they both die…



Rebel Without a Cause (1955)

Chicken Game

- Two players, two actions => 2 by 2 game
- First player has following preferences:
  (Stay,Turn)>(Turn,Turn)>(Turn,Stay)>(Stay,Stay)
  Situation for second player is analogical
- Assign payoff correspondingly: 20,5,0,-100
  (e.g. 4,3,2,1 would work just as well)
  Chicken game:

Jim \ Buzz	Turn	Stay
Turn	5,5	0,20
Stay	20,0	-100,-100

Chicken Game - BR

Chicken game:

Jim \ Buzz	Turn	Stay
Turn	5,5	0,20
Stay	20,0	-100,-100

Two Nash Equilibria: {Stay,Turn}, {Turn,Stay}

Best response:

$BR_1(T)=S$; $BR_1(S)=T$; $BR_2(T)=S$; $BR_2(S)=T$;

Best Response Function

Why does the “method of circles” work?
Because “circles” are best response functions!

$BR_i(a_{-i}) = \{a_i \text{ in } A_i : u_i(a_i,a_{-i}) \geq u_i(a'_i,a_{-i}) \text{ for all } a'_i \text{ in } A_i\}$

Every member of the set $BR_i(a_{-i})$ is a best response of player i to a_{-i}:
if each of the other players adheres to a_{-i} then player i can do no better than choose a member of $BR_i(a_{-i})$

Best Response Function - NE

The action profile a^* is a Nash equilibrium if and only if every player's action is a best response to the other players' actions:

$$a^*_i \text{ is in } B_i(a^*_{-i}) \text{ for every player } i$$

This is why "method of circles", i.e. looking for best responses leads to NE

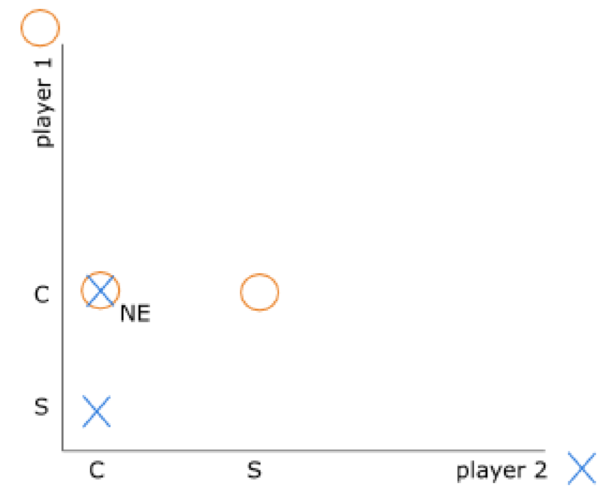
NE - Review

- Nash Equilibrium is a concept of a *steady state* in given situation
- No one can *unilaterally* improve their payoff, therefore no one has incentive to deviate from equilibrium action
- Nash Equilibrium is an action profile in which every player's action is best response to every other player's action

Best Response Function

Prisoners' Dilemma Game:

1 \ 2	Confess	Silent
Confess	1,1	3,0
Silent	0,3	2,2

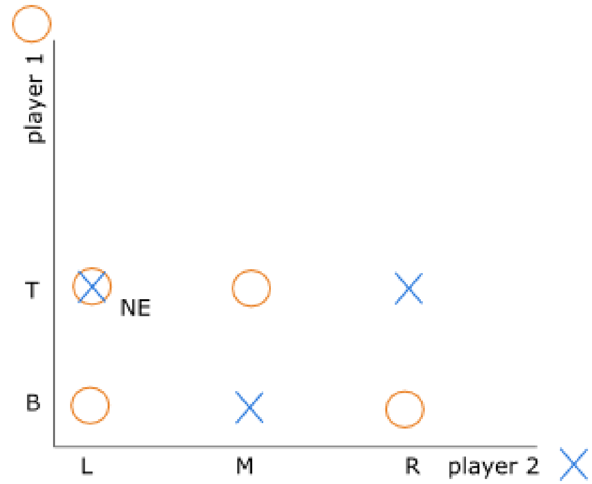


$BR_1(C) = \{C\}$ -> single optimal action
$BR_1(RS) = \{C\}$
$BR_2(C) = \{C\}$ -> single optimal action
$BR_2(RS) = \{C\}$

Best Response Function

Yet another game:

1 \ 2	L	M	R
T	1,1	1,0	0,1
B	1,0	0,1	1,0



$BR_1(L) = \{T,B\}$ -> more optimal actions

$BR_1(M) = \{T\}$, $BR_1(R) = \{B\}$

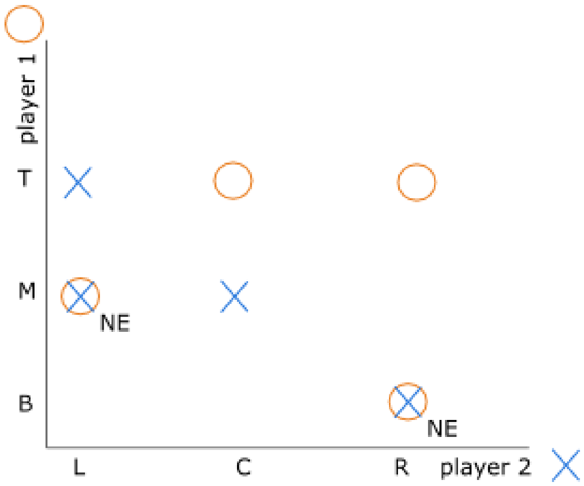
$BR_2(T) = \{L,R\}$ -> more optimal actions

$BR_2(B) = \{M\}$

Best Response Function

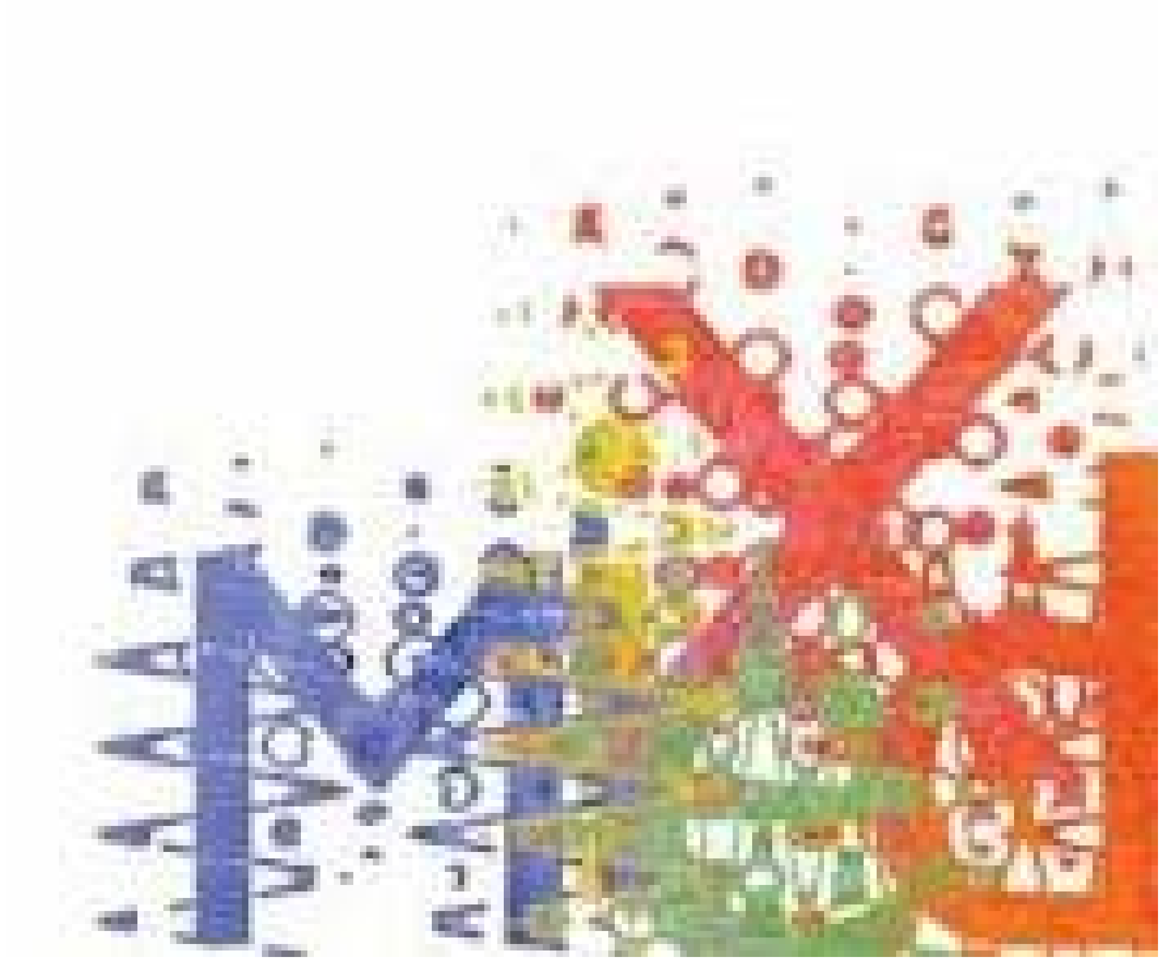
Yet another another game:

2 / 1	L	C	R
T	1,2	2,1	1,0
M	2,1	0,1	0,0
B	0,1	0,0	1,2



$BR_1(L)=\{M\}$, $BR_1(C)=\{T\}$, $BR_1(R)=\{T,B\}$
$BR_2(T)=\{L\}$, $BR_2(M)=\{L,C\}$, $BR_2(B)=\{R\}$

Mixed Strategies



Preview

- So far, in NE, behavior of each player is simply one action that she always plays

Today – “mixing things up”

- Players’ choices may vary:
    - different members of a population choose different actions
    - each member of a population chooses her action according to a probabilistic distribution

Mixed Strategies - Examples

- penalty kick
- rock paper scissor game
- matching pennies
- price wars (duopoly)
- card games

- need for making oneself unpredictable leads to mixing strategies

Mixed Strategies - Example

Matching Pennies:

1 \ 2	Head	Tail
Head	$1,-$1	-$1,$1
Tail	-$1,$1	$1,-$1

- no Nash Equilibria, no pair of actions is compatible with a steady state
- there exists steady state in which each player chooses each action with probability ½

Mixed Strategies - Example

- player 2 plays H and T with probability ½:
- for player 1:
  - expected payoff from playing H:
    ½ (1) + ½ (-1) = 0
  - expected payoff from playing T:
    ½ (-1) + ½ (1) = 0

Player 1: playing H and T with probability ½ is her best response (she can not do any better)

The same holds for Player 2

Mixed Strategies - Example

- both players play H and T with probability ½
- both play their best response given the action of their opponent
- none of them wants to change their strategy

=> {(½,½);(½,½)} is MSNE
numbers in brackets correspond to probabilities of playing H and T respectively

Mixed Strategies - Definition

Mixed strategy: player chooses a probability distribution $(p_1,p_2,..,p_N)$ over her set of actions

- e.g. (½,¼,¼) is mixed strategy where player plays "L" with probability ½ and "M" and "R" with probability ¼

- probabilities have to sum up to 1!
- mixed strategy may assign probability 1 to a single action – pure strategy
  - e.g. (0,0,1) is pure strategy where player always plays "R"

Mixed Strategy NE

Notation:

- a_i – action of i^{th} player
- a – action profile (set of all players' actions)
- $a_{-i} = (a_1, a_2, a_3, \dots, a_{i-1}, a_{i+1}, \dots, a_{N-2}, a_{N-1}, a_N)$

- $\alpha_i = (p_1, p_2, p_3, .., p_N)$ - mixed strategy of player i
- α – mixed strategy profile (set of all players' mixed strategies)
- $\alpha_{-i} = (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{N-2}, \alpha_{N-1}, \alpha_N)$

Mixed Strategy NE

- A Mixed Strategy Nash Equilibrium (MSNE) is a mixed strategy profile α^* such that no player i has a mixed strategy α_i such that she prefers $(\alpha_i, \alpha^*_{-i})$ to α^*
- i.e. expected payoff of α^* is at least as large as expected payoff of $(\alpha_i, \alpha^*_{-i})$ for every α_i:

  $EU(\alpha_i) \geq EU(\alpha_i, \alpha^*_{-i})$ for every α_i of player i

- α^* is a MSNE if and only if α^*_i is in $B_i(\alpha^*_{-i})$ for every player i

Mixed Strategy NE – How to Find

1 \ 2	H (q)	T (1-q)
H (p)	$1,-$1	-$1,$1
T (1-p)	-$1,$1	$1,-$1

- 1 playing H: q*1+(1-q)*(-1)=2q-1
- 1 playing T: q*(-1)+(1-q)*1=1-2q

- If q<½: T is better than H
- If q>½: H is better than T
- If q=½: H is equally good as T
- same holds for p => {(½,½);(½,½)} is MSNE

Mixed Strategy NE

- If there is no NE without mixing, we will find at least one MSNE (Nash - proof)
- If NE without mixing exists, we may find additional MSNE

vNM Preferences

- so far, payoff function had only ordinal meaning, now there is more…
- von Neumann and Morgenstern (vNM preferences)
- preferences regarding lotteries can be represented by the expected value of the payoff function
- each player prefers lottery with higher expected value of a payoff function
- so far – players were maximizing their payoff; now – maximize expected payoff

vNM Preferences

- consider the following tables:

1 \ 2	C	RS
C	1,1	3,0
RS	0,3	2,2

1 \ 2	C	RS
C	1,1	4,0
RS	0,4	3,3

- these tables represent the same game with ordinal preferences: (C,RS)>(RS,RS)>(C,C)>(RS,C)
- these tables represent different games with vNM preferences: e.g. compare sure outcome (RS,RS) with lottery ½*(C,C) + ½*(C,RS)

  2 = ½*1 + ½*3 3 > ½*1 + ½*4

vNM Preferences

Strategic game with vNM preferences:

- set of players
- for each player, a set of actions
- for each player, preferences regarding lotteries over action profiles can be represented by the expected value of the payoff function over action profiles

Note: if NE – ordinal; if MSNE – vNM preferences

Summary

- von Neumann and Morgenstern preferences
- preferences regarding lotteries can be represented by the expected value of the payoff function
- Mixed strategy: player chooses a probability distribution $(p_1,p_2,..,p_N)$ over her set of actions rather than a single action
- α^* is a MSNE if and only if $EU(\alpha_i) \geq EU(\alpha_i,\alpha^*_{-i})$ for every α_i of player *I*